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NEIGHBORHOODS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. By making use of the familiar concept of neighborhoods of analytic functions, the authors prove several inclusion relations associated with the (n, δ)-neighborhoods of certain subclasses of analytic functions of complex order, which are introduced here by means of the Ruscheweyh derivatives. Special cases of some of these inclusion relations are shown to yield known results.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}(n)$ denote the class of functions f of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0;\ k \in \mathbb{N} \setminus \{1\};\ n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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Following the works of Goodman [9] and Ruscheweyh [14], we define the (n,δ)-neighborhood of a function $f \in \mathcal{A}(n)$ by (see also [2], [3], [4], and [16])

$$N_{n,\delta}(f) := \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k\, z^k \quad \text{and} \quad \sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leq \delta \right\}. \tag{1.2}$$

In particular, for the *identity* function

$$e(z) = z, \tag{1.3}$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k\, z^k \text{ and } \sum_{k=n+1}^{\infty} k\,|b_k| \leq \delta \right\}. \tag{1.4}$$

The above concept of (n,δ)-neighborhoods was extended and applied recently to families of *analytically* multivalent functions by Altintaş *et al.* [6] and to families of *meromorphically* multivalent functions by Liu and Srivastava ([10] and [11]). The main object of the present paper is to investigate the (n,δ)-neighborhoods of several subclasses of the class $\mathcal{A}(n)$ of *normalized* analytic functions in $\mathbb{U}$ with negative *and* missing coefficients, which are introduced below by making use of the Ruscheweyh derivatives.

First of all, we say that a function $f \in \mathcal{A}(n)$ is *starlike of complex order* γ $(\gamma \in \mathbb{C} \setminus \{0\})$, that is, $f \in \mathcal{S}_n^*(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C} \setminus \{0\}). \tag{1.5}$$

Furthermore, a function $f \in \mathcal{A}(n)$ is said to be *convex of complex order* γ $(\gamma \in \mathbb{C} \setminus \{0\})$, that is, $f \in \mathcal{C}_n(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C} \setminus \{0\}). \tag{1.6}$$

The classes $\mathcal{S}_n^*(\gamma)$ and $\mathcal{C}_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [12] and Wiatrowski [18], respectively (see also [5] and [7]).

Next, for the functions f_j $(j = 1,2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}\, z^k \qquad (j = 1,2), \tag{1.7}$$

let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2, defined by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1}\, a_{k,2}\, z^k =: (f_2 * f_1)(z). \tag{1.8}$$

Thus the Ruscheweyh derivative operator $D^\lambda : \mathcal{A} \longrightarrow \mathcal{A}$ is defined for $\mathcal{A} := \mathcal{A}(1)$ by

$$D^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1;\ f \in \mathcal{A}) \tag{1.9}$$

or, equivalently, by

$$D^\lambda f(z) := z - \sum_{k=2}^{\infty} \binom{\lambda + k - 1}{k - 1} a_k\, z^k \qquad (\lambda > -1;\ f \in \mathcal{A}) \tag{1.10}$$

for a function $f \in \mathcal{A}$ of the form (1.1). Here, *and in what follows*, we make use of the following standard notation:

$$\binom{\kappa}{k} := \frac{\kappa(\kappa-1)\cdots(\kappa-k+1)}{k!} \qquad (\kappa \in \mathbb{C};\ k \in \mathbb{N}_0) \tag{1.11}$$

for a binomial coefficient. In particular, we have

$$D^n f(z) = \frac{z\big(z^{n-1} f(z)\big)^{(n)}}{n!} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.12}$$

Finally, in terms of the Ruscheweyh derivative D^λ $(\lambda > -1)$ defined by (1.9) or (1.10) above, let $\mathcal{S}_n(\gamma, \lambda, \beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions f which satisfy the following inequality:

$$\left| \frac{1}{\gamma} \left(\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)} - 1 \right) \right| < \beta \tag{1.13}$$

$$(z \in \mathbb{U};\ \gamma \in \mathbb{C} \setminus \{0\};\ \lambda > -1;\ 0 < \beta \leq 1).$$

Also let $\mathcal{R}_n(\gamma, \lambda, \beta; \mu)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions f which satisfy the following inequality:

$$\left| \frac{1}{\gamma} \left((1-\mu)\frac{D^\lambda f(z)}{z} + \mu\left(D^\lambda f(z)\right)' - 1 \right) \right| < \beta \tag{1.14}$$

$$(z \in \mathbb{U};\ \gamma \in \mathbb{C} \setminus \{0\};\ \lambda > -1;\ 0 < \beta \leq 1;\ 0 \leq \mu \leq 1).$$

Various further subclasses of the classes $\mathcal{S}_n(\gamma, \lambda, \beta)$ and $\mathcal{R}_n(\gamma, \lambda, \beta; \mu)$ *with* $\gamma = 1$ were studied in many earlier works (*cf.*, *e.g.*, [8] and [17]; see also the references cited in these earlier works). Clearly, in the case of (for example) the class $\mathcal{S}_n(\gamma, \lambda, \beta)$, we have

$$\mathcal{S}_n(\gamma, 0, 1) \subset \mathcal{S}_n^*(\gamma) \quad \text{and} \quad \mathcal{S}_n(\gamma, 1, 1) \subset \mathcal{C}_n(\gamma) \tag{1.15}$$

$$(n \in \mathbb{N};\ \gamma \in \mathbb{C} \setminus \{0\}).$$

2. INCLUSION RELATIONS INVOLVING $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we shall require Lemma 1 and Lemma 2 below.

Lemma 1. *Let the function* $f \in \mathcal{A}(n)$ *be defined by* (1.1). *Then* f *is in the class* $\mathcal{S}_n(\gamma, \lambda, \beta)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} (\beta|\gamma| + k - 1)\, a_k \leq \beta|\gamma|. \tag{2.1}$$

Proof. We first suppose that $f \in \mathcal{S}_n(\gamma, \lambda, \beta)$. Then, by appealing to the condition (1.13), we readily obtain

$$\Re\left(\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)} - 1 \right) > -\beta|\gamma| \quad (z \in \mathbb{U}) \tag{2.2}$$

or, equivalently,

$$\Re\left(\frac{-\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1}(k-1)a_k z^k}{z - \sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} a_k z^k} \right) > -\beta|\gamma| \quad (z \in \mathbb{U}), \tag{2.3}$$

where we have made use of (1.10) *and* the definition (1.1). We now choose values of z on the real axis and let $z \to 1-$ through real values. Then the inequality (2.3) immediately yields the desired condition (2.1). Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find that

$$\begin{aligned}\left|\frac{z\left(D^{\lambda} f(z)\right)'}{D^{\lambda} f(z)} - 1\right| &= \left|\frac{\sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}(k-1)a_k z^k}{z - \sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}a_k z^k}\right| \\ &\leqq \frac{\beta|\gamma|\left(1 - \sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}a_k\right)}{1 - \sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}a_k} \\ &\leqq \beta|\gamma|. \end{aligned} \tag{2.4}$$

Hence, by the maximum modulus theorem, we have

$$f \in \mathcal{S}_n(\gamma, \lambda, \beta),$$

which evidently completes the proof of Lemma 1. □

Similarly, we can prove the following result.

Lemma 2. *Let the function* $f \in \mathcal{A}(n)$ *be defined by* (1.1). *Then* f *is in the class* $\mathcal{R}(\gamma, \lambda, \beta; \mu)$ *if and only if*

$$\sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}\left[\mu(k-1)+1\right]a_k \leqq \beta|\gamma|. \tag{2.5}$$

Remark 1. *A special case of Lemma* 1 *when*

$$n = 1, \qquad \gamma = 1, \qquad \textit{and} \qquad \beta = 1-\alpha \qquad (0 \leqq \alpha < 1)$$

was given earlier by Ahuja [1]. *Furthermore, if in Lemma* 1 *with*

$$n = 1, \qquad \gamma = 1, \qquad \textit{and} \qquad \beta = 1-\alpha \qquad (0 \leqq \alpha < 1),$$

we set $\lambda = 0$ *and* $\lambda = 1$, *we shall obtain the familiar results of Silverman* [15].

Our first inclusion relation involving $N_{n,\delta}(e)$ is given by Theorem 1 below.

Theorem 1. *If*

$$\delta := \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\binom{\lambda+n}{n}} \qquad (|\gamma| < 1), \tag{2.6}$$

then

$$\mathcal{S}_n(\gamma, \lambda, \beta) \subset N_{n,\delta}(e). \tag{2.7}$$

Proof. For a function $f \in \mathcal{S}_n(\gamma, \lambda, \beta)$ of the form (1.1), Lemma 1 immediately yields

$$(\beta|\gamma| + n)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} a_k \leqq \beta|\gamma|,$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\beta\,|\gamma|+n)\binom{\lambda+n}{n}}. \tag{2.8}$$

On the other hand, we also find from (2.1) and (2.8) that

$$\begin{aligned}\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} k a_k &\leq \beta|\gamma| + (1-\beta|\gamma|)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} a_k \\ &\leq \beta|\gamma| + (1-\beta|\gamma|)\binom{\lambda+n}{n}\frac{\beta|\gamma|}{(\beta\,|\gamma|+n)\binom{\lambda+n}{n}} \\ &\leq \frac{(n+1)\beta|\gamma|}{\beta|\gamma|+n} \quad (|\gamma|<1),\end{aligned}$$

that is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|\gamma|}{(\beta\,|\gamma|+n)\binom{\lambda+n}{n}} := \delta, \tag{2.9}$$

which, in view of the definition (1.4), proves Theorem 1. □

By similarly applying Lemma 2 instead of Lemma 1, we now prove Theorem 2 below.

Theorem 2. *If*

$$\delta := \frac{(n+1)\,\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}}, \tag{2.10}$$

then

$$\mathcal{R}_n(\gamma,\lambda,\beta;\mu) \subset N_{n,\delta}(e). \tag{2.11}$$

Proof. Suppose that a function $f \in \mathcal{R}\,(\gamma,\lambda,\beta;\mu)$ is of the form (1.1). Then we find from the assertion (2.5) of Lemma 2 that

$$\binom{\lambda+n}{n}(\mu n+1)\sum_{k=n+1}^{\infty} a_k \leq \beta\,|\gamma|\,,$$

which yields the following coefficient inequality:

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}}. \tag{2.12}$$

Making use of (2.5) in conjunction with (2.12), we also have

$$\begin{aligned}\mu\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} k a_k &\leq \beta\,|\gamma| + (\mu-1)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} a_k \\ &\leq \beta\,|\gamma| + (\mu-1)\binom{\lambda+n}{n}\frac{\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}},\end{aligned} \tag{2.13}$$

that is,

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n+1)\,\beta\,|\gamma|}{(\mu n+1)\dbinom{\lambda+n}{n}} =: \delta,$$

which, in light of the definition (1.4), completes the proof of Theorem 2. □

Remark 2. *By suitably specializing the various parameters involved in Theorem* 1 *and Theorem* 2*, we can derive the corresponding inclusion relations for many relatively more familiar function classes* $($*see also Equation* (1.15) *and Remark* 1 *above*$)$.

3. Neighborhoods for the Classes $\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta)$ and $\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu)$

In this section we determine the neighborhood for each of the classes

$$\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta) \quad \text{and} \quad \mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu),$$

which we define as follows. A function $f \in \mathcal{A}(n)$ is said to be in the class $\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta)$ if there exists a function $g \in \mathcal{S}_n(\gamma,\lambda,\beta)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\alpha \quad (z\in\mathbb{U};\ 0\leq\alpha<1)\,. \tag{3.1}$$

Analogously, a function $f \in \mathcal{A}(n)$ is said to be in the class $\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu)$ if there exists a function $g \in \mathcal{R}_n(\gamma,\lambda,\beta;\mu)$ such that the inequality (3.1) holds true.

Theorem 3. *If* $g \in \mathcal{S}_n(\gamma,\lambda,\beta)$ *and*

$$\alpha = 1 - \frac{(\beta\,|\gamma|+n)\,\delta\dbinom{\lambda+n}{n}}{(n+1)\left[(\beta\,|\gamma|+n)\dbinom{\lambda+n}{n}-\beta\,|\gamma|\right]}, \tag{3.2}$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta). \tag{3.3}$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from the definition (1.2) that

$$\sum_{k=n+1}^{\infty} k\,|a_k-b_k| \leq \delta, \tag{3.4}$$

which readily implies the coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k-b_k| \leq \frac{\delta}{n+1} \quad (n\in\mathbb{N}). \tag{3.5}$$

Next, since $g \in \mathcal{S}_n(\gamma,\lambda,\beta)$, we have [*cf.* Equation (2.8)]

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{(\beta\,|\gamma|+n)\dbinom{\lambda+n}{n}}, \tag{3.6}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}\\ &\leq \frac{\delta}{n+1}\cdot\frac{(\beta|\gamma|+n)\binom{\lambda+n}{n}}{(\beta|\gamma|+n)\binom{\lambda+n}{n}-\beta|\gamma|}\\ &= 1-\alpha, \end{aligned}\tag{3.7}$$

provided that α is given precisely by (3.2). Thus, by definition, $f\in\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta)$ for α given by (3.2). This evidently completes our proof of Theorem 3. □

Our proof of Theorem 4 below is much akin to that of Theorem 3.

Theorem 4. *If* $g\in\mathcal{R}_n(\gamma,\lambda,\beta;\mu)$ *and*

$$\alpha = 1-\frac{(\mu n+1)\,\delta\binom{\lambda+n}{n}}{(n+1)\left[(\mu n+1)\binom{\lambda+n}{n}-\beta|\gamma|\right]},\tag{3.8}$$

then

$$N_{n,\delta}(g)\subset\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu).\tag{3.9}$$

Remark 3. *Just as we indicated already in Section* 1 *and Remark* 2*, Theorem* 3 *and Theorem* 4 *can readily be specialized to deduce the corresponding neighborhood results for many simpler function classes.*

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