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SOME HARDY TYPE INEQUALITIES IN THE HEISENBERG GROUP

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Abstract

Some Hardy type inequalities on the domain in the Heisenberg group are established by using the Picone type identity and constructing suitable auxiliary functions.

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Contents

1. Introduction

The Hardy inequality in the Euclidean space (see [3], [4], [7]) has been established using many methods. In [1], Allegretto and Huang found a Picone's identity for the p-Laplacian and pointed out that one can prove the Hardy inequality via the identity. Niu, Zhang and Wang in [6] obtained a Picone type identity for the p-sub-Laplacian in the Heisenberg group and then established a Hardy type inequality. When $p = 2$, the result of [6] coincides with the inequality in [2]. As stated in [1], the Picone type identity allows us to avoid postulating regularity conditions on the boundary of the domain under consideration. Since there is a presence of characteristic points in the sub-Laplacian Dirichlet problem in the Heisenberg group (see [2]), we understand that such an identity is especially useful.

We recall that the Heisenberg group $\mathcal{H}_n$ of real dimension $N = 2n+1$, $n \in \mathcal{N}$, is the nilpotent Lie group of step two whose underlying manifold is $\mathcal{R}^{2n+1}$. A basis for the Lie algebra of left invariant vector fields on $\mathcal{H}_n$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \qquad j = 1, 2, \dots, n.$$

The number $Q = 2n + 2$ is the homogeneous dimension of $\mathcal{H}_n$. There exists a Heisenberg distance

$$d\left((z,t),(z',t')\right) = \left\{\left[(x-x')^2 + (y-y')^2\right]^2 + \left[t - t' - 2(x\cdot y' - x'\cdot y)\right]^2\right\}^{\frac{1}{4}}$$

between (z,t) and (z',t'). We denote the Heisenberg gradient by

$$\nabla_{\mathcal{H}_n} = (X_1, \dots, X_n, Y_1, \dots, Y_n).$$

In this note we give some Hardy type inequalities on the domain in the Heisenberg group by considering different auxiliary functions.

2. Hardy Inequalities

First we state two lemmas given in [6] which will be needed in the sequel.

Lemma 2.1. *Let Ω be a domain in $\mathcal{H}_n$, $v > 0$, $u \geq 0$ be differentiable in Ω. Then*

$$L(u,v) = R(u,v) \geq 0, \tag{2.1}$$

where

$$L(u,v) = |\nabla_{H_n} u|^p + (p-1)\frac{u^p}{v^p}|\nabla_{H_n} v|^p - p\frac{u^{p-1}}{v^{p-2}}\nabla_{H_n} \cdot |\nabla_{H_n} v|^{p-2}\nabla_{H_n} v,$$

$$R(u,v) = |\nabla_{H_n} u|^p - \nabla_{H_n}\left(\frac{u^p}{v^{p-1}}\right) \cdot |\nabla_{H_n} v|^{p-2}\nabla_{H_n} v.$$

Denote the p-sub-Laplacian by $\Delta_{H_n,p} v = \nabla_{H_n} \cdot (|\nabla_{H_n} v|^{p-2}\nabla_{H_n} v)$.

Lemma 2.2. *Assume that the differentiable function $v > 0$ satisfies the condition $-\Delta_{H_n,p} v \geq \lambda g v^{p-1}$, for some $\lambda > 0$ and nonnegative function g. Then for every $u \in C_0^\infty(\Omega)$, $u \geq 0$,*

$$\int_\Omega |\nabla_{H_n} u|^p \geq \lambda \int_\Omega g|u|^p. \tag{2.2}$$

Let $B_R = \{(z,t) \in H_n |\, d((z,t),(0,0)) < R\}$ be the Heisenberg group and $\delta(z,t) = \operatorname{dist}((z,t), \partial B_R)\,, (z,t) \in B_R$, in the sense of distance functions on the Heisenberg group.

Theorem 2.3. *Let* $\Omega = B_R \backslash \{(0,0\}, p > 1$. *Then for every* $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla_{H_n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|z|^p}{d^p}\frac{|u|^p}{\delta^p}, \tag{2.3}$$

where $|z| = \sqrt{x^2+y^2}$, $d = d\left((z,t),(0,0)\right)$.

Proof. We first consider $u \geq 0$. The following equations are evident:

$$\begin{cases} X_j d = d^{-3}\left(|z|^2 x_j + y_j t\right), \; Y_j d = d^{-3}\left(|z|^2 y_j - x_j t\right), \\ X_j^2 d = -3d^{-7}\left(|z|^2 x_j + y_j t\right)^2 + d^{-3}\left(|z|^2 + 2x_j^2 + 2y_j^2\right), \\ Y_j^2 d = -3d^{-7}\left(|z|^2 y_j - x_j t\right)^2 + d^{-3}\left(|z|^2 + 2x_j^2 + 2y_j^2\right), \\ \hfill j = 1,\dots,n \end{cases} \tag{2.4}$$

and

$$|\nabla_{H_n} d| = |z|d^{-1}, \quad \Delta_{H_n} d = (Q-1)d^{-3}|z|^2. \tag{2.5}$$

Choose $v(z,t) = \delta(z,t)^\beta = (R-d)^\beta$, in which $\beta = \frac{p-1}{p}$, one has

$$X_j v = -\beta\delta^{\beta-1}X_j d, \quad Y_j v = -\beta\delta^{\beta-1}Y_j d, \quad j = 1,\dots,n,$$
$$\nabla_{H_n} v = -\beta\delta^{\beta-1}\nabla_{H_n} d, \quad |\nabla_{H_n} v| = |\beta|\delta^{\beta-1}|z|d^{-1},$$

and

$$\begin{aligned} -\Delta_{H_n} v &= -\nabla_{H_n}\cdot\left(|\nabla_{H_n} v|^{p-2}\nabla_{H_n} v\right) \\ &= -\nabla_{H_n}\cdot\left(-\beta|\beta|^{p-2}\delta^{(\beta-1)(p-1)}|z|^{p-2}d^{2-p}\nabla_{H_n} d\right) \end{aligned}$$

$$
\begin{aligned}
&= \beta|\beta|^{p-2}\Big\{ -(\beta-1)(p-1)\delta^{(\beta-1)(p-1)-1}|z|^{p-2}d^{2-p}|\nabla_{H_n}d|^2 \\
&\qquad + \delta^{(\beta-1)(p-1)}d^{2-p}\nabla_{H_n}\left(|z|^{p-2}\right)\cdot\nabla_{H_n}d \\
&\qquad + (2-p)\delta^{(\beta-1)(p-1)}|z|^{p-2}d^{1-p}|\nabla_{H_n}d|^2 \\
&\qquad + \delta^{(\beta-1)(p-1)}|z|^{p-2}d^{2-p}\triangle_{H_n}d\Big\}.
\end{aligned}
$$

From the fact $\nabla_{H_n}(|z|^{p-2})\cdot\nabla_{H_n}d = (p-2)|z|^{p-4}d^{-3}|z|^4 = (p-2)|z|^p d^{-3}$ and (2.5), it follows that

$$
\begin{aligned}
-\Delta_{H_n}v &= \beta|\beta|^{p-2}\Big\{ -(\beta-1)(p-1)\delta^{(\beta-1)(p-1)-1}|z|^p d^{-p} \\
&\qquad + (p-2)\delta^{(\beta-1)(p-1)}|z|^p d^{-1-p} \\
&\qquad - (p-2)\delta^{(\beta-1)(p-1)}|z|^p d^{-1-p} \\
&\qquad + (Q-1)\delta^{(\beta-1)(p-1)}|z|^p d^{-1-p}\Big\} \\
&= \beta|\beta|^{p-2}\left\{-(\beta-1)(p-1)+(Q-1)\frac{\delta}{d}\right\}\frac{|z|^p}{d^p}\frac{v^{p-1}}{\delta^p} \\
&= \left(\frac{p-1}{p}\right)^{p-1}\left\{\frac{p-1}{p}+(Q-1)\frac{\delta}{d}\right\}\frac{|z|^p}{d^p}\frac{v^{p-1}}{\delta^p} \\
&\geq \left(\frac{p-1}{p}\right)^{p}\frac{|z|^p}{d^p}\frac{v^{p-1}}{\delta^p}.
\end{aligned}
$$

The desired inequality (2.3) is obtained by Lemma 2.2. For general u, by letting $u = u^+ - u^-$, we directly obtain (2.3). □

Theorem 2.4. *Let $\Omega = H_n \backslash \{B_{H_n,R}\}$, $Q > p > 1$. Then for every $u \in C_0^\infty(\Omega)$, there exists a constant $C > 0$, such that*

$$\int_\Omega |\nabla_{H_n} u|^p \geq C \int_\Omega \frac{|z|^p}{d^p} \frac{|u|^p}{d^{2p}}. \tag{2.6}$$

Proof. Suppose that $u \geq 0$. Take $v = \log\left(\frac{d}{R}\right)^\alpha$, $R < d = d((z,t),(0,0)) < +\infty$, $\alpha < 0$. Using (2.4) and (2.5) show that

$$\nabla_{H_n} v = \left(\frac{R}{d}\right)^\alpha \alpha \left(\frac{d}{R}\right)^{\alpha-1} \frac{1}{R} \nabla_{H_n} d = \frac{\alpha}{d} \nabla_{H_n} d,$$
$$|\nabla_{H_n} v| = |\alpha||z|d^{-2},$$

$$\begin{aligned}
-\triangle_{H_n} v &= -\nabla_{H_n} \cdot \left(|\nabla_{H_n} v|^{p-2} \nabla_{H_n} v\right) \\
&= -\alpha|\alpha|^{p-2} \nabla_{H_n} \cdot \left(|z|^{p-2} d^{-2(p-2)-1} \nabla_{H_n} d\right) \\
&= -\alpha|\alpha|^{p-2} \Big\{ (p-2)|z|^{p-3} d^{2(2-p)-1} \nabla_{H_n}(|z|) \cdot \nabla_{H_n} d \\
&\qquad + (2(2-p)-1)\, |z|^{p-2} d^{2(1-p)} |\nabla_{H_n} d|^2 \\
&\qquad + |z|^{p-2} d^{2(2-p)-1} \triangle_{H_n} d \Big\}.
\end{aligned}$$

Since $\nabla_{H_n}(|z|) \cdot \nabla_{H_n} d = |z|^3 d^{-3}$, the last equation above becomes

$$\begin{aligned}
-\triangle_{H_n} v = -\alpha|\alpha|^{p-2} \Big\{ &(p-2)|z|^{p-3} d^{2(2-p)-1} |z|^3 d^{-3} \\
&+ (3-2p)|z|^{p-2} d^{2(1-p)} |z|^2 d^{-2} \\
&+ (Q-1)|z|^{p-2} d^{3-2p} |z|^2 d^{-3} \Big\}
\end{aligned}$$

$$= -\alpha|\alpha|^{p-2}|z|^p d^{-2p}(p - 2 + 3 - 2p + Q - 1)$$

(2.7) $$= -\alpha|\alpha|^{p-2}(Q - p)|z|^p d^{-2p}.$$

Noting

$$\lim_{d\to+\infty} \frac{v^{p-1}}{d^p} = 0,$$

there exists a positive number $M \geq R$, such that $\frac{v^{p-1}}{d^p} < 1$, for $d > M$. Since $\frac{v^{p-1}}{d^p}$ is continuous on the interval $[R, M]$, we find a constant $C' > 0$, such that $\frac{v^{p-1}}{d^p} < C'$. Pick out $C'' = \max\{C', 1\}$ and one has $v^{p-1} < C''d^p$ in Ω. This leads to the following

$$-\Delta_{H_n} v \geq C \frac{|z|^p}{d^{2p}} \frac{v^{p-1}}{d^p},$$

where $C = \frac{-\alpha|\alpha|^{p-2}(Q-p)}{C''}$, and to (2.6) by Lemma 2.2. A similar treatment for general u completes the proof. □

In particular, $\alpha = p - Q$ $(1 < p < Q)$ satisfies the assumption in the proof above.

Theorem 2.5. *Let Ω be as defined in Theorem 2.4 and $p \geq Q$. Then there exists a constant $C > 0$, such that for every $u \in C_0^\infty(\Omega)$,*

(2.8) $$\int_\Omega |\nabla_{H_n} u|^p \geq C \int_\Omega \frac{|z|^p}{d^p \left(\log\left(\frac{d}{R}\right)\right)^p} \frac{|u|^p}{d^p}.$$

Proof. It is sufficient to show that (2.8) holds for $u \geq 0$. Choose $v = \phi^\alpha$, $\phi = \log \frac{d}{R}$, where $R < d < +\infty$, $0 < \alpha < 1$. We know that from (2.4) and (2.5),

$$\nabla_{H_n}\phi = d^{-1}\nabla_{H_n} d,\ |\nabla_{H_n}\phi| = d^{-2}|z|,$$
$$\Delta_{H_n}\phi = d^{-1}\Delta_{H_n} d - d^{-2}|\nabla_{H_n} d|^2 = (Q - 2)|z|^2 d^{-4}.$$

This allows us to obtain

$$\begin{aligned}
-\Delta_{H_n} v &= -\nabla_{H_n}\cdot\left(|\nabla_{H_n} v|^{p-2}\nabla_{H_n} v\right)\\
&= -\nabla_{H_n}\cdot\left(|\alpha\phi^{\alpha-1}\nabla_{H_n}\phi|^{p-2}\alpha\phi^{\alpha-1}\nabla_{H_n}\phi\right)\\
&= -\alpha|\alpha|^{p-2}\nabla_{H_n}\cdot\left(\phi^{(\alpha-1)(p-1)}|z|^{p-2}d^{2(2-p)}\nabla_{H_n}\phi\right)\\
&= -\alpha|\alpha|^{p-2}\Big\{(\alpha-1)(p-1)\phi^{(\alpha-1)(p-1)-1}|z|^{p-2}d^{2(2-p)}|\nabla_{H_n}\phi|^2\\
&\qquad + (p-2)\phi^{(\alpha-1)(p-1)}|z|^{p-3}d^{2(2-p)}\nabla_{H_n}(|z|)\cdot\nabla_{H_n}\phi\\
&\qquad + 2(2-p)\phi^{(\alpha-1)(p-1)}|z|^{p-2}d^{2(2-p)-1}\nabla_{H_n} d\cdot\nabla_{H_n}\phi\\
&\qquad + \phi^{(\alpha-1)(p-1)}|z|^{p-2}d^{2(2-p)}\triangle_{H_n}\phi\Big\}\\
&= -\alpha|\alpha|^{p-2}\Big\{(\alpha-1)(p-1)\phi^{(\alpha-1)(p-1)-1}|z|^{p-2}d^{2(2-p)}|z|^2 d^{-4}\\
&\qquad + (p-2)\phi^{(\alpha-1)(p-1)}|z|^{p-3}d^{2(2-p)}|z|^3 d^{-4}\\
&\qquad + 2(2-p)\phi^{(\alpha-1)(p-1)}|z|^{p-2}d^{2(2-p)-1}|z|^2 d^{-3}\\
&\qquad + \phi^{(\alpha-1)(p-1)}|z|^{p-2}d^{2(2-p)}(Q-2)|z|^2 d^{-4}\Big\}\\
&= -\alpha|\alpha|^{p-2}\frac{v^{p-1}}{\phi^p}\frac{|z|^p}{d^{2p}}\left\{(\alpha-1)(p-1)+(p-2)\phi\right.\\
&\qquad \left.+2(2-p)\phi+(Q-2)\phi\right\}\\
&= -\alpha|\alpha|^{p-2}\frac{v^{p-1}}{\phi^p}\frac{|z|^p}{d^{2p}}\left\{(\alpha-1)(p-1)+(Q-p)\phi\right\},
\end{aligned}$$

Taking into account that $0 < \alpha < 1$ and $p \geq Q$, we have

$$-\alpha|\alpha|^{p-2}(Q-p)\phi \geq 0,$$

and therefore

$$-\triangle_{H_n} v \geq -\alpha|\alpha|^{p-2}(\alpha-1)(p-1)\frac{v^{p-1}}{\phi^p}\frac{|z|^p}{d^{2p}} = C\frac{v^{p-1}}{\phi^p}\frac{|z|^p}{d^{2p}},$$

where $C = -\alpha|\alpha|^{p-2}(\alpha-1)(p-1)$. An application of Lemma 2.2 completes the proof of Theorem 2.5. □

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