journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 4, Article 117, 11 pp.

ARITHMETIC-GEOMETRIC-HARMONIC MEAN OF THREE POSITIVE OPERATORS

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Received 18 December, 2007; accepted 08 July, 2009
Communicated by F. Kittaneh

ABSTRACT. In this paper, we introduce the geometric mean of several positive operators defined from a simple and practical recursive algorithm. This approach allows us to construct the arithmetic-geometric-harmonic mean of three positive operators which has many of the properties of the standard one.

Key words and phrases: Positive operator, Geometric operator mean, Arithmetic-geometric-harmonic operator mean.

2000 *Mathematics Subject Classification.* 15A48, 47A64.

1. INTRODUCTION

The geometric mean of two positive linear operators arises naturally in several areas and can be used as a tool for solving many scientific problems. Researchers have recently tried to differently define such operator means because of their useful properties and applications. Let H be a Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. We denote by $\mathcal{L}(H)$ the Banach space of continuous linear operators defined from H into itself. For $A, B \in \mathcal{L}(H)$, we write $A \leq B$ if A and B are self-adjoint and $B - A$ is positive (semi-definite). The geometric mean $\mathrm{g}_2(A, B)$ of two positive operators A and B was introduced as the solution of the matrix optimization problem, [1]

$$\mathrm{g}_2(A,B) := \max\left\{X;\ X^* = X,\quad \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\}. \tag{1.1}$$

This operator mean can be also characterized as the strong limit of the arithmetic-harmonic sequence $\{\Phi_n(A,B)\}$ defined by, [2, 3]

$$\begin{cases} \Phi_0(A,B) = \frac{1}{2}A + \frac{1}{2}B \\ \Phi_{n+1}(A,B) = \frac{1}{2}\Phi_n(A,B) + \frac{1}{2}A\left(\Phi_n(A,B)\right)^{-1}B \qquad (n \geq 0). \end{cases} \tag{1.2}$$

The authors wish to thank the anonymous referee for his suggestions and comments, which helped improve an earlier version of this paper, and for bringing Example 2.3 to our attention.

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As is well known, the explicit form of $\mathbf{g}_2(A,B)$ is given by

$$\mathbf{g}_2(A,B)=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}. \tag{1.3}$$

An interesting question arises from the previous approaches defining $\mathbf{g}_2(A,B)$: what should be the analogue of the above algorithm from two positive operators to three or more ones?

We first describe an extended algorithm of (1.2) involving several positive operators. The key idea of such an extension comes from the fact that the arithmetic, harmonic and geometric means of m positive real numbers $a_1,\ a_2,\ldots,\ a_m$ can be written recursively as follows

$$\mathbf{a}_m(a_1,\ldots,a_m):=\frac{1}{m}\sum_{i=1}^m a_i=\frac{1}{m}a_1+\frac{m-1}{m}\mathbf{a}_{m-1}(a_2,\ldots,a_m), \tag{1.4}$$

$$\mathbf{h}_m(a_1,\ldots,a_m):=\left(\frac{1}{m}\sum_{i=1}^m a_i^{-1}\right)^{-1}=\left(\frac{1}{m}a_1^{-1}+\frac{m-1}{m}\left(\mathbf{h}_{m-1}(a_2,\ldots,a_m)^{-1}\right)\right)^{-1}, \tag{1.5}$$

$$\mathbf{g}_m(a_1,\ldots,a_m):=\sqrt[m]{a_1a_2\cdots a_m}=a_1^{\frac{1}{m}}\left(\mathbf{g}_{m-1}(a_2,\ldots,a_m)\right)^{\frac{m-1}{m}}. \tag{1.6}$$

The extensions of (1.4) and (1.5) when the scalar variables $a_1,a_2,\ldots,a_m$ are positive operators can be immediately given, by setting $A^{-1}=\lim\limits_{\epsilon\downarrow 0}(A+\epsilon I)^{-1}$. By virtue of the induction relation (1.6), the extension of the geometric mean $\mathbf{g}_m(a_1,\ a_2,\ldots,\ a_m)$ from the scalar case to the operator one can be reduced to the following question: what should be the analogue of $a^{1/m}b^{1-1/m}$ when the variables a and b are positive operators? As well known, a reasonable analogue of $a^{1/m}b^{1-1/m}$ for operators is the power geometric mean of A and B, namely

$$\Phi_{1/m}(A,B):=B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1/m}B^{1/2}. \tag{1.7}$$

The appearance of the term $\left(B^{-1/2}AB^{-1/2}\right)^{1/m}$ in (1.7) imposes many difficulties in the computation context when A and B are two given matrices. To remove this difficulty, in this paper we introduce a simple and practical algorithm involving two positive operators A and B converging to

$$B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1/m}B^{1/2},$$

in the strong operator topology. Numerical examples, throughout this paper, show the interest of this work. Afterwards, inspired by the above algorithm we define recursively the geometric mean of several positive operators. Our approach has a convex concept and so allows us to introduce the arithmetic-geometric-harmonic operator mean which possesses many of the properties of the scalar one.

2. GEOMETRIC OPERATOR MEAN OF SEVERAL VARIABLES

Let $m\geq 2$ be an integer and $A_1,A_2,\ldots,A_m\in\mathcal{L}(H)$ be m positive operators. As already mentioned, this section is devoted to introducing the geometric mean of $A_1,A_2,\ldots,A_m$. Let $A,\ B\in\mathcal{L}(H)$ be two positive operators. Inspired by the algorithm (1.2), we define the recursive sequence $\{T_n\}:=\{T_n(A,B)\}$

$$\begin{cases} T_0=\dfrac{1}{m}A+\dfrac{m-1}{m}B; \\[2ex] T_{n+1}=\dfrac{m-1}{m}T_n+\dfrac{1}{m}A\left(T_n^{-1}B\right)^{m-1} & (n\geq 0). \end{cases}$$

In what follows, for simplicity we write $\{T_n\}$ instead of $\{T_n(A,B)\}$ and we set

$$T_n^{(-1)} = \left(T_n(A^{-1}, B^{-1})\right)^{-1}.$$

Clearly, for $m = 2$ the above recursive scheme coincides with the algorithm (1.2). The convergence of the operator sequence $\{T_n\}$ is given by the following main result.

Theorem 2.1. *With the above, the sequence $\{T_n\} := \{T_n(A,B)\}$ converges decreasingly in $\mathcal{L}(H)$, with the limit*

$$\lim_{n\uparrow+\infty} T_n := \Phi_{1/m}(A,B) = B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1/m} B^{1/2}. \tag{2.1}$$

Further, the next estimation holds

$$\forall n \geq 0 \qquad 0 \leq T_n - \Phi_{1/m}(A,B) \leq \left(1 - \frac{1}{m}\right)^n \left(T_0 - T_0^{(-1)}\right). \tag{2.2}$$

Proof. We divide it into three steps:

Step 1: Let $a > 0$ be a real number and consider the scheme

$$\begin{cases} x_0 = \dfrac{1}{m}a + \dfrac{m-1}{m}; \\ \\ x_{n+1} = \dfrac{m-1}{m}x_n + \dfrac{1}{m}\dfrac{a}{x_n^{m-1}} \qquad\qquad (n \geq 0). \end{cases} \tag{2.3}$$

This is a formal Newton's algorithm to calculate $\sqrt[m]{a}$ with a chosen initial data $x_0 > 0$. We wish to establish its convergence. By induction, it is easy to see that $x_n > 0$ for all $n \geq 0$. Using the concavity of the function $t \longrightarrow \mathrm{Log} t$ $(t > 0)$, we can write

$$\mathrm{Log} x_{n+1} \geq \frac{m-1}{m}\,\mathrm{Log}\, x_n + \frac{1}{m}\,\mathrm{Log}\,\frac{a}{x_n^{m-1}},$$

or again

$$\mathrm{Log} x_{n+1} \geq \frac{m-1}{m}\,\mathrm{Log}\, x_n + \frac{1}{m}\,(\mathrm{Log} a - (m-1)\mathrm{Log} x_n).$$

It follows that, after reduction

$$\forall n \geq 0 \qquad x_n \geq \sqrt[m]{a},$$

which, with a simple manipulation, yields

$$\forall n \geq 0 \qquad \frac{a}{x_n^{m-1}} \leq \sqrt[m]{a}.$$

Now, writing

$$x_{n+1} - \sqrt[m]{a} = \frac{m-1}{m}\left(x_n - \sqrt[m]{a}\right) + \frac{1}{m}\left(\frac{a}{x_n^{m-1}} - \sqrt[m]{a}\right),$$

we can deduce that

$$0 \leq x_{n+1} - \sqrt[m]{a} \leq \frac{m-1}{m}(x_n - \sqrt[m]{a}),$$

and by induction

$$0 \leq x_{n+1} - \sqrt[m]{a} \leq \left(\frac{m-1}{m}\right)^{n+1}\left(x_0 - \sqrt[m]{a}\right),$$

from which we conclude that the real sequence $\{x_n\}$ converges to $\sqrt[m]{a}$.

Step 2: Let $A \in \mathcal{L}(H)$ be a positive definite operator and define the following iterative process

$$(2.4)\qquad \begin{cases} X_0 = \dfrac{1}{m}A + \dfrac{m-1}{m}I; \\ \\ X_{n+1} = \dfrac{m-1}{m}X_n + \dfrac{1}{m}AX_n^{1-m} \qquad\qquad (n \geq 0). \end{cases}$$

It is clear that A commutes with X_n for each $n \geq 0$. By Guelfand's representation, the convergence of the matrix algorithm (2.4) is reduced to the number case (2.3) discussed in the previous step. It follows that $\{X_n\}$ converges in $\mathcal{L}(H)$ to $A^{1/m}$. Further, one can easily deduce that

$$\forall n \geq 0 \qquad 0 \leq X_n - A^{1/m} \leq \left(\frac{m-1}{m}\right)^n \left(X_0 - A^{1/m}\right) \leq \left(\frac{m-1}{m}\right)^n \left(X_0 - X_0^{(-1)}\right).$$

Step 3: By virtue of the second step, the next sequence $\{Y_n\}$

$$(2.5)\qquad \begin{cases} Y_0 = \dfrac{1}{m}B^{-1/2}AB^{-1/2} + \dfrac{m-1}{m}I; \\ \\ Y_{n+1} = \dfrac{m-1}{m}Y_n + \dfrac{1}{m}B^{-1/2}AB^{-1/2}Y_n^{1-m} \qquad\qquad (n \geq 0), \end{cases}$$

converges in $\mathcal{L}(H)$ to $\left(B^{-1/2}AB^{-1/2}\right)^{1/m}$ and

$$\forall n \geq 0 \qquad 0 \leq Y_n - \left(B^{-1/2}A^{1/m}B^{-1/2}\right)^{1/m} \leq \left(\frac{m-1}{m}\right)^n \left(Y_0 - Y_0^{(-1)}\right).$$

It is clear that the algorithm (2.5) is equivalent to

$$\begin{cases} B^{1/2}Y_0B^{1/2} = \dfrac{1}{m}A + \dfrac{m-1}{m}B; \\ \\ B^{1/2}Y_{n+1}B^{1/2} = \dfrac{m-1}{m}B^{1/2}Y_nB^{1/2} + \dfrac{1}{m}AB^{-1/2}Y_n^{1-m}B^{1/2} \qquad\qquad (n \geq 0). \end{cases}$$

Now, writing

$$B^{-1/2}Y_n^{1-m}B^{1/2} = \left(B^{-1/2}Y_n^{-1}B^{-1/2}\right)B\left(B^{-1/2}Y_n^{-1}B^{-1/2}\right)B \cdots \left(B^{-1/2}Y_n^{-1}B^{-1/2}\right)B,$$

and setting

$$T_n = B^{1/2}Y_nB^{1/2},$$

we obtain the desired conclusion. $\square$

Let us remark that we have $\Phi_{1/m}(A,B) = A^{1/m}B^{1-1/m}$ when A and B are two commuting positive operators and so, $\Phi_{1/m}(A,I) = A^{1/m}$, $\Phi_{1/m}(I,B) = B^{1-1/m}$ for all positive operators A and B. Let us also note the following remark that will be needed later.

Remark 1. The map $(A,B) \longmapsto \Phi_{1/m}(A,B)$ satisfies the conjugate symmetry relation, i.e

$$(2.6)\qquad \Phi_{1/m}(A,B) = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\frac{m-1}{m}}A^{1/2} = \Phi_{\frac{m-1}{m}}(B,A),$$

which is not directly obvious.

Further properties of $(A,B) \longmapsto \Phi_{1/m}(A,B)$ are summarized in the following corollary.

Corollary 2.2. *With the above conditions, the following assertions are met:*

(i) *For a fixed positive operator B, the map $X \longmapsto \Phi_{1/m}(X,B)$ is operator increasing and concave.*

(ii) *For every invertible operator $L \in \mathcal{L}(H)$ there holds*

$$\Phi_{1/m}(L^*AL, L^*BL) = L^*\Phi_{1/m}(A, B)L.$$

(iii) *For a fixed positive operator A, the map $X \longmapsto \Phi_{1/m}(A, X)$ is operator increasing and concave.*

Proof. (i) Follows from the fact that the map $X \longmapsto X^{1/m}$, with $m \geq 1$, is operator increasing and concave, see [4] for instance.

(ii) Since the sequence $\{T_n\}$ of Theorem 2.1 depends on A, B, we can set $T_n := T_n(A, B)$. We verify, by induction on n, that

$$T_n(L^*AL, L^*BL) = L^*T_n(A, B)L,$$

for all $n \geq 0$. Letting $n \to +\infty$ in this last relation we obtain, by an argument of continuity and the definition of $\Phi_{1/m}(A, B)$, the desired result.

(iii) By (2.6) and similarly to (i), we deduce the desired result. □

Now, we are in a position to state the following central definition.

Definition 2.1. With the above notations, the geometric operator mean of $A_1,\ A_2,\ \dots,\ A_m$ is recursively defined by the relationship

$$\mathbf{g}_m(A_1,\ A_2,\ \dots,\ A_m) = \Phi_{1/m}\left(A_1, \mathbf{g}_{m-1}(A_2,\ \dots,\ A_m)\right). \tag{2.7}$$

From this definition, it is easy to verify that, if $A_1, A_2, \dots, A_m$ are commuting, then

$$\mathbf{g}_m(A_1, A_2, \dots, A_m) = (A_1A_2 \cdots A_m)^{1/m}.$$

In particular, for all positive operators $A \in \mathcal{L}(H)$ one has

$$\mathbf{g}_m(A, A, \dots, A) = A \ \text{ and } \ \mathbf{g}_m(I, I, \dots, A, I, \dots, I) = A^{1/m}.$$

It is well known that $(A, B) \longmapsto \mathbf{g}_2(A, B)$ is symmetric. However, $\mathbf{g}_m$ is not symmetric for $m \geq 3$ as shown by Example 2.3 below.

Now, we will study the properties of the operator mean $\mathbf{g}_m(A_1, A_2, \dots, A_m)$.

Proposition 2.3. *The operator mean $\mathbf{g}_m(A_1, A_2, \dots, A_m)$ satisfies the following properties:*

(i) *Self-duality relation, i.e*

$$\left(\mathbf{g}_m(A_1, A_2, \dots, A_m)\right)^{-1} = \mathbf{g}_m(A_1^{-1}, A_2^{-1}, \dots, A_m^{-1}).$$

(ii) *The arithmetic-geometric-harmonic mean inequality, i.e*

$$\mathbf{h}_m(A_1, A_2, \dots, A_m) \leq \mathbf{g}_m(A_1, A_2, \dots, A_m) \leq \mathbf{a}_m(A_1, A_2, \dots, A_m).$$

(iii) *The algebraic equation: find a positive operator X such that $X(BX)^{m-1} = A$, has one and only one solution given by $X = \mathbf{g}_m(A, B^{-1}, \dots, B^{-1})$.*

Proof. (i) Follows by a simple induction on $m \geq 2$ with the duality relation:

$$\left(\Phi_{1/m}(A, B)\right)^{-1} = \Phi_{1/m}(A^{-1}, B^{-1}).$$

(ii) By induction on $m \geq 2$: the double inequality is well known for $m = 2$. Assume that it holds true for $m - 1$ and show that it holds for m. According to (2.2) with $n = 0$, we obtain

$$\Phi_{1/m}(A, B) \leq \frac{1}{m}A + \frac{m-1}{m}B,$$

from which we deduce, using the definition of $\mathbf{g}_m(A_1, A_2, \dots, A_m)$,

$$\mathbf{g}_m(A_1, A_2, \dots, A_m) \leq \frac{1}{m}A_1 + \frac{m-1}{m}\mathbf{g}_{m-1}(A_2, A_3, \dots, A_m),$$

which, with the induction hypothesis, gives the arithmetic-geometric mean inequality, i.e

$$\mathbf{g}_m(A_1, A_2, \ldots, A_m) \leq \mathbf{a}_m(A_1, A_2, \ldots, A_m).$$

This last inequality is valid for all positive operators $A_1, A_2, \ldots, A_m$, hence

$$\mathbf{g}_m(A_1^{-1}, A_2^{-1}, \ldots, A_m^{-1}) \leq \mathbf{a}_m(A_1^{-1}, A_2^{-1}, \ldots, A_m^{-1}),$$

and by (i) and the fact that the map $X \longmapsto X^{-1}$ is operator decreasing, we obtain the geometric-harmonic mean inequality.

(iii) Follows by essentially the same arguments used to prove the previous properties. Details are left to the reader. □

Proposition 2.4. *Let $A_1, A_2, \ldots, A_m \in \mathcal{L}(H)$ be positive operators. Then the following assertions are met:*

(i) *For all positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ one has*

$$\mathbf{g}_m(\alpha_1 A_1, \alpha_2 A_2, \ldots, \alpha_m A_m) = \mathbf{g}_m(\alpha_1, \alpha_2, \ldots, \alpha_m)\mathbf{g}_m(A_1, A_2, \ldots, A_m),$$

where

$$\mathbf{g}_m(\alpha_1, \alpha_2, \ldots, \alpha_m) = \sqrt[m]{\alpha_1\alpha_2 \cdots \alpha_m},$$

is the standard geometric mean of $\alpha_1, \alpha_2, \ldots, \alpha_m$.

(ii) *The map $X \to \mathbf{g}_m(X, A_2, \ldots, A_m)$ is operator increasing and concave, i.e.*

$$X \leq Y \implies \mathbf{g}_m(X, A_2, \ldots, A_m) \leq \mathbf{g}_m(Y, A_2, \ldots, A_m)$$

and,

$$\mathbf{g}_m(\lambda X + (1-\lambda)Y, A_2, \ldots, A_m) \geq \lambda \mathbf{g}_m(X, A_2, \ldots, A_m) + (1-\lambda)\mathbf{g}_m(Y, A_2, \ldots, A_m),$$

for all positive operators $X, Y \in \mathcal{L}(H)$ and all $\lambda \in [0,1]$.

(iii) *For every invertible operator $L \in \mathcal{L}(H)$ there holds*

$$\mathbf{g}_m(L^* A_1 L, L^* A_2 L, \ldots, L^* A_m L) = L^* \left(\mathbf{g}_m(A_1, A_2, \ldots, A_m)\right) L. \tag{2.8}$$

(iv) *If H is a finite dimensional Hilbert space then*

$$\det \mathbf{g}_m(A_1, A_2, \ldots, A_m) = \mathbf{g}_m(\det A_1, \det A_2, \ldots, \det A_m).$$

Proof. (i) Follows immediately from the definition of $\mathbf{g}_m$.

(ii) Follows from Corollary 2.2, (i).

(iii) This follows from the definition and Corollary 2.2, (ii).

(iv) By the properties of the determinant, it is easy to see that, for all positive operators A and B,

$$\det \Phi_{1/m}(A, B) = \Phi_{1/m}(\det A, \det B).$$

This, with the definition of $\mathbf{g}_m(A_1, A_2, \ldots, A_m)$ and a simple induction on $m \geq 2$, implies the desired result. □

We note that, as for all monotone operator means [5], if the operator L is not invertible then the transformer equality (2.8) is an inequality. Otherwise, we have the following.

Corollary 2.5. *The map $X \longmapsto \mathbf{g}_m(A_1, A_2, \ldots, X, \ldots, A_m)$ is operator increasing and concave.*

Proof. The desired result is well known for $m = 2$. For the map $X \longmapsto \mathbf{g}_m(X, A_2, \dots, A_m)$, it is the statement of Proposition 2.4, (ii). Now, by Remark 1 it is easy to see that if $X \longmapsto \Psi(X)$ is an operator increasing concave map, then so is $X \longmapsto \Phi_{1/m}(A_1, \Psi(X))$. Setting $\Psi(X) = \mathbf{g}_{m-1}(A_2, A_3, \dots, X, \dots, A_m)$ and again by Proposition 2.4, (ii), the desired result follows by a simple induction on $m \geq 2$. This completes the proof. □

Now, we state the following remark that will be needed in the sequel.

Remark 2. Let us take $m = 3$. Then the equation: find $X \in \mathcal{L}(H)$ such that $X = \mathbf{g}_3(A, X, C)$, has one and only one positive solution given by $X = \mathbf{g}_2(A, C)$. In fact, it is easy to see that $\mathbf{g}_3(A, I, C) = I$ if and only if $A = C^{-1}$. Further, by Proposition 2.4, (iii), we can write

$$X = \mathbf{g}_3(A, X, C) \Longleftrightarrow X = X^{1/2}\mathbf{g}_3\left(X^{-1/2}AX^{-1/2}, I, X^{-1/2}CX^{-1/2}\right)X^{1/2},$$

which implies that

$$\mathbf{g}_3\left(X^{-1/2}AX^{-1/2}, I, X^{-1/2}CX^{-1/2}\right) = I,$$

or again

$$X^{-1/2}AX^{-1/2} = X^{1/2}C^{-1}X^{1/2}.$$

The desired result follows by a simple manipulation.

We end this section by noting an interesting relationship given by the following proposition.

Proposition 2.6. *Let $\{A_n\}$ be a sequence of positive operators converging in $\mathcal{L}(H)$ to A. Assume that A is positive definite, then*

$$\lim_{n\uparrow+\infty} \mathbf{g}_n(A_1,\ A_2, \dots,\ A_n) = A. \tag{2.9}$$

Proof. Under the hypothesis of the proposition, it is not hard to show that

$$\lim_{n\uparrow+\infty} \mathbf{a}_n(A_1,\ A_2, \dots,\ A_n) = A, \tag{2.10}$$

and

$$\lim_{n\uparrow+\infty} \mathbf{h}_n(A_1,\ A_2, \dots,\ A_n) = A. \tag{2.11}$$

Indeed, (2.10) is well-known for the scalar case (Cesaro's theorem) and the same method works for the operator one. We deduce (2.11) by recalling that the map $A \to A^{-1}$ is continuous on the open cone of positive definite operators. Relation (2.9) follows then from the arithmetic-geometric-harmonic mean inequality (Proposition 2.3, (ii)), with (2.10) and (2.11). The proof is complete. □

Now, we wish to illustrate the above theoretical results with three numerical matrix examples. For a square matrix A, we denote by $\|\cdot\|$ the Schur's norm of A defined by

$$\|A\| = \sqrt{\mathrm{Trace}(A^*A)}.$$

Example 2.1. Let us consider the following matrices:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 9 & 3 & 1 \\ 3 & 8 & 2 \\ 1 & 2 & 6 \end{pmatrix}.$$

In order to compute some iterations of the sequence $\{T_n\}$, we compute $\mathbf{g}_2(B,C)$ by algorithm (1.2). Using MATLAB, we obtain numerical iterations $T_2, T_3, \ldots, T_6$ satisfying the following estimations:

$$\begin{aligned}
\| T_3 - T_2 \| &= 8.894903423045612 \times 10^{-4},\\
\| T_4 - T_3 \| &= 2.762580836245787 \times 10^{-7},\\
\| T_5 - T_4 \| &= 2.660171405523615 \times 10^{-14},\\
\| T_6 - T_5 \| &= 4.974909261937442 \times 10^{-16},
\end{aligned}$$

and good approximations are obtained from the first iterations.

Example 2.2. In this example, we will solve numerically the algebraic equation: for given positive matrices A and B, find a positive matrix X such that $XBXBX = A$. Consider,

$$A = \begin{pmatrix} 7 & 3 & 0 & 1 \\ 3 & 4 & -2 & 1 \\ 0 & -2 & 4 & -1 \\ 1 & 1 & -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 6 & -1 & 2 \\ 2 & -1 & 5 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}.$$

By Proposition 2.3, (iii), the unique solution of the above equation is $X = \mathbf{g}_3(A, B^{-1}, B^{-1})$. Numerically, we obtain the iterations $T_5, T_6, \ldots, T_9$ with the following estimations:

$$\begin{aligned}
\| T_6 - T_5 \| &= 0.01369442620176,\\
\| T_7 - T_6 \| &= 2.933841711132645 \times 10^{-4},\\
\| T_8 - T_7 \| &= 1.329143009263914 \times 10^{-7},\\
\| T_9 - T_8 \| &= 3.063703619940987 \times 10^{-13}.
\end{aligned}$$

Example 2.3. As already demonstrated, this example shows the non-symmetry of $\mathbf{g}_m$ for $m \geq 3$. Take

$$A = \begin{pmatrix} 1.8597 & 1.0365 & 1.9048 \\ 1.0365 & 0.7265 & 0.9889 \\ 1.9048 & 09889 & 2.0084 \end{pmatrix}, \qquad B = \begin{pmatrix} 1.0740 & 0.2386 & 1.1999 \\ 0.2386 & 0.0548 & 0.2826 \\ 1.1999 & 0.2826 & 1.4894 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.4407 & 0.6183 & 0.1982 \\ 0.6183 & 0.9995 & 0.4150 \\ 0.1982 & 0.4150 & 0.2718 \end{pmatrix}, \qquad D = \begin{pmatrix} 1.0076 & 0.4516 & 0.5909 \\ 0.4516 & 0.4177 & 0.7656 \\ 0.5909 & 0.7656 & 1.8679 \end{pmatrix}.$$

Executing a program in MATLAB, we obtain the following.

$$\mathbf{g}_4(A,D,B,C) = \begin{pmatrix} 0.3259 & 0.1187 & 0.2833 \\ 0.1187 & 0.0736 & 0.1282 \\ 0.2833 & 0.1282 & 0.4220 \end{pmatrix},$$

$$\mathbf{g}_4(A,B,C,D) = \begin{pmatrix} 0.3174 & 0.0982 & 0.2832 \\ 0.0.982 & 0.0584 & 0.1058 \\ 0.2832 & 0.1058 & 0.4371 \end{pmatrix},$$

$$\mathbf{g}_4(A,C,D,B) = \begin{pmatrix} 0.2847 & 0.0948 & 0.2381 \\ 0.0948 & 0.0643 & 0.0967 \\ 0.2381 & 0.0967 & 0.3733 \end{pmatrix}.$$

Therefore

$$\mathbf{g}_4(A,D,B,C) \neq \mathbf{g}_4(A,B,C,D) \neq \mathbf{g}_4(A,C,D,B),$$

and so $\mathbf{g}_m$ is not symmetric for $m \geq 3$.

3. ARITHMETIC-GEOMETRIC-HARMONIC OPERATOR MEAN

As already mentioned, in this section we introduce the arithmetic-geometric-harmonic operator mean which possesses many of the properties of the standard one. More precisely, given three positive real numbers a, b, c, consider the sequences

$$\begin{cases} a_0 = a, & \dfrac{3}{a_{n+1}} = \dfrac{1}{a_n} + \dfrac{1}{b_n} + \dfrac{1}{c_n}; \\ b_0 = b, & b_{n+1} = \sqrt[3]{a_n b_n c_n} \qquad\qquad (n \geq 0); \\ c_0 = c, & c_{n+1} = \dfrac{a_n + b_n + c_n}{3}. \end{cases}$$

It is well known that the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ converge to the same positive limit, called the arithmetic-geometric-harmonic mean of $a,\ b$ and c. In what follows, we extend the above algorithm from positive real numbers to positive operators. We start with some additional notions that are needed below. An operator sequence $\{A_n\}$ is called quadratic convergent if there is a self-adjoint operator $A \in \mathcal{L}(H)$ such that $\lim\limits_{n\to+\infty} \langle A_n x, x\rangle = \langle Ax, x\rangle$, for all $x \in H$. It is known that if $\{A_n\}$ is a sequence of positive operators, the quadratic convergence is equivalent to the strong convergence, i.e $\lim\limits_{n\to+\infty} A_n x = Ax$ if and only if $\lim\limits_{n\to+\infty} \langle A_n x, x\rangle = \langle Ax, x\rangle$, for all $x \in H$.

The sequence $\{A_n\}$ is said to be operator-increasing (resp. decreasing) if for all $x \in H$ the real sequence $\{\langle A_n x, x\rangle\}$ is scalar-increasing (resp. decreasing). The sequence $\{A_n\}$ is upper bounded (resp. lower bounded) if there is a self-adjoint operator $M \in \mathcal{L}(H)$ such that $A_n \leq M$ (resp. $M \leq A_n$), for all $n \geq 0$. With this, it is not hard to verify the following lemma that will be needed in the sequel.

Lemma 3.1. *Let $\{A_n\} \in \mathcal{L}(H)$ be a sequence of positive operators such that $\{A_n\}$ is operator-increasing (resp. decreasing) and upper bounded (resp. lower bounded). Then $\{A_n\}$ converges, in the strong operator topology, to a positive operator.*

Now, we will discuss our aim in more detail. Let $A, B, C \in \mathcal{L}(H)$ be three positive operators and define the following sequences:

$$\begin{cases} A_0 = A, & A_{n+1} = \mathbf{h}_3(A_n, B_n, C_n); \\ B_0 = B, & B_{n+1} = \mathbf{g}_3(A_n, B_n, C_n) \qquad\qquad (n \geq 0); \\ C_0 = C, & C_{n+1} = \mathbf{a}_3(A_n, B_n, C_n). \end{cases}$$

By induction on $n \in \mathbb{N}$, it is easy to see that the sequences $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ have positive operator arguments.

Theorem 3.2. *The sequences $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ converge strongly to the same positive operator $\mathbf{agh}(A, B, C)$ satisfying the following inequality*

$$\mathbf{h}_3(A, B, C) \leq \mathbf{agh}(A, B, C) \leq \mathbf{a}_3(A, B, C). \tag{3.1}$$

Proof. By the arithmetic-geometric-harmonic mean inequality, we obtain

$$\forall n \geq 0 \qquad A_{n+1} \leq B_{n+1} \leq C_{n+1},$$

which, with the monotonicity of $\mathbf{a}_3$ and $\mathbf{h}_3$, yields

$$A_{n+1} \geq \mathbf{h}_3(A_n, A_n, A_n) = A_n \ \text{ and } \ C_{n+1} \leq \mathbf{a}_3(C_n, C_n, C_n) = C_n.$$

In summary, we have established that, for all $n \geq 1$,

$$\mathbf{h}_3(A,B,C) := A_1 \leq \cdots \leq A_n \leq B_n \leq C_n \leq \cdots \leq C_1 := \mathbf{a}_3(A,B,C). \tag{3.2}$$

We conclude that $\{A_n\}$ (resp.$\{C_n\}$) is operator-increasing and upper bounded by $\mathbf{a}_3(A,B,C)$ (resp. operator-decreasing and lower bounded by $\mathbf{h}_3(A,B,C)$). By Lemma 3.1, we deduce that the two sequences $\{A_n\}$ and $\{C_n\}$ both converge strongly and so there exist two positive operators $P,\ Q \in \mathcal{L}(H)$ such that

$$\lim_{n\uparrow+\infty} \langle A_n x, x\rangle = \langle Px, x\rangle \quad \text{and} \quad \lim_{n\uparrow+\infty} \langle C_n x, x\rangle = \langle Qx, x\rangle,$$

for all $x \in H$. If we write the relation

$$C_{n+1} = \mathbf{a}_3(A_n, B_n, C_n)$$

in the equivalent form

$$B_n = 3C_{n+1} - A_n - C_n,$$

we can deduce that $\{B_n\}$ converges strongly to $2Q - P := R$. Letting $n \to +\infty$ in relationship (3.2), we obtain $P \leq R \leq Q$. Moreover, the recursive relation

$$B_{n+1} = \mathbf{g}_3(A_n, B_n, C_n),$$

with an argument of continuity, gives when $n \to +\infty$,

$$R = \mathbf{g}_3(P, R, Q),$$

which, by Remark 2, yields

$$R = \mathbf{g}_2(P, Q).$$

Due to relations

$$R = 2Q - P, \qquad R = \mathbf{g}_2(P, Q)$$

and the arithmetic-geometric mean inequality, we get

$$R = 2Q - P = \mathbf{g}_2(P, Q) \leq \frac{1}{2}P + \frac{1}{2}Q,$$

which, after reduction, implies that $Q \leq P$. Since P, Q and R are self-adjoint we conclude, by summarizing, that $P = Q = R$. Inequalities (3.1) follow from (3.2) by letting $n \to +\infty$, and the proof is complete. □

Definition 3.1. The operator $\mathbf{agh}(A,B,C)$, defined by Theorem 3.2, will be called the arithmetic-geometric-harmonic mean of A, B and C.

Remark 3. Theorem 3.2 can be written in the following equivalent form: Let $A,\ B,\ C \in \mathcal{L}(H)$ be three positive operators and define the map

$$\Theta(A,B,C) = \left(\mathbf{h}_3(A,B,C),\ \mathbf{g}_3(A,B,C),\ \mathbf{a}_3(A,B,C)\right).$$

If $\Theta^n := \Theta \circ \Theta \circ \cdots \circ \Theta$ denotes the n^{th} iterate of Θ, then there exists a positive operator $M := \mathbf{agh}(A,B,C)$ satisfying

$$\lim_{n\uparrow+\infty} \Theta^n(A,B,C) = (M,M,M).$$

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