Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 2, Issue 1, Article 3, 2001

ON MULTIPLICATIVELY PERFECT NUMBERS

J. SÁNDOR

BABEŞ-BOLYAI UNIVERSITY, 3400 CLUJ-NAPOCA, ROMANIA.
jsandor@math.ubbcluj.ro

Received 7 January, 2000; accepted 18 August, 2000
Communicated by L. Toth

ABSTRACT. We study multiplicatively perfect, superperfect and analogous numbers . Connection to various arithmetic functions is pointed out. New concepts, inequalities and asymptotic evaluations are introduced.

Key words and phrases: Perfect numbers, arithmetic functions, inequalities in Number theory.

2000 *Mathematics Subject Classification.* 11A25,11N56, 26D15.

1. INTRODUCTION

It is well-known that a number n is said to be perfect if the **sum** of aliquot divisors of n is equal to n. By introducing the function σ (sum of divisors), this can be written equivalently as

$$\sigma(n) = 2n. \tag{1.1}$$

The Euclid-Euler theorem gives the form of even perfect numbers: $n = 2^k p$, where $p = 2^{k+1} - 1$ is prime ("Mersenne prime"). No odd perfect numbers are known. The number n is said to be super-perfect if

$$\sigma(\sigma(n)) = 2n. \tag{1.2}$$

The Suryanarayana-Kanold theorem [16], [4] gives the general form of even super-perfect numbers: $n = 2^k$, where $2^{k+1} - 1 = p$ is a prime. No odd super-perfect numbers are known. For new proofs of these results, see [10], [11]. Many open problems are stated e.g. in [1], [10].

2. m-PERFECT NUMBERS

Let $T(n)$ denote the **product** of all divisors of n. There are many numbers n with the property $T(n) = n^2$, but none satisfying $T(T(n)) = n^2$. Let us call the number $n > 1$ **multiplicatively perfect** (or, for short, m-perfect) if

$$T(n) = n^2, \tag{2.1}$$

ISSN (electronic): 1443-5756

019-99

and **multiplicatively super-perfect** (m-super-perfect), if

$$T(T(n)) = n^2. \tag{2.2}$$

To begin with, we prove the following little result:

Theorem 2.1. *All m-perfect numbers n have one of the following forms: $n = p_1p_2$ or $n = p_1^3$, where p_1, p_2 are arbitrary, distinct primes. There are no m-super-perfect numbers.*

Proof. Firstly, we note that if $d_1, d_2, \ldots, d_s$ are all divisors of n, then

$$\{d_1, d_2, \ldots, d_s\} = \left\{\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_s}\right\},$$

implying that

$$d_1d_2 \ldots d_s = \frac{n}{d_1} \cdot \frac{n}{d_2} \ldots \frac{n}{d_s},$$

i.e.

$$T(n) = n^{s/2}, \tag{2.3}$$

where $s = d(n)$ denotes the number of (distinct) divisors of n.

Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ be the prime factorisation of $n > 1$. It is well-known that $d(n) = (\alpha_1 + 1) \ldots (\alpha_r + 1)$, so equation (2.1) combined with (2.3) gives

$$(\alpha_1 + 1) \ldots (\alpha_r + 1) = 4. \tag{2.4}$$

Since $\alpha_i + 1 > 1$, for $r \geq 2$ we can have only $\alpha_1 + 1 = 2$, $\alpha_2 + 1 = 2$, implying $\alpha_1 = \alpha_2 = 1$, i.e. $n = p_1p_2$. For $r = 1$ we have $\alpha_1 + 1 = 4$, i.e. $\alpha_1 = 3$, giving $n = p^3$. There are no other solutions $n > 1$ ($n = 1$ is a trivial solution) of equation (2.1).

On the other hand, let us remark that for $n \geq 2$ one has $d(n) \geq 2$, so

$$T(n) \geq n \tag{2.5}$$

with equality only for $n =$ prime. If $n \neq$ prime, then it is immediate that $d(n) \geq 3$, giving

$$T(n) \geq n^{3/2} \quad \text{for} \quad n \neq \text{prime}. \tag{2.6}$$

Now, relations (2.5) and (2.6) together give

$$T(T(n)) \geq n^{9/4}, \quad n \neq \text{prime}. \tag{2.7}$$

Thus, by $9/4 > 2$, there are no non-trivial (i.e. $n \neq 1$) m-super-perfect numbers. In fact, we have found that the equation

$$T(T(n)) = n^a, \quad a \in \left(1, \frac{9}{4}\right) \tag{2.8}$$

has no nontrivial solutions. □

Note. According to the referee the notion of "m-perfect numbers", as well as Theorem 2.1 appears in [3].

Corollary 2.2. *$n = 6$ is the only perfect number, which is also m-perfect.*

Indeed, n cannot be odd, since by a result of Sylvester, an odd perfect number must have at least five prime divisors. If n is even, then $n = 2^kp = p_1p_2 \Leftrightarrow k = 1$, when $2 = p_1$ and $2^2 - 1 = 3$, when $3 = p_2$. Thus $n = 2 \cdot 3 = 6$.

3. k-m-PERFECT NUMBERS

In a similar manner, one can define k-m-perfect numbers by

$$T(n) = n^k \tag{3.1}$$

where $k \geq 2$ is given. Since the equation $(\alpha_1 + 1) \dots (\alpha_r + 1) = 2k$ has a finite number of solutions, the general form of k-multiply perfect numbers can be determined. We collect certain particular cases in the following.

Theorem 3.1. 1) *All tri-m-perfect numbers have the forms $n = p_1 p_2^2$ or $n = p_1^5$;*
2) *All 4-m-perfect numbers have the forms $n = p_1 p_2^3$ or $n = p_1 p_2 p_3$ or $n = p_1^7$;*
3) *All 5-m-perfect numbers have the forms $n = p_1 p_2^4$ or $n = p_1^9$;*
4) *All 6-m-perfect numbers have the forms $n = p_1 p_2 p_3^2$, $n = p_1 p_2^5$, $n = p_1^{11}$;*
5) *All 7-m-perfect numbers have the forms $n = p_1 p_2^6$, or $n = p_1^{13}$;*
6) *All 8-m-perfect numbers have the forms $n = p_1 p_2 p_3 p_4$ or $n = p_1 p_2 p_3^3$ or $n = p_1^3 p_2^3$, $n = p_1^{15}$;*
7) *All 9-m-perfect numbers have the forms $n = p_1 p_2^2 p_3^2$, $n = p_1 p_2^8$, $n = p_1^{17}$;*
8) *All 10-m-perfect numbers have the forms $n = p_1 p_2 p_3^4$, $n = p_1 p_2^9$, $n = p_1^{19}$, etc.*
(Here p_i denote certain distinct primes.)

Proof. We prove only the case 6). By relation (2.3) we must solve the equation

$$(\alpha_1 + 1) \dots (\alpha_r + 1) = 16 \tag{3.2}$$

in α_r and r. It is easy to see that the following four cases are possible:

i) $\alpha_1 + 1 = 2$, $\alpha_2 + 1 = 2$, $\alpha_3 + 1 = 2$, $\alpha_4 + 1 = 2$;
ii) $\alpha_1 + 1 = 2$, $\alpha_2 + 1 = 2$, $\alpha_3 + 1 = 4$, $\alpha_4 + 1 = 4$;
iii) $\alpha_1 + 1 = 4$, $\alpha_2 + 1 = 4$;
iv) $\alpha_1 + 1 = 16$.

This gives the general forms of all 8-m-perfect numbers, namely $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1)$ $n = p_1 p_2 p_3 p_4$; $(\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 3)$ $n = p_1 p_2 p_3^3$; $(\alpha_1 = 3, \alpha_2 = 3)$ $n = p_1^3 p_2^3$; $(\alpha_1 = 15)$ $n = p_1^{15}$. □

Corollary 3.2. 1) $n = 28$ *is the single perfect and tri-perfect number.*
2) *There are no perfect and 4-perfect numbers;*
3) $n = 496$ *is the only perfect number which is 5-m-perfect;*
4) *There are no perfect numbers which are 6-m-perfect;*
5) $n = 8128$ *is the only perfect number which is 7-m-perfect.*

In fact, we have:

Theorem 3.3. *Let p be a prime, with $2^p - 1$ prime too (i.e. $2^p - 1$ is a Mersenne prime). Then $2^{p-1}(2^p - 1)$ is the only perfect number, which is p-m-perfect.*

Proof. By writing $(\alpha_1 + 1) \dots (\alpha_r + 1) = 2p$ (p prime), the following cases are only possible:

i) $\alpha_1 + 1 = 2$, $\alpha_2 + 1 = p$;
ii) $\alpha_1 + 1 = 2p$.

Then $n = p_1 p_2^{p-1}$ or $n = p_1^{2p-1}$ are the general forms of p-m-perfect numbers. By the Euclid-Euler theorem $p_1 p_2^{p-1} = 2^{p-1}(2^p - 1)$ iff $p_2 = 2$ and $p_1 = 2^p - 1$ is prime. □

Remark 3.4. For $p < 10000$ the following Mersenne primes are known; namely for $p = 2$, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941. It is an open problem to show the existence of infinitely many Mersenne primes ([1]).

4. Some Results for k-m-Perfect Numbers

As we have seen, the equation (2.2), i.e. $T(T(n)) = n^2$ has no nontrivial solutions. A similar problem arises for the equation

$$T(T(n)) = n^k; \quad n > 1 \tag{4.1}$$

($k \geq 2$, fixed). By (2.3) we can see that this is equivalent to

$$\frac{d(n)d(T(n))}{4} = k. \tag{4.2}$$

Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r} > 1$ be the canonical representation of n. By $d(n) = (\alpha_1+1)\dots(\alpha_r+1)$, and (2.3) we have

$$T(n) = p_1^{\alpha_1(\alpha_1+1)\dots(\alpha_r+1)/2} \dots p_r^{\alpha_r(\alpha_1+1)\dots(\alpha_r+1)/2},$$

so (4.2) becomes equivalent to

(4.3)

$$(\alpha_1+1)\dots(\alpha_r+1)\left[\frac{\alpha_1(\alpha_1+1)\dots(\alpha_r+1)}{2}+1\right]\dots\left[\frac{\alpha_r(\alpha_1+1)\dots(\alpha_r+1)}{2}+1\right] = 4k,$$

and this, clearly has at most a finite number of solutions.

Theorem 4.1. 1) *Equation (4.1) is not solvable for* $k = 4, 5, 6$*;*

2) *For* $k = 3$ *the general solutions are* $n = p_1^2$*;*

3) *For* $k = 7$ *the solutions are* $n = p_1^3$*;*

4) *For* $k = 9$ *the solutions are:* $n = p_1 p_2$ *(*$p_1 \neq p_2$ *primes).*

Proof. For $k = 4, 5, 6$, from (4.3) we must solve the corresponding equations for 16, 20, 24. It is a simple exercise to verify these impossibilities. For $k = 3$ we have the single equality $12 = 3 \cdot 4$, when $\alpha_1 = 2$, $\frac{\alpha_1(\alpha_1+1)}{2} + 1 = 4$. For $k = 7$, $\alpha_1 = 3$ by $\frac{3\cdot 4}{2} + 1 = 7$ and $4 \cdot 7 = 28$. For $k = 9$ we have $2 \cdot 2 \cdot 3 \cdot 3 = 36$ and $\alpha_1 = \alpha_2 = 1$. □

Corollary 4.2. $n = 6$ *is the single perfect number which is also 9-super-*m*-perfect.*

Indeed, $p_1 p_2 = 2 \cdot (2^2 - 1) = 2 \cdot 3 = 6$ by Theorem 4.1 and the Euclid-Euler theorem.

Remark 4.3. By relation (2.6), by consecutive iteration we can deduce

$$\underbrace{T(T(\dots T(n)\dots))}_{k} \geq n^{3^k/2^k}$$

for $n \neq$ prime. Since $3^k > 2^k \cdot k$ for all $k \geq 1$ (induction: $3^{k+1} = 3 \cdot 3^k > 3 \cdot 2^k \cdot k > 2 \cdot 2^k(k+1) = 2^{k+1}(k+1)$) we can obtain the following generalization of equation (2.2):

$$\underbrace{T(T(\dots T(n)\dots))}_{k} = n^k$$

has no nontrivial solutions.

5. Other Results

By relation (2.3) we have

$$\frac{\log T(n)}{\log n} = \frac{d(n)}{2}. \tag{5.1}$$

Clearly, this implies

$$\lim_{n\to\infty} \inf \frac{\log T(n)}{\log n} = 1, \quad \lim_{n\to\infty} \sup \frac{\log T(n)}{\log n} = +\infty$$

(take e.g. $n = p$ (prime); $n = 2^k$ $(k \in \mathbb{N})$). Since $2 \le d(n) \le 2\sqrt{n}$ (see e.g. [13]) for $n \ge 2$ we get

$$1 \le \frac{\log T(n)}{\log n} \le \sqrt{n}.$$

By $2^{\omega(n)} \le d(n) \le 2^{\Omega(n)}$ (see e.g. [12]) we can deduce:

$$2^{\omega(n)-1} \le \frac{\log T(n)}{\log n} \le 2^{\Omega(n)-1} \quad (n \ge 2).$$

Since it is known by a theorem of Hardy and Ramanujan [2] that the normal order of magnitude of $\omega(n)$ and $\Omega(n)$ is $\log\log n$, the above double inequality implies that:
the normal order of magnitude of

(5.2) $$\log\log T(n) - \log\log n is (\log 2)(\log\log n - 1).$$

By a theorem of Wiegert ([17]) we have

$$\lim_{n\to\infty} \sup \frac{\log d(n) \log\log n}{\log n} = \log 2,$$

so by (5.1) we get:

(5.3) $$\lim_{n\to\infty} \sup \frac{(\log\log T(n))(\log\log n)}{\log n} = \log 2.$$

In fact, by a result of Nicolas and Robin ([7]), for $n \ge 3$ one has

$$\frac{\log d(n)}{\log 2} \le c\frac{\log n}{\log\log n} \quad (c \approx 1,5379\dots),$$

we can obtain the following inequality:

(5.4) $$\log\log T(n) \le \log\log n + \frac{k \log n}{\log\log n} - \log 2,$$

where $k = c\log 2$ and $n \ge 3$. This gives

(5.5) $$\lim_{n\to\infty} \frac{\log\log T(n)}{f(n)} = 0$$

for any positive function with $\frac{\log n}{f(n)\log\log n} \to 0$ $(n \to \infty)$.

By $\varphi(n)d(n) \ge n$ (see [14]) and $\varphi(n)d^2(n) \le n^2$ for $n \ne 4$ (see [8]) we get

$$\frac{n}{\varphi(n)} \le d(n) \le \frac{n}{\sqrt{\varphi(n)}} \quad \text{for} \quad n > 4,$$

and this, by (5.1) yields

(5.6) $$\frac{n}{2\varphi(n)} \le \frac{\log T(n)}{\log n} \le \frac{n}{2\sqrt{\varphi(n)}}.$$

Here φ is the usual Euler totient function.

Hence, the arithmetic function T is connected to the other classical arithmetic functions.

By $\sqrt{n} \le \frac{\sigma(n)}{d(n)} \le \frac{n+1}{2}$ (see [14], [5], [6]), we get

(5.7) $$\frac{\sigma(n)}{n+1} \le \frac{\log T(n)}{\log n} \le \frac{\sigma(n)}{2\sqrt{n}}.$$

For infinitely many primes p we have

$$d(p-1) > \exp\left(c\frac{\log p}{\log\log p}\right)$$

($c > 0$, constant, see [9]), so we have:

$$\log\log T(p-1) > \log\log(p-1) + \frac{c\log p}{\log\log p} - \log 2 \tag{5.8}$$

for infinitely many primes p, implying, e.g.

$$\lim_{\substack{p\to\infty \\ p\,prime}} \sup \frac{\log\log T(p-1)}{\log\log p} = +\infty \tag{5.9}$$

and

$$\lim_{n\to\infty} \inf \frac{(\log\log T(n))(\log\log n)}{\log n} > 0. \tag{5.10}$$

REFERENCES

[1] R.K. GUY, *Unsolved Problems in Number Theory*, Springer Verlag, 2nd Ed., 1994.

[2] G.H. HARDY AND S. RAMANUJAN, The normal number of prime factors of a number n, *Quart. J. Math.*, **48** (1920), 76–92.

[3] K. IRELAND AND M. ROSEN, *A Classical Introduction to Modern Number Theory*, Springer, 1982, Chapter 2.

[4] H.J. HANOLD, Über super-perfect numbers, *Elem. Math.*, **24** (1969), 61–62.

[5] E.S. LANGFORD, *See D.S. Mitrinović and M.S. Popadić, Inequalities in Number Theory*, Niš, 1978 (p. 44).

[6] D.S. MITRINOVIĆ AND J. SÁNDOR (in coop. with B. CRSTICI), *Handbook of Number Theory*, Kluwer Acad. Publ. 1995.

[7] J.L. NICOLAS AND G. ROBIN, Majorantions explicites pour le nombre de diviseurs de n, *Canad. Math. Bull.*, **26** (1983), 485–492.

[8] S. PORUBSKY, Problem E2351, *Amer. Math. Monthly*, **79** (1972), 394.

[9] K. PRACHER, *Primzahlverteilung*, Berlin, Göttingen, Heidelberg, 1957 (p. 370).

[10] J. SÁNDOR, On the composition of some arithmetic functions, *Studia Univ. Babeş-Bolyai, Math.*, **34**(1) (1989), 7–14.

[11] J. SÁNDOR, On perfect and super-perfect numbers (Romanian), *Lucr. Semin. Didactica Mat.*, **8** (1992), 167–168.

[12] J. SÁNDOR, On certain inequalities for arithmetic functions, *Notes Numb. Th. Discr. Math.*, **1** (1995), 27–32.

[13] W. SIERPINSKI, *Elementary theory of numbers*, Warsawa, 1964.

[14] R. SIVARAMAKRISHNAN AND C.S. VENKATARAMAN, Problem 5326, *Amer. Math. Monthly*, **72** (1965), 915.

[15] R. SIVARAMAKRISHNAN, Problem E1962, *Amer. Math. Monthly*, **74** (1967), 198.

[16] D. SURYANARAYANA, Super-perfect numbers, *Elem. Math.*, **24** (1969), 16–17.

[17] S. WIEGERT, Sur l'ordre de grandeur du nombre des diviseurs d'une entière, *Arkiv. für Math.*, **3** (1907), 1–9.