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ON HEISENBERG AND LOCAL UNCERTAINTY PRINCIPLES FOR THE q-DUNKL TRANSFORM

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ABSTRACT. In this paper, we provide, for the q-Dunkl transform studied in [2], a Heisenberg uncertainty principle and two local uncertainty principles leading to a new Heisenberg-Weyl type inequality.

Key words and phrases: q-Dunkl transform, Heisenberg-Weyl inequality, Local uncertainty principles.

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1. INTRODUCTION

In harmonic analysis, the uncertainty principle states that a function and its Fourier transform cannot be simultaneously sharply localized. A quantitative formulation of this fact is provided by the Heisenberg uncertainty principle, which asserts that every square integrable function f on $\mathbb{R}$ verifies the following inequality

$$\left(\int_{-\infty}^{+\infty} x^2|f(x)|^2dx\right)\left(\int_{-\infty}^{+\infty} \lambda^2|\widehat{f}(\lambda)|^2d\lambda\right) \geq \frac{1}{4}\left(\int_{-\infty}^{+\infty} x^2|f(x)|^2dx\right)^2, \tag{1.1}$$

where

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}dx$$

is the classical Fourier transform.

Generalizations of this result in both classical and quantum analysis have been treated and many versions of Heisenberg-Weyl type uncertainty inequalities were obtained for several generalized Fourier transforms (see [1], [14], [10]).

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In [2], by the use of the q^2-analogue differential operator studied in [11], Bettaibi et al. introduced a new q-analogue of the classical Dunkl operator and studied its related Fourier transform, which is a q-analogue of the classical Bessel-Dunkl one and called the q-Dunkl transform.

The aim of this paper is twofold: first, we prove a Heisenberg uncertainty principle for the q-Dunkl transform and next, we state for this transform two local uncertainty principles leading to a new q-Heisenberg-Weyl type inequality.

This paper is organized as follows: in Section 2, we present some preliminary notions and notations useful in the sequel. In Section 3, we recall some results and properties from the theory of the q-Dunkl operator and the q-Dunkl transform (see [2]). Section 4 is devoted to proving a Heisenberg uncertainty principle for the q-Dunkl transform and as consequences, we obtain Heisenberg uncertainty principles for the q^2-analogue Fourier transform [12, 11] and for the q-Bessel transform [2]. Finally, in Section 5, we state, for the q-Dunkl transform, two local uncertainty principles, which give a new Heisenberg-Weyl type inequality for the q-Dunkl transform.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, we assume $q \in]0,1[$, and refer to the general reference [6] for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.

We write $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$,

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

The q^2-analogue differential operator is (see [11, 12])

$$\partial_q(f)(z) = \begin{cases} \frac{f\left(q^{-1}z\right)+f\left(-q^{-1}z\right)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z} & \text{if } z \neq 0 \\ \lim_{x\to 0} \partial_q(f)(x) \qquad (\text{in } \mathbb{R}_q) & \text{if } z = 0. \end{cases} \tag{2.1}$$

We remark that if f is differentiable at z, then $\lim_{q\to 1} \partial_q(f)(z) = f'(z)$.

A repeated application of the q^2-analogue differential operator is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

The following lemma lists some useful computational properties of ∂_q.

Lemma 2.1.

(1) *For all functions f on $\mathbb{R}_q$,*

$$\partial_q f(z) = \frac{f_e(q^{-1}z) - f_e(z)}{(1-q)z} + \frac{f_o(z) - f_o(qz)}{(1-q)z},$$

where, f_e and f_o are, respectively, the even and the odd parts of f.

(2) *For two functions f and g on $\mathbb{R}_q$, we have*

- *if f is even and g is odd,*

$$\begin{aligned}\partial_q(fg)(z) &= q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) \\ &= \partial_q(g)(z)f(z) + qg(qz)\partial_q(f)(qz);\end{aligned}$$

- *if f and g are even,*

$$\partial_q(fg)(z) = \partial_q(f)(z)g(q^{-1}z) + f(z)\partial_q(g)(z).$$

The operator ∂_q induces a q-analogue of the classical exponential function (see [11, 12])

$$e(z;q^2) = \sum_{n=0}^{\infty} a_n \frac{z^n}{[n]_q!}, \quad \text{with} \quad a_{2n} = a_{2n+1} = q^{n(n+1)}. \tag{2.2}$$

The q-Jackson integrals are defined by (see [8])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n),$$
$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$
$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

and

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n),$$

provided the sums converge absolutely.

The q-Gamma function is given by (see [8])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0, -1, -2, ...$$

- $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n, m \in \mathbb{N}, \qquad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} \left| x^m \partial_q^n f(x) \right| < +\infty$$

  and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists;}$$

- $L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\};$
- $L_{\alpha,q}^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_qx \right)^{\frac{1}{p}} < \infty \right\};$
- $L_{\alpha,q}^p([-a,a]) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-a}^{a} |f(x)|^p |x|^{2\alpha+1} d_qx \right)^{\frac{1}{p}} < \infty \right\}.$

For the particular case $p = 2$, we denote by $\langle\cdot;\cdot\rangle$ the inner product of the Hilbert space $L_{\alpha,q}^2(\mathbb{R}_q)$.

3. THE q-DUNKL OPERATOR AND THE q-DUNKL TRANSFORM

In this section, we collect some basic properties of the q-Dunkl operator and the q-Dunkl transform introduced in [2] which will useful in the sequel.

For $\alpha \geq -\frac{1}{2}$, the q-Dunkl operator is defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q \left[H_{\alpha,q}(f) \right](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1} f_o.$$

It satisfies the following relations:

- For $\alpha = -\frac{1}{2}$, $\Lambda_{\alpha,q} = \partial_q$.

- $\Lambda_{\alpha,q}$ lives $\mathcal{S}_q(\mathbb{R}_q)$ invariant.
- If f is odd then $\Lambda_{\alpha,q}(f)(x) = q^{2\alpha+1}\partial_q f(x) + [2\alpha+1]_q \frac{f(x)}{x}$ and if f is even then $\Lambda_{\alpha,q}(f)(x) = \partial_q f(x)$.
- For all $a \in \mathbb{C}$, $\Lambda_{\alpha,q}[f(ax)] = a\Lambda_{\alpha,q}(f)(ax)$.
- For all f and g such that $\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx$ exists, we have

$$\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = -\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1}d_qx. \tag{3.1}$$

It was shown in [2] that for each $\lambda \in \mathbb{C}$, the function

$$\psi_\lambda^{\alpha,q} : x \longmapsto j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2) \tag{3.2}$$

is the unique solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{\alpha,q}(f) = i\lambda f \\ f(0) = 1, \end{cases}$$

where $j_\alpha(\cdot; q^2)$ is the normalized third Jackson's q-Bessel function given by

$$j_\alpha(x; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2; q^2)_n (q^{2(\alpha+1)}; q^2)_n} ((1-q)x)^{2n}. \tag{3.3}$$

The function $\psi_\lambda^{\alpha,q}(x)$, has a unique extension to $\mathbb{C} \times \mathbb{C}$ and verifies the following properties.

- $\psi_{a\lambda}^{\alpha,q}(x) = \psi_\lambda^{\alpha,q}(ax) = \psi_{ax}^{\alpha,q}(\lambda), \quad \forall a, x, \lambda \in \mathbb{C}$.
- For all $x, \lambda \in \mathbb{R}_q$,

$$|\psi_\lambda^{\alpha,q}(x)| \le \frac{4}{(q;q)_\infty}. \tag{3.4}$$

The q-Dunkl transform $F_D^{\alpha,q}$ is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ (see [2]) by

$$F_D^{\alpha,q}(f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1}d_qx,$$

where

$$c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}.$$

It satisfies the following properties:

- For $\alpha = -\frac{1}{2}$, $F_D^{\alpha,q}$ is the q^2-analogue Fourier transform $\widehat{f}(\cdot; q^2)$ given by (see [12, 11])

$$\widehat{f}(\lambda; q^2) = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_{-\infty}^{+\infty} f(x)e(-i\lambda x; q^2)d_qx.$$

- On the even functions space, $F_D^{\alpha,q}$ coincides with the q-Bessel transform given by (see [2])

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_0^{+\infty} f(x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} d_qx.$$

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have:

$$\|F_D^{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,\alpha,q}. \tag{3.5}$$

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, such that $xf \in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$F_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F_D^{\alpha,q}(f)(\lambda) \tag{3.6}$$

  and

$$\Lambda_{\alpha,q}(F_D^{\alpha,q}(f)) = -iF_D^{\alpha,q}(xf). \tag{3.7}$$

- The q-Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) onto itself and satisfies the following Plancherel formula:

$$\|F_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}, \quad f \in L^2_{\alpha,q}(\mathbb{R}_q). \tag{3.8}$$

4. q-ANALOGUE OF THE HEISENBERG INEQUALITY

In this section, we provide a Heisenberg uncertainty principle for the q-Dunkl transform. For this purpose, inspired by the approach given in [10], we follow the steps of [1], using the operator $\Lambda_{\alpha,q}$ instead of the operator ∂_q, and consider the operators

$$L_{\alpha,q}(f)(x) = f_e(x) + q^{2\alpha+2}f_o(qx) \quad \text{and} \quad Qf(x) = xf(x),$$

and the q-commutator:

$$[D_{\alpha,q}, Q]_q = D_{\alpha,q}Q - qQD_{\alpha,q},$$

where

$$D_{\alpha,q} = L_{\alpha,q}\Lambda_{\alpha,q}.$$

The following theorem gives a Heisenberg uncertainty principle for the q-Dunkl transform $F_D^{\alpha,q}$.

Theorem 4.1. *For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have*

$$\frac{q^{2\alpha+1}}{1+q+q^{\alpha-1}+q^{\alpha}}\left|q\|f\|^2_{2,\alpha,q} + \left(1-q-\frac{[2\alpha+1]_q}{q^{2\alpha}}\right)\|f_o\|^2_{2,\alpha,q}\right| \leq \|xf\|_{2,\alpha,q}\|xF_D^{\alpha,q}(f)(x)\|_{2,\alpha,q}. \tag{4.1}$$

Proof. By Lemma 2.1 and simple calculus, we obtain

$$[D_{\alpha,q}, Q]_q f = q^{2\alpha+2}f_e + q^{2\alpha+1}\left(1-\frac{[2\alpha+1]_q}{q^{2\alpha}}\right)f_o.$$

Then, using the Cauchy-Schwarz inequality and the properties of the q-Dunkl operator, one can write

$$\begin{aligned}
&\left|q^{2\alpha+2}\|f_e\|^2_{2,\alpha,q} + q^{2\alpha+1}\left(1-\frac{[2\alpha+1]_q}{q^{2\alpha}}\right)\|f_o\|^2_{2,\alpha,q}\right| \\
&= |\langle [D_{\alpha,q}, Q]_q f; f\rangle| \\
&= |\langle D_{\alpha,q}Qf - qQD_{\alpha,q}f; f\rangle| \leq |\langle D_{\alpha,q}Qf; f\rangle| + q\,|\langle QD_{\alpha,q}f; f\rangle| \\
&= |\langle D_{\alpha,q}(xf_e + xf_o); f\rangle| + q\,|\langle D_{\alpha,q}f; xf\rangle| \\
&= \left|\langle \Lambda_{\alpha,q}(xf_e) + q^{2\alpha+2}\Lambda_{\alpha,q}(xf_o)(qx); f\rangle\right| \\
&\qquad + q\left|\langle q^{2\alpha+2}\Lambda_{\alpha,q}(f_e)(qx) + \Lambda_{\alpha,q}(f_o)(x); xf\rangle\right|
\end{aligned}$$

$$\begin{aligned}
&\leq |\langle \Lambda_{\alpha,q}(xf_e); f\rangle| + q^{2\alpha+2}\,|\langle \Lambda_{\alpha,q}(xf_o)(qx); f\rangle| \\
&\qquad + q^{2\alpha+3}\,|\langle \Lambda_{\alpha,q}(f_e)(qx); xf\rangle| + q\,|\langle \Lambda_{\alpha,q}(f_o)(x); xf\rangle| \\
&= |\langle \Lambda_{\alpha,q}(xf_e); f\rangle| + q^{2\alpha+1}\,|\langle \Lambda_{\alpha,q}(xf_o(q.)); f\rangle| \\
&\qquad + q^{2\alpha+2}\,|\langle \Lambda_{\alpha,q}(f_e(q.)); xf\rangle| + q\,|\langle \Lambda_{\alpha,q}(f_o)(x); xf\rangle| \\
&\leq \|xf_e\|_{2,\alpha,q}\|xF_D^{\alpha,q}(f)\|_{2,\alpha,q} + q^{\alpha-1}\|xf_o\|_{2,\alpha,q}\|xF_D^{\alpha,q}(f)\|_{2,\alpha,q} \\
&\qquad + q^{\alpha}\|xf_e\|_{2,\alpha,q}\|xF_D^{\alpha,q}(f)\|_{2,\alpha,q} + q\|xf_o\|_{2,\alpha,q}\|xF_D^{\alpha,q}(f)\|_{2,\alpha,q} \\
&\leq (1+q+q^{\alpha-1}+q^{\alpha})\|xf\|_{2,\alpha,q}\|xF_D^{\alpha,q}(f)\|_{2,\alpha,q},
\end{aligned}$$

which achieves the proof. □

As a consequence, we obtain a Heisenberg-Weyl uncertainty principle for the q^2-analogue Fourier transform (by taking $\alpha = -1/2$) and the q-Bessel transform (in the even case).

Corollary 4.2.

(1) *For* $f \in \mathcal{S}_q(\mathbb{R}_q)$, *we have*

$$\frac{q}{1+q+q^{-3/2}+q^{-1/2}}\|f\|_{2,q}^2 \leq \|xf\|_{2,q}\|\lambda\widehat{f}(\lambda; q^2)\|_{2,q}. \tag{4.2}$$

(2) *For an even function* $f \in \mathcal{S}_q(\mathbb{R}_q)$, *we have*

$$\frac{q^{2\alpha+2}}{1+q+q^{\alpha-1}+q^{\alpha}}\|f\|_{2,\alpha,q}^2 \leq \|xf\|_{2,\alpha,q}\|\lambda\mathcal{F}_{\alpha,q}(f)(\lambda)\|_{2,\alpha,q}. \tag{4.3}$$

We remark that when q tends to 1^-, (4.2) tends at least formally to the classical Heisenberg uncertainty principle given by (1.1).

5. LOCAL UNCERTAINTY PRINCIPLES

In this section, we will state, for the q-Dunkl transform, two local uncertainty principles leading to a new Heisenberg-Weyl type inequality.

Notations: For $E \subset \mathbb{R}_q$ and f defined on $\mathbb{R}_q$, we write

$$\int_E f(t)d_qt = \int_{-\infty}^{\infty} f(t)\chi_E(t)d_qt \quad \text{and} \quad |E|_\alpha = \int_E |t|^{2\alpha+1}d_qt,$$

where χ_E is the characteristic function of E.

Theorem 5.1. *If* $0 < a < \alpha+1$, *then for all bounded subsets* E *of* $\mathbb{R}_q$ *and all* $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\int_E |F_D^{\alpha,q}(f)(\lambda)|^2|\lambda|^{2\alpha+1}\,d_q\lambda \leq K_{a,\alpha}|E|^{\frac{a}{\alpha+1}}\|x^a f\|_{2,\alpha,q}^2, \tag{5.1}$$

where

$$K_{a,\alpha} = \left(\frac{2\widetilde{c}_{\alpha,q}}{\sqrt{[2(\alpha+1-a)]_q}}\left(\frac{\alpha+1-a}{a}\right)\right)^{\frac{2a}{\alpha+1}}\left(\frac{\alpha+1}{\alpha+1-a}\right)^2$$

and $\widetilde{c}_{\alpha,q} = \frac{2c_{\alpha,q}}{(q;q)_\infty}$.

Proof. For $r > 0$, let $\chi_r = \chi_{[-r,r]}$ the characteristic function of $[-r, r]$ and $\widetilde{\chi}_r = 1 - \chi_r$. Then for $r > 0$, we have, since $f \cdot \chi_r \in L^1_q(\mathbb{R}_q)$,

$$\begin{aligned}\left(\int_E |F_D^{\alpha,q}(f)(\lambda;q^2)|^2|\lambda|^{2\alpha+1}\,d_q\lambda\right)^{1/2} &= \|F_D^{\alpha,q}(f)\cdot\chi_E\|_{2,\alpha,q}\\ &\le \|F_D^{\alpha,q}(f\cdot\chi_r)\chi_E\|_{2,\alpha,q} + \|F_D^{\alpha,q}(f\cdot\widetilde{\chi}_r)\chi_E\|_{2,\alpha,q}\\ &\le |E|_\alpha^{1/2}\|F_D^{\alpha,q}(f\cdot\chi_r)\|_{\infty,q} + \|F_D^{\alpha,q}(f\cdot\widetilde{\chi}_r)\|_{2,\alpha,q}.\end{aligned}$$

Now, on the one hand, we have by the relation (3.5) and the Cauchy-Schwartz inequality,

$$\begin{aligned}\|F_D^{\alpha,q}(f.\chi_r)\|_{\infty,q} &\le \widetilde{c}_{\alpha,q}\|f\cdot\chi_r\|_{1,\alpha,q}\\ &= \widetilde{c}_{\alpha,q}\|x^{-a}\chi_r\cdot x^a f\|_{1,\alpha,q}\\ &\le \widetilde{c}_{\alpha,q}\|x^{-a}\chi_r\|_{2,\alpha,q}\|x^a f\|_{2,\alpha,q}\\ &\le \frac{2\widetilde{c}_{\alpha,q}}{\sqrt{[2(\alpha+1-a)]_q}}r^{(\alpha+1)-a}\|x^a f\|_{2,\alpha,q}.\end{aligned}$$

On the other hand, since $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have $f\cdot\widetilde{\chi}_r \in L^2_{\alpha,q}(\mathbb{R}_q)$ and by the Plancherel formula, we obtain

$$\begin{aligned}\|F_D^{\alpha,q}(f.\widetilde{\chi}_r)\|_{2,\alpha,q} &= \|f\cdot\widetilde{\chi}_r\|_{2,\alpha,q}\\ &= \|x^{-a}\widetilde{\chi}_r\cdot x^a f\|_{2,\alpha,q}\\ &\le \|x^{-a}\widetilde{\chi}_r\|_{\infty,q}\|x^a f\|_{2,\alpha,q}\\ &\le r^{-a}\|x^a f\|_{2,\alpha,q}.\end{aligned}$$

So,

$$\left(\int_E |F_D^{\alpha,q}(f)(\lambda)|^2|\lambda|^{2\alpha+1}\,d_q\lambda\right)^{\frac12} \le \left(\frac{2\widetilde{c}_{\alpha,q}}{\sqrt{[2(\alpha+1-a)]_q}}|E|_\alpha^{\frac12}r^{\alpha+1-a} + r^{-a}\right)\|x^a f\|_{2,\alpha,q}.$$

The desired result is obtained by minimizing the right hand side of the previous inequality over $r > 0$. □

Corollary 5.2. *For $\alpha \ge -\frac12$, $\quad 0 < a < \alpha+1$ and $b > 0$, we have for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$,*

$$\|f\|_{2,\alpha,q}^{(a+b)} \le K_{a,b,\alpha}\|x^a f\|_{2,\alpha,q}^b\|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^a, \tag{5.2}$$

with

$$K_{a,b,\alpha} = \left[\left(\frac{b}{a}\right)^{\frac{a}{a+b}} + \left(\frac{a}{b}\right)^{\frac{b}{a+b}}\right]^{\frac{a+b}{2}} (2K_{a,\alpha})^{\frac{b}{2}}\frac{q^{-(2\alpha+1)(a+b)}}{([2\alpha+2]_q)^{\frac{ab}{2(\alpha+1)}}}$$

where $K_{a,\alpha}$ is the constant given in Theorem 5.1.

Proof. For $r > 0$, we put $E_r =]-r, r[\cap\mathbb{R}_q$ and $\widetilde{E}_r$ the supplementary of E_r in $\mathbb{R}_q$.
We have E_r is a bounded subset of $\mathbb{R}_q$ and $|E_r|_\alpha \le 2\frac{r^{2\alpha+2}}{[2\alpha+2]_q}$. Then the Plancherel formula and the previous theorem lead to

$$\begin{aligned}\|f\|_{2,\alpha,q}^2 &= \|F_D^{\alpha,q}(f)\|_{2,\alpha,q}^2\\ &= \int_{E_r}|F_D^{\alpha,q}(f)|^2(\lambda)|\lambda|^{2\alpha+1}d_q\lambda + \int_{\widetilde{E}_r}|F_D^{\alpha,q}(f)|^2(\lambda)|\lambda|^{2\alpha+1}d_q\lambda\end{aligned}$$

$$\leq 2K_{a,\alpha}|E_r|_\alpha^{\frac{a}{\alpha+1}}\|x^a f\|_{2,\alpha,q}^2 + r^{-2b}\|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^2$$
$$\leq 2\frac{K_{a,\alpha}}{[2\alpha+2]_q^{\frac{a}{\alpha+1}}} r^{2a}\|x^a f\|_{2,\alpha,q}^2 + r^{-2b}\|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^2.$$

The desired result follows by minimizing the right expressions over $r > 0$. □

Theorem 5.3. *For $\alpha \geq -\frac{1}{2}$ and $a > \alpha + 1$, there exists a constant $K'_{a,\alpha,q}$ such that for all bounded subsets E of $\mathbb{R}_q$ and all f in $L^2_{\alpha,q}(\mathbb{R}_q)$, we have*

$$\int_E |F_D^{\alpha;q}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda \leq K'_{a,\alpha,q}|E|_\alpha \, \|f\|_{2,\alpha,q}^{2(1-\frac{\alpha+1}{a})} \, \|x^a f\|_{2,\alpha,q}^{2\frac{\alpha+1}{a}}. \tag{5.3}$$

The proof of this result needs the following lemmas.

Lemma 5.4. *Suppose $a > \alpha + 1$, then for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ such that $x^a f \in L^2_{\alpha,q}(\mathbb{R}_q)$,*

$$\|f\|_{1,\alpha,q}^2 \leq K_2 \left[\|f\|_{2,\alpha,q}^2 + \|x^a \, f\|_{2,\alpha,q}^2\right], \tag{5.4}$$

where

$$K_2 = 2(1-q)\frac{(q^{2a}, q^{2a}, -q^{2\alpha+2}, -q^{2(a-\alpha-1)}; q^{2a})_\infty}{(q^{2\alpha+2}, q^{2(a-\alpha-1)}, -q^{2a}, -1; q^{2a})_\infty}.$$

Proof. From ([4, Example 1]) and Hölder's inequality, we have

$$\|f\|_{1,\alpha,q}^2 = \left[\int_{-\infty}^{+\infty}(1+|x|^{2a})^{\frac{1}{2}}|f(x)|(1+|x|^{2a})^{-\frac{1}{2}}|x|^{2\alpha+1}d_q x\right]^2$$
$$\leq K_2\left[\|f\|_{2,\alpha,q}^2 + \|x^a \, f\|_{2,\alpha,q}^2\right],$$

where

$$K_2 = 2\int_0^{+\infty}\frac{x^{2\alpha+1}}{1+x^{2a}}d_q x$$
$$= 2(1-q)\frac{(q^{2a}, q^{2a}, -q^{2\alpha+2}, -q^{2(a-\alpha-1)}; q^{2a})_\infty}{(q^{2\alpha+2}, q^{2(a-\alpha-1)}, -q^{2a}, -1; q^{2a})_\infty}.$$

□

Lemma 5.5. *Suppose $a > \alpha+1$, then for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ such that $x^a f \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have*

$$\|f\|_{1,\alpha,q} \leq K_3\|f\|_{2,\alpha,q}^{(1-\frac{\alpha+1}{a})} \, \|x^a \, f\|_{2,\alpha,q}^{\frac{\alpha+1}{a}}, \tag{5.5}$$

where

$$K_3 = K_3(a,\alpha,q) = \left[\left(q^{2(\alpha+1)}(\frac{a}{\alpha+1}-1)\right)^{\frac{\alpha+1}{a}} q^{-2(\alpha+1)}(1+\frac{\alpha+1}{a-\alpha-1}) \, K_2\right]^{\frac{1}{2}}.$$

Proof. For $s \in \mathbb{R}_q$, define the function f_s by $f_s(x) = f(sx), x \in \mathbb{R}_q$.
We have

$$\|f_s\|_{1,\alpha,q} = s^{-2(\alpha+1)}\|f\|_{1,\alpha,q}, \qquad \|x^a \, f_s\|_{2,\alpha,q}^2 = s^{-2(\alpha+a+1)}\|x^a \, f\|_{2,\alpha,q}^2.$$

Replacement of f by f_s in Lemma 5.4 gives:

$$\|f\|_{1,\alpha,q}^2 \leq K_2\left[s^{2(\alpha+1)}\|f\|_{2,\alpha,q}^2 + s^{2(\alpha-a+1)}\|x^a \, f\|_{2,\alpha,q}^2\right].$$

Now, for all $r > 0$, put $\alpha(r) = \frac{\mathrm{Log}(r)}{\mathrm{Log}(q)} - E\left(\frac{\mathrm{Log}(r)}{\mathrm{Log}(q)}\right)$. We have $s = \frac{r}{q^{\alpha(r)}} \in \mathbb{R}_q$ and $r \leq s < \frac{r}{q}$. Then, for all $r > 0$,

$$\|f\|^2_{1,\alpha,q} \leq K_2 \left[\left(\frac{r}{q}\right)^{2(\alpha+1)} \|f\|^2_{2,\alpha,q} + r^{2(\alpha-a+1)} \|x^a\ f\|^2_{2,\alpha,q} \right].$$

The right hand side of this inequality is minimized by choosing

$$r = \left(\frac{a}{\alpha+1} - 1\right)^{\frac{1}{2a}} q^{\frac{\alpha+1}{a}} \|f\|^{-\frac{1}{a}}_{2,\alpha,q} \|x^a\ f\|^{\frac{1}{a}}_{2,\alpha,q}.$$

When this is done we obtain the result. □

Proof of Theorem 5.3. Let E be a bounded subset of $\mathbb{R}_q$. When the right hand side of the inequality is finite, Lemma 5.4 implies that $f \in L^1_q(\mathbb{R}_q)$, so, $F_D^{\alpha,q}(f)$ is defined and bounded on $\mathbb{R}_q$. Using Lemma 5.5, the relation (3.5) and the fact that

$$\int_E |F_D^{\alpha,q}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q\lambda \leq |\ E\ |_\alpha\ \|F_D^{\alpha,q}(f)\|^2_{\infty,q},$$

we obtain the result with

$$\begin{aligned} K'_{a,\alpha,q} &= \frac{4(1+q)^{-2\alpha}}{\Gamma^2_{q^2}(\alpha+1)(q;q)^2_\infty} K_3^2 \\ &= \frac{8(1-q)(1+q)^{-2\alpha}}{\Gamma^2_{q^2}(\alpha+1)(q;q)^2_\infty} \left(q^{2(\alpha+1)}\left(\frac{a}{\alpha+1} - 1\right)\right)^{\frac{\alpha+1}{a}} q^{-2(\alpha+1)} \left(1 + \frac{\alpha+1}{a-\alpha-1}\right) \\ &\quad \times \frac{(q^{2a}, q^{2a}, -q^{2\alpha+2}, -q^{2(a-\alpha-1)}; q^{2a})_\infty}{(q^{2\alpha+2}, q^{2(a-\alpha-1)}, -q^{2a}, -1; q^{2a})_\infty}. \end{aligned}$$

□

Corollary 5.6. *For $\alpha \geq -\frac{1}{2}$, $a > \alpha + 1$ and $b > 0$, we have for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$,*

(5.6)
$$\|f\|^{(a+b)}_{2,\alpha,q} \leq K'_{a,b,\alpha} \|x^a f\|^b_{2,\alpha,q} \|\lambda^b F_D^{\alpha,q}(f)\|^a_{2,\alpha,q},$$

with

$$K'_{a,b,\alpha} = \left(\frac{K'_{a,\alpha,q}}{[2\alpha+2]_q}\right)^{\frac{ab}{2\alpha+2}} \left(q^{-(4\alpha+2)}\left[\left(\frac{b}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha+b+1}} + \left(\frac{b}{\alpha+1}\right)^{-\frac{b}{\alpha+b+1}}\right]\right)^{\frac{a(\alpha+b+1)}{2(\alpha+1)}},$$

where $K'_{a,\alpha,q}$ is the constant given in the previous theorem.

Proof. The same techniques as in Corollary 5.2 give the result. □

The following result gives a new Heisenberg-Weyl type inequality for the q-Dunkl transform.

Theorem 5.7. *For $\alpha \geq -\frac{1}{2}$, $\alpha \neq 0$, we have for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$,*

(5.7)
$$\|f\|^2_{2,\alpha,q} \leq K_\alpha \|xf\|_{2,\alpha,q} \|\lambda F_D^{\alpha,q}(f)\|_{2,\alpha,q},$$

with

$$K_\alpha = \begin{cases} K_{1,1,\alpha} & \textit{if} \quad \alpha > 0 \\ K'_{1,1,\alpha} & \textit{if} \quad \alpha < 0. \end{cases}$$

Proof. The result follows from Corollaries 5.2 and 5.6, by taking $a = b = 1$. □

Remark 1. Note that Theorem 4.1 and Theorem 5.7 are both Heisenberg-Weyl type inequalities for the q-Dunkl transform. However, the constants in the two theorems are different, the first one seems to be more optimal. Moreover, Theorem 4.1 is true for every $\alpha > -\frac{1}{2}$ and uses both f and f_0, in contrast to Theorem 5.7, which is true only for $\alpha \neq 0$ and uses only f.

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