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## SOME HARDY TYPE INEQUALITIES IN THE HEISENBERG GROUP

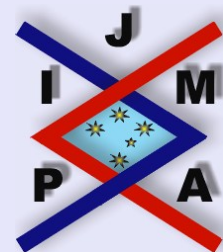
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Abstract

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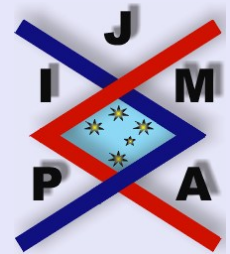
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## Abstract

Some Hardy type inequalities on the domain in the Heisenberg group are established by using the Picone type identity and constructing suitable auxiliary functions.

*2000 Mathematics Subject Classification:* 35H20.

*Key words:* Hardy inequality, Picone identity, Heisenberg group.

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# 1. Introduction

The Hardy inequality in the Euclidean space (see [3], [4], [7]) has been established using many methods. In [1], Allegretto and Huang found a Picone's identity for the  $p$ -Laplacian and pointed out that one can prove the Hardy inequality via the identity. Niu, Zhang and Wang in [6] obtained a Picone type identity for the  $p$ -sub-Laplacian in the Heisenberg group and then established a Hardy type inequality. When  $p = 2$ , the result of [6] coincides with the inequality in [2]. As stated in [1], the Picone type identity allows us to avoid postulating regularity conditions on the boundary of the domain under consideration. Since there is a presence of characteristic points in the sub-Laplacian Dirichlet problem in the Heisenberg group (see [2]), we understand that such an identity is especially useful.

We recall that the Heisenberg group  $\mathcal{H}_n$  of real dimension  $N = 2n + 1$ ,  $n \in \mathcal{N}$ , is the nilpotent Lie group of step two whose underlying manifold is  $\mathcal{R}^{2n+1}$ . A basis for the Lie algebra of left invariant vector fields on  $\mathcal{H}_n$  is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \dots, n.$$

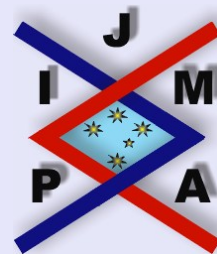
The number  $Q = 2n + 2$  is the homogeneous dimension of  $\mathcal{H}_n$ . There exists a Heisenberg distance

$$d((z, t), (z', t')) = \left\{ [(x - x')^2 + (y - y')^2]^2 + [t - t' - 2(x \cdot y' - x' \cdot y)]^2 \right\}^{\frac{1}{4}}$$

between  $(z, t)$  and  $(z', t')$ . We denote the Heisenberg gradient by

$$\nabla_{\mathcal{H}_n} = (X_1, \dots, X_n, Y_1, \dots, Y_n).$$

In this note we give some Hardy type inequalities on the domain in the Heisenberg group by considering different auxiliary functions.



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## 2. Hardy Inequalities

First we state two lemmas given in [6] which will be needed in the sequel.

**Lemma 2.1.** *Let  $\Omega$  be a domain in  $\mathcal{H}_n$ ,  $v > 0$ ,  $u \geq 0$  be differentiable in  $\Omega$ . Then*

$$(2.1) \quad L(u, v) = R(u, v) \geq 0,$$

where

$$L(u, v) = |\nabla_{H_n} u|^p + (p-1) \frac{u^p}{v^p} |\nabla_{H_n} v|^p - p \frac{u^{p-1}}{v^{p-2}} \nabla_{H_n} \cdot |\nabla_{H_n} v|^{p-2} \nabla_{H_n} v,$$

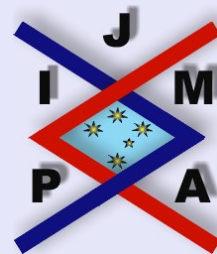
$$R(u, v) = |\nabla_{H_n} u|^p - \nabla_{H_n} \left( \frac{u^p}{v^{p-1}} \right) \cdot |\nabla_{H_n} v|^{p-2} \nabla_{H_n} v.$$

Denote the  $p$ -sub-Laplacian by  $\Delta_{H_n, p} v = \nabla_{H_n} \cdot (|\nabla_{H_n} v|^{p-2} \nabla_{H_n} v)$ .

**Lemma 2.2.** *Assume that the differentiable function  $v > 0$  satisfies the condition  $-\Delta_{H_n, p} v \geq \lambda g v^{p-1}$ , for some  $\lambda > 0$  and nonnegative function  $g$ . Then for every  $u \in C_0^\infty(\Omega)$ ,  $u \geq 0$ ,*

$$(2.2) \quad \int_{\Omega} |\nabla_{H_n} u|^p \geq \lambda \int_{\Omega} g |u|^p.$$

Let  $B_R = \{(z, t) \in H_n \mid d((z, t), (0, 0)) < R\}$  be the Heisenberg group and  $\delta(z, t) = \text{dist}((z, t), \partial B_R)$ ,  $(z, t) \in B_R$ , in the sense of distance functions on the Heisenberg group.



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**Theorem 2.3.** Let  $\Omega = B_R \setminus \{(0, 0)\}$ ,  $p > 1$ . Then for every  $u \in C_0^\infty(\Omega)$ ,

$$(2.3) \quad \int_{\Omega} |\nabla_{H_n} u|^p \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|z|^p |u|^p}{d^p \delta^p},$$

where  $|z| = \sqrt{x^2 + y^2}$ ,  $d = d((z, t), (0, 0))$ .

*Proof.* We first consider  $u \geq 0$ . The following equations are evident:

$$(2.4) \quad \begin{cases} X_j d = d^{-3} (|z|^2 x_j + y_j t), & Y_j d = d^{-3} (|z|^2 y_j - x_j t), \\ X_j^2 d = -3d^{-7} (|z|^2 x_j + y_j t)^2 + d^{-3} (|z|^2 + 2x_j^2 + 2y_j^2), \\ Y_j^2 d = -3d^{-7} (|z|^2 y_j - x_j t)^2 + d^{-3} (|z|^2 + 2x_j^2 + 2y_j^2), \\ j = 1, \dots, n \end{cases}$$

and

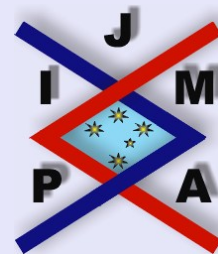
$$(2.5) \quad |\nabla_{H_n} d| = |z|d^{-1}, \quad \Delta_{H_n} d = (Q-1)d^{-3}|z|^2.$$

Choose  $v(z, t) = \delta(z, t)^\beta = (R-d)^\beta$ , in which  $\beta = \frac{p-1}{p}$ , one has

$$\begin{aligned} X_j v &= -\beta \delta^{\beta-1} X_j d, & Y_j v &= -\beta \delta^{\beta-1} Y_j d, & j &= 1, \dots, n, \\ \nabla_{H_n} v &= -\beta \delta^{\beta-1} \nabla_{H_n} d, & |\nabla_{H_n} v| &= |\beta| \delta^{\beta-1} |z| d^{-1}, \end{aligned}$$

and

$$\begin{aligned} -\Delta_{H_n} v &= -\nabla_{H_n} \cdot (|\nabla_{H_n} v|^{p-2} \nabla_{H_n} v) \\ &= -\nabla_{H_n} \cdot (-\beta |\beta|^{p-2} \delta^{(\beta-1)(p-1)} |z|^{p-2} d^{2-p} \nabla_{H_n} d) \end{aligned}$$



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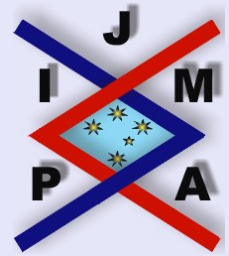


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$$\begin{aligned}
 &= \beta|\beta|^{p-2} \left\{ -(\beta-1)(p-1)\delta^{(\beta-1)(p-1)-1}|z|^{p-2}d^{2-p}|\nabla_{H_n}d|^2 \right. \\
 &\quad + \delta^{(\beta-1)(p-1)}d^{2-p}\nabla_{H_n}(|z|^{p-2}) \cdot \nabla_{H_n}d \\
 &\quad + (2-p)\delta^{(\beta-1)(p-1)}|z|^{p-2}d^{1-p}|\nabla_{H_n}d|^2 \\
 &\quad \left. + \delta^{(\beta-1)(p-1)}|z|^{p-2}d^{2-p}\Delta_{H_n}d \right\}.
 \end{aligned}$$

From the fact  $\nabla_{H_n}(|z|^{p-2}) \cdot \nabla_{H_n}d = (p-2)|z|^{p-4}d^{-3}|z|^4 = (p-2)|z|^p d^{-3}$  and (2.5), it follows that

$$\begin{aligned}
 -\Delta_{H_n}v &= \beta|\beta|^{p-2} \left\{ -(\beta-1)(p-1)\delta^{(\beta-1)(p-1)-1}|z|^p d^{-p} \right. \\
 &\quad + (p-2)\delta^{(\beta-1)(p-1)}|z|^p d^{-1-p} \\
 &\quad - (p-2)\delta^{(\beta-1)(p-1)}|z|^p d^{-1-p} \\
 &\quad \left. + (Q-1)\delta^{(\beta-1)(p-1)}|z|^p d^{-1-p} \right\} \\
 &= \beta|\beta|^{p-2} \left\{ -(\beta-1)(p-1) + (Q-1)\frac{\delta}{d} \right\} \frac{|z|^p v^{p-1}}{d^p \delta^p} \\
 &= \left(\frac{p-1}{p}\right)^{p-1} \left\{ \frac{p-1}{p} + (Q-1)\frac{\delta}{d} \right\} \frac{|z|^p v^{p-1}}{d^p \delta^p} \\
 &\geq \left(\frac{p-1}{p}\right)^p \frac{|z|^p v^{p-1}}{d^p \delta^p}.
 \end{aligned}$$

The desired inequality (2.3) is obtained by Lemma 2.2. For general  $u$ , by letting  $u = u^+ - u^-$ , we directly obtain (2.3).  $\square$

**Theorem 2.4.** Let  $\Omega = H_n \setminus \{B_{H_n, R}\}$ ,  $Q > p > 1$ . Then for every  $u \in C_0^\infty(\Omega)$ , there exists a constant  $C > 0$ , such that

$$(2.6) \quad \int_{\Omega} |\nabla_{H_n} u|^p \geq C \int_{\Omega} \frac{|z|^p}{d^p} \frac{|u|^p}{d^{2p}}.$$

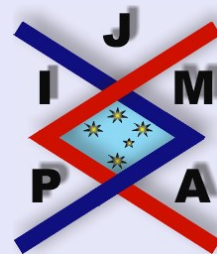
*Proof.* Suppose that  $u \geq 0$ . Take  $v = \log\left(\frac{d}{R}\right)^\alpha$ ,  $R < d = d((z, t), (0, 0)) < +\infty$ ,  $\alpha < 0$ . Using (2.4) and (2.5) show that

$$\begin{aligned} \nabla_{H_n} v &= \left(\frac{R}{d}\right)^\alpha \alpha \left(\frac{d}{R}\right)^{\alpha-1} \frac{1}{R} \nabla_{H_n} d = \frac{\alpha}{d} \nabla_{H_n} d, \\ |\nabla_{H_n} v| &= |\alpha| |z| d^{-2}, \end{aligned}$$

$$\begin{aligned} -\Delta_{H_n} v &= -\nabla_{H_n} \cdot (|\nabla_{H_n} v|^{p-2} \nabla_{H_n} v) \\ &= -\alpha |\alpha|^{p-2} \nabla_{H_n} \cdot (|z|^{p-2} d^{2(p-2)-1} \nabla_{H_n} d) \\ &= -\alpha |\alpha|^{p-2} \left\{ (p-2) |z|^{p-3} d^{2(2-p)-1} \nabla_{H_n} (|z|) \cdot \nabla_{H_n} d \right. \\ &\quad \left. + (2(2-p) - 1) |z|^{p-2} d^{2(1-p)} |\nabla_{H_n} d|^2 \right. \\ &\quad \left. + |z|^{p-2} d^{2(2-p)-1} \Delta_{H_n} d \right\}. \end{aligned}$$

Since  $\nabla_{H_n} (|z|) \cdot \nabla_{H_n} d = |z|^3 d^{-3}$ , the last equation above becomes

$$\begin{aligned} -\Delta_{H_n} v &= -\alpha |\alpha|^{p-2} \left\{ (p-2) |z|^{p-3} d^{2(2-p)-1} |z|^3 d^{-3} \right. \\ &\quad \left. + (3-2p) |z|^{p-2} d^{2(1-p)} |z|^2 d^{-2} \right. \\ &\quad \left. + (Q-1) |z|^{p-2} d^{3-2p} |z|^2 d^{-3} \right\} \end{aligned}$$



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$$\begin{aligned}
 &= -\alpha|\alpha|^{p-2}|z|^p d^{-2p}(p-2+3-2p+Q-1) \\
 (2.7) \quad &= -\alpha|\alpha|^{p-2}(Q-p)|z|^p d^{-2p}.
 \end{aligned}$$

Noting

$$\lim_{d \rightarrow +\infty} \frac{v^{p-1}}{d^p} = 0,$$

there exists a positive number  $M \geq R$ , such that  $\frac{v^{p-1}}{d^p} < 1$ , for  $d > M$ . Since  $\frac{v^{p-1}}{d^p}$  is continuous on the interval  $[R, M]$ , we find a constant  $C' > 0$ , such that  $\frac{v^{p-1}}{d^p} < C'$ . Pick out  $C'' = \max\{C', 1\}$  and one has  $v^{p-1} < C'' d^p$  in  $\Omega$ . This leads to the following

$$-\Delta_{H_n} v \geq C \frac{|z|^p v^{p-1}}{d^{2p} d^p},$$

where  $C = \frac{-\alpha|\alpha|^{p-2}(Q-p)}{C''}$ , and to (2.6) by Lemma 2.2. A similar treatment for general  $u$  completes the proof.  $\square$

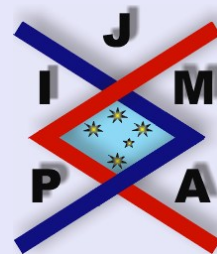
In particular,  $\alpha = p - Q$  ( $1 < p < Q$ ) satisfies the assumption in the proof above.

**Theorem 2.5.** *Let  $\Omega$  be as defined in Theorem 2.4 and  $p \geq Q$ . Then there exists a constant  $C > 0$ , such that for every  $u \in C_0^\infty(\Omega)$ ,*

$$(2.8) \quad \int_{\Omega} |\nabla_{H_n} u|^p \geq C \int_{\Omega} \frac{|z|^p}{d^p \left(\log\left(\frac{d}{R}\right)\right)^p} \frac{|u|^p}{d^p}.$$

*Proof.* It is sufficient to show that (2.8) holds for  $u \geq 0$ . Choose  $v = \phi^\alpha$ ,  $\phi = \log \frac{d}{R}$ , where  $R < d < +\infty$ ,  $0 < \alpha < 1$ . We know that from (2.4) and (2.5),

$$\begin{aligned}
 \nabla_{H_n} \phi &= d^{-1} \nabla_{H_n} d, \quad |\nabla_{H_n} \phi| = d^{-2} |z|, \\
 \Delta_{H_n} \phi &= d^{-1} \Delta_{H_n} d - d^{-2} |\nabla_{H_n} d|^2 = (Q-2)|z|^2 d^{-4}.
 \end{aligned}$$



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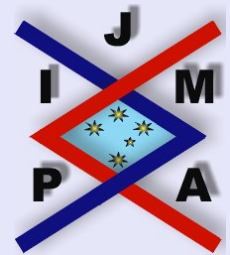
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This allows us to obtain

$$\begin{aligned}
 -\Delta_{H_n} v &= -\nabla_{H_n} \cdot (|\nabla_{H_n} v|^{p-2} \nabla_{H_n} v) \\
 &= -\nabla_{H_n} \cdot (|\alpha \phi^{\alpha-1} \nabla_{H_n} \phi|^{p-2} \alpha \phi^{\alpha-1} \nabla_{H_n} \phi) \\
 &= -\alpha |\alpha|^{p-2} \nabla_{H_n} \cdot (\phi^{(\alpha-1)(p-1)} |z|^{p-2} d^{2(2-p)} \nabla_{H_n} \phi) \\
 &= -\alpha |\alpha|^{p-2} \left\{ (\alpha-1)(p-1) \phi^{(\alpha-1)(p-1)-1} |z|^{p-2} d^{2(2-p)} |\nabla_{H_n} \phi|^2 \right. \\
 &\quad + (p-2) \phi^{(\alpha-1)(p-1)} |z|^{p-3} d^{2(2-p)} \nabla_{H_n} (|z|) \cdot \nabla_{H_n} \phi \\
 &\quad + 2(2-p) \phi^{(\alpha-1)(p-1)} |z|^{p-2} d^{2(2-p)-1} \nabla_{H_n} d \cdot \nabla_{H_n} \phi \\
 &\quad \left. + \phi^{(\alpha-1)(p-1)} |z|^{p-2} d^{2(2-p)} \Delta_{H_n} \phi \right\} \\
 &= -\alpha |\alpha|^{p-2} \left\{ (\alpha-1)(p-1) \phi^{(\alpha-1)(p-1)-1} |z|^{p-2} d^{2(2-p)} |z|^2 d^{-4} \right. \\
 &\quad + (p-2) \phi^{(\alpha-1)(p-1)} |z|^{p-3} d^{2(2-p)} |z|^3 d^{-4} \\
 &\quad + 2(2-p) \phi^{(\alpha-1)(p-1)} |z|^{p-2} d^{2(2-p)-1} |z|^2 d^{-3} \\
 &\quad \left. + \phi^{(\alpha-1)(p-1)} |z|^{p-2} d^{2(2-p)} (Q-2) |z|^2 d^{-4} \right\} \\
 &= -\alpha |\alpha|^{p-2} \frac{v^{p-1}}{\phi^p} \frac{|z|^p}{d^{2p}} \left\{ (\alpha-1)(p-1) + (p-2)\phi \right. \\
 &\quad \left. + 2(2-p)\phi + (Q-2)\phi \right\} \\
 &= -\alpha |\alpha|^{p-2} \frac{v^{p-1}}{\phi^p} \frac{|z|^p}{d^{2p}} \left\{ (\alpha-1)(p-1) + (Q-p)\phi \right\},
 \end{aligned}$$




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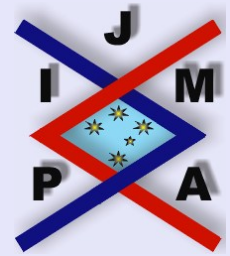
Taking into account that  $0 < \alpha < 1$  and  $p \geq Q$ , we have

$$-\alpha|\alpha|^{p-2}(Q-p)\phi \geq 0,$$

and therefore

$$-\Delta_{H_n} v \geq -\alpha|\alpha|^{p-2}(\alpha-1)(p-1) \frac{v^{p-1}}{\phi^p} \frac{|z|^p}{d^{2p}} = C \frac{v^{p-1}}{\phi^p} \frac{|z|^p}{d^{2p}},$$

where  $C = -\alpha|\alpha|^{p-2}(\alpha-1)(p-1)$ . An application of Lemma 2.2 completes the proof of Theorem 2.5.  $\square$



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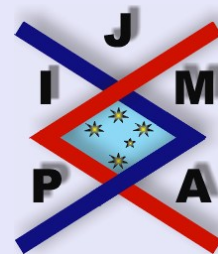
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