ON AN OPEN QUESTION REGARDING AN INTEGRAL INEQUALITY

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Abstract: In the paper "Notes on an integral inequality" published in *J. Inequal. Pure & Appl. Math.*, **7**(4) (2006), Art. 120, an open question was posed. In this short paper, we give the solution and we generalize the results of the mentioned paper.

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1. Introduction

The following open question was proposed in the paper [1]:

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\beta} f^{\alpha}(x)\,dx \tag{1.1}$$

hold for α and β?

In the above paper, the authors established some integral inequalities and derived their results using an analytic approach.

In the present paper, we give a solution and further generalization of the integral inequalities presented in [1].

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2. The Answer to the Posed Question

Throughout this paper, we suppose that $f(x)$ is a continuous and nonnegative function on $[0,1]$.

In [1]], the following lemma was proved.

Lemma 2.1. *If f satisfies*

$$\int_x^1 f(t)\,dt \geq \frac{1-x^2}{2}, \quad \forall x \in [0,1], \tag{2.1}$$

then

$$\int_0^1 x^{\alpha+1} f(x)\,dx \geq \frac{1}{\alpha+3}, \quad \forall \alpha > 0. \tag{2.2}$$

Theorem 2.2. *If the function f satisfies (2.1), then the inequality*

$$\int_0^1 x^{\beta} f^{\alpha}(x)\,dx \geq \frac{1}{\alpha+\beta+1} \tag{2.3}$$

holds for every real $\alpha \geq 1$ and $\beta > 0$.

Proof. Applying the AG inequality, we get

$$\frac{1}{\alpha} f^{\alpha}(x) + \frac{\alpha-1}{\alpha} x^{\alpha} \geq f(x)\, x^{\alpha-1}. \tag{2.4}$$

Multiplying both sides of (2.4) by x^{β} and integrating the resultant inequality from 0 to 1, we obtain

$$\int_0^1 x^{\beta} f^{\alpha}(x)\,dx + \frac{\alpha-1}{\alpha+\beta+1} \geq \alpha \int_0^1 x^{\alpha+\beta-1} f(x)\,dx. \tag{2.5}$$

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Taking into account Lemma 2.1, we have

$$\int_0^1 x^\beta f^\alpha(x)\,dx + \frac{\alpha-1}{\alpha+\beta+1} \geq \frac{\alpha}{\alpha+\beta+1}.$$

That is,

$$\int_0^1 x^\beta f^\alpha(x)\,dx \geq \frac{1}{\alpha+\beta+1}.$$

This completes the proof. □

Theorem 2.3. *If the function f satisfies (2.1), then*

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^\beta f^\alpha(x)\,dx \tag{2.6}$$

for every real $\alpha \geq 1$ *and* $\beta > 0$.

Proof. Using the AG inequality, we obtain

$$\frac{\alpha}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\beta}{\alpha+\beta} x^{\alpha+\beta} \geq x^\beta f^\alpha(x). \tag{2.7}$$

Integrating both sides of (2.7), we get

$$\frac{\alpha}{\alpha+\beta}\int_0^1 f^{\alpha+\beta}(x)\,dx + \frac{\beta}{(\alpha+\beta)(\alpha+\beta+1)} \geq \int_0^1 x^\beta f^\alpha(x)\,dx. \tag{2.8}$$

From

$$\int_0^1 x^\beta f^\alpha(x)\,dx = \frac{\alpha}{\alpha+\beta}\int_0^1 x^\beta f^\alpha(x)\,dx + \frac{\beta}{\alpha+\beta}\int_0^1 x^\beta f^\alpha(x)\,dx$$

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and by virtue of Theorem 2.3, it follows that

$$\int_0^1 x^\beta f^\alpha(x)\,dx \geq \frac{\alpha}{\alpha+\beta}\int_0^1 x^\beta f^\alpha(x)\,dx + \frac{\beta}{(\alpha+\beta)(\alpha+\beta+1)}. \tag{2.9}$$

From this inequality and using (2.8) we have,

$$\frac{\alpha}{\alpha+\beta}\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \frac{\alpha}{\alpha+\beta}\int_0^1 x^\beta f^\alpha(x)\,dx.$$

Thus (2.6) is proved. □

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