Journal of Inequalities in Pure and Applied Mathematics

AN EXTENDED HARDY-HILBERT INEQUALITY AND ITS APPLICATIONS

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volume 7, issue 1, article 30, 2006.

Received 17 February, 2004; accepted 10 November, 2005.

Communicated by: L. Pick

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ISSN (electronic): 1443-5756
032-04

Abstract

In this paper, it is shown that an extended Hardy-Hilbert's integral inequality with weights can be established by introducing a power-exponent function of the form $ax^{1+x}(a > 0, x \in [0, +\infty))$, and the coefficient $\frac{\pi}{(a)^{1/q}(b)^{1/p} \sin \pi/p}$ is shown to be the best possible constant in the inequality. In particular, for the case $p = 2$, some extensions on the classical Hilbert's integral inequality are obtained. As applications, generalizations of Hardy-Littlewood's integral inequality are given.

2000 Mathematics Subject Classification: 26D15.

Key words: Power-exponent function, Weight function, Hardy-Hilbert's integral inequality, Hardy-Littlewood's integral inequality.

The author is grateful to the referees for valuable suggestions and helpful comments in this subject.

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1. Introduction

The famous Hardy-Hilbert's integral inequality is

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}} \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}}, \tag{1.1}$$

where $p > 1$, $q = p/(p-1)$ and the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is best possible (see [1]). In particular, when $p = q = 2$, the inequality (1.1) is reduced to the classical Hilbert integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \le \pi \left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}}, \tag{1.2}$$

where the coefficient π is best possible.

Recently, the following result was given by introducing the power function in [2]:

$$\begin{aligned}\int_a^b \int_a^b \frac{f(x)\,g(y)}{x^t+y^t}dxdy &\le \left\{\omega(t,p,q)\int_a^b x^{1-t} f^p(x)\,dx\right\}^{\frac{1}{p}} \\ &\quad\times \left\{\omega(t,q,p)\int_a^b x^{1-t} g^q(x)\,dx\right\}^{\frac{1}{q}},\end{aligned} \tag{1.3}$$

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where t is a parameter which is independent of x and y, $\omega(t,p,q) = \frac{\pi}{t\sin\frac{\pi}{pt}} - \varphi(q)$ and here the function φ is defined by

$$\varphi(r) = \int_0^{a/b} \frac{u^{t-2+1/r}}{1+u^t} du, \quad r = p, q.$$

However, in [2] the best constant for (1.3) was not determined.

Afterwards, various extensions on the inequalities (1.1) and (1.2) have appeared in some papers (such as [3, 4] etc.). The purpose of the present paper is to show that if the denominator $x+y$ of the function on the left-hand side of (1.1) is replaced by the power-exponent function $ax^{1+x} + by^{1+y}$, then we can obtain a new inequality and show that the coefficient $\frac{\pi}{(a)^{1/q}(b)^{1/p}\sin\pi/p}$ is the best constant in the new inequality. In particular if $p = 2$ then several extensions of (1.2) follow. As its applications, it is shown that extensions on the Hardy-Littlewood integral inequality can be established.

Throughout this paper we stipulate that $a > 0$ and $b > 0$.

For convenience, we give the following lemma which will be used later.

Lemma 1.1. *Let* $h(x) = \frac{x}{1+x+x\ln x}$, $x \in (0,+\infty)$, *then there exists a function* $\varphi(x)$ $\left(0 \le \varphi(x) < \frac{1}{2}\right)$, *such that* $h(x) = \frac{1}{2} - \varphi(x)$.

Proof. Consider the function defined by

$$s(x) = \frac{1+x}{x} + \ln x, \quad x \in (0,+\infty).$$

It is easy to see that the minimum of $s(x)$ is 2. Hence $s(x) \ge 2$, and $h(x) = s^{-1}(x) \le \frac{1}{2}$. Obviously $h(x) = \frac{1}{s(x)} > 0$. We can define a nonnegative function

φ by

$$\varphi(x) = \frac{1 - x + x\ln x}{2(1 + x + x\ln x)} \quad x \in (0, +\infty). \tag{1.4}$$

Hence we have $h(x) = \frac{1}{2} - \varphi(x)$. The lemma follows. $\square$

2. Main Results

Define a function by

$$\omega(r,x) = x^{(1+x)(1-r)}\left(\frac{1}{2}-\varphi(x)\right)^{r-1} \quad x \in (0,+\infty), \tag{2.1}$$

where $r > 1$ and $\varphi(x)$ is defined by (1.4).

Theorem 2.1. *Let*

$$0 < \int_0^\infty \omega(p,x) f^p(x)\,dx < +\infty, \quad 0 < \int_0^\infty \omega(q,x) g^q(x)\,dx < +\infty,$$

the weight function $\omega(r,x)$ is defined by (2.1), $\frac{1}{p}+\frac{1}{q}=1$, and $p \geq q > 1$. Then

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x}+by^{1+y}}dxdy \\ \leq \frac{\mu\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty \omega(p,x) f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega(q,x) g^q(x)\,dx\right\}^{\frac{1}{q}}, \end{aligned} \tag{2.2}$$

where $\mu = (1/a)^{1/q}(1/b)^{1/p}$and the constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ is best possible.

Proof. Let $f(x) = F(x)\left\{(ax^{1+x})'\right\}^{\frac{1}{q}}$ and $g(y) = G(y)\left\{(by^{1+y})'\right\}^{\frac{1}{p}}$. Define two functions by

$$\alpha=\frac{F(x)\left\{\left(by^{1+y}\right)'\right\}^{\frac{1}{p}}}{\left(ax^{1+x}+by^{1+y}\right)^{\frac{1}{p}}}\left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{pq}}\quad\text{and}\tag{2.3}$$

$$\beta=\frac{G(y)\left\{\left(ax^{1+x}\right)'\right\}^{\frac{1}{q}}}{\left(ax^{1+x}+by^{1+y}\right)^{\frac{1}{q}}}\left(\frac{by^{1+y}}{ax^{1+x}}\right)^{\frac{1}{pq}}.$$

Let us apply Hölder's inequality to estimate the right hand side of (2.2) as follows:

$$\begin{aligned}\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{ax^{1+x}+by^{1+y}}dxdy&=\int_0^\infty\int_0^\infty\alpha\beta dxdy\\&\le\left\{\int_0^\infty\int_0^\infty\alpha^p dxdy\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty\beta^q dxdy\right\}^{\frac{1}{q}}.\end{aligned}\tag{2.4}$$

It is easy to deduce that

$$\begin{aligned}\int_0^\infty\int_0^\infty\alpha^p dxdy&=\int_0^\infty\int_0^\infty\frac{\left(by^{1+y}\right)'}{ax^{1+x}+by^{1+y}}\left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}}F^p(x)\,dxdy\\&=\int_0^\infty\omega_qF^p(x)\,dx.\end{aligned}$$

We compute the weight function ω_q as follows:

$$\omega_q = \int_0^\infty \frac{\left(by^{1+y}\right)'}{ax^{1+x}+by^{1+y}} \left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}} dy$$
$$= \int_0^\infty \frac{1}{ax^{1+x}+by^{1+y}} \left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}} d\left(by^{1+y}\right).$$

Let $t = by^{1+y}/ax^{1+x}$. Then we have

$$\omega_q = \int_0^\infty \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{q}} dt = \frac{\pi}{\sin\frac{\pi}{q}} = \frac{\pi}{\sin\frac{\pi}{p}}.$$

Notice that $F(x) = \left\{\left(ax^{1+x}\right)'\right\}^{-1/q} f(x)$. Hence we have

$$\int_0^\infty \int_0^\infty \alpha^p dx dy = \frac{\pi}{\sin\frac{\pi}{p}} \int_0^\infty \left(\left(ax^{1+x}\right)'\right)^{1-p} f^p(x)\, dx, \tag{2.5}$$

and ,similarly,

$$\int_0^\infty \int_0^\infty \beta^q dx dy = \frac{\pi}{\sin\frac{\pi}{p}} \int_0^\infty \left(\left(by^{1+y}\right)'\right)^{1-q} g^q(y)\, dy. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.3), we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{ax^{1+x}+by^{1+y}} dx dy \tag{2.7}$$

$$\leq \frac{\pi}{\sin \frac{\pi}{p}} \left\{ \int_0^\infty \left(\left(a x^{1+x} \right)' \right)^{1-p} f^p (x) \, dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty \left(\left(b y^{1+y} \right)' \right)^{1-q} g^q (y) \, dy \right\}^{\frac{1}{q}} .$$

We need to show that the constant factor $\frac{\pi}{\sin \frac{\pi}{p}}$ contained in (2.7) is best possible.

Define two functions by

$$\tilde{f}(x) = \begin{cases} 0, & x \in (0,1) \\ \left(a x^{1+x}\right)^{-(1+\varepsilon)/p} \left(a x^{1+x}\right)', & x \in [1, +\infty) \end{cases}$$

and

$$\tilde{g}(y) = \begin{cases} 0, & y \in (0,1) \\ \left(b y^{1+y}\right)^{-(1+\varepsilon)/q} \left(b y^{1+y}\right)', & y \in [1, +\infty) \end{cases} .$$

Assume that $0 < \varepsilon < \frac{q}{2p} \quad (p \geq q > 1)$. Then

$$\int_0^{+\infty} \left(\left(a x^{1+x} \right)' \right)^{1-p} \tilde{f}^p (x) \, dx = \int_1^{+\infty} \left(a x^{1+x} \right)^{-1-\varepsilon} d \left(a x^{1+x} \right) = \frac{1}{\varepsilon}.$$

Similarly, we have

$$\int_0^\infty \left(\left(b y^{1+y} \right)' \right)^{1-q} \tilde{g}^q (y) \, dy = \frac{1}{\varepsilon}.$$

If $\frac{\pi}{\sin\frac{\pi}{p}}$ is not best possible, then there exists $k > 0$, $k < \frac{\pi}{\sin\frac{\pi}{p}}$ such that

$$\begin{aligned}(2.8)\quad \int_0^\infty\int_0^\infty \frac{\tilde f(x)\,\tilde g(y)}{ax^{1+x}+by^{1+y}}dxdy &< k\left(\int_0^\infty \left(\left(ax^{1+x}\right)'\right)^{1-p}\tilde f^p(x)\,dx\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_0^\infty \left(\left(by^{1+y}\right)'\right)^{1-q}\tilde g^q(y)\,dy\right)^{\frac{1}{q}} = \frac{k}{\varepsilon}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{\tilde f(x)\,\tilde g(y)}{ax^{1+x}+by^{1+y}}dxdy \\ &= \int_1^\infty\int_1^\infty \frac{\left\{\left(ax^{1+x}\right)^{-\frac{1+\varepsilon}{p}}\left(ax^{1+x}\right)'\right\}\left\{\left(by^{1+y}\right)^{-\frac{1+\varepsilon}{q}}\left(by^{1+y}\right)'\right\}}{ax^{1+x}+by^{1+y}}dxdy \\ &= \int_1^\infty\left\{\int_1^\infty \frac{\left(by^{1+y}\right)^{-\frac{1+\varepsilon}{q}}}{ax^{1+x}+by^{1+y}}d\left(by^{1+y}\right)\right\}\left\{\left(ax^{1+x}\right)^{-\frac{1+\varepsilon}{p}}\left(ax^{1+x}\right)'\right\}dx \\ &= \int_1^\infty\left\{\int_{b/ax^{1+x}}^\infty \frac{1}{1+t}\left(\frac{1}{t}\right)^{-\frac{1+\varepsilon}{q}}dt\right\}\left(ax^{1+x}\right)^{-1-\varepsilon}d\left(ax^{1+x}\right) \\ &= \frac{1}{\varepsilon}\int_{b/ax^{1+x}}^\infty \frac{1}{1+t}\left(\frac{1}{t}\right)^{-\frac{1+\varepsilon}{q}}dt.\end{aligned}$$

If the lower limit b/ax^{1+x} of this integral is replaced by zero, then the resulting error is smaller than $\frac{\left(b/ax^{1+x}\right)^{\alpha}}{\alpha}$, where α is positive and independent of ε. In fact, we have

$$\int_0^{b/ax^{1+x}} \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1+\varepsilon}{q}} dt < \int_0^{b/ax^{1+x}} t^{-(1+\varepsilon)/q} dt = \frac{(b/ax^{1+x})^{\beta}}{\beta}$$

where $\beta = 1-(1+\varepsilon)/q$. If $0<\varepsilon<\frac{q}{2p}$, then we may take α such that

$$\alpha = 1-\frac{1+q/2p}{q} = \frac{1}{2p}.$$

Consequently, we get

$$\int_0^{\infty}\int_0^{\infty} \frac{\tilde{f}(x)\,\tilde{g}(y)}{ax^{1+x}+by^{1+y}} dxdy > \frac{1}{\varepsilon}\left\{\frac{\pi}{\sin\frac{\pi}{p}} + o(1)\right\} \quad (\varepsilon \to 0). \tag{2.9}$$

Clearly, when ε is small enough, the inequality (2.7) is in contradiction with (2.9). Therefore, $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible value for which the inequality (2.7) is valid.

Let $u = ax^{1+x}$ and $v = by^{1+y}$. Then

$$u' = ax^{1+x}\left(\frac{1+x}{x} + \ln x\right) = ax^{1+x}h^{-1}(x).$$

Similarly, we have $v' = by^{1+y}h^{-1}(y)$. Substituting them into (2.7) and then using Lemma 1.1, the inequality (2.2) yields after simplifications. The constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ is best possible, where $\mu = (1/a)^{1/q}(1/b)^{1/p}$. Thus the proof of the theorem is completed. □

It is known from (2.1) that

$$\omega(r,x) = x^{(1+x)(1-r)}\left(\frac{1}{2}-\varphi(x)\right)^{r-1} = \left(\frac{1}{2}\right)^{r-1} x^{(1+x)(1-r)}\left(1-2\varphi(x)\right)^{r-1}.$$

The following result is equivalent to Theorem 2.1.

Theorem 2.2. *Let $\varphi(x)$ be a function defined by (1.4), $\frac{1}{p}+\frac{1}{q}=1$ and $p \geq q > 1$. If*

$$0 < \int_0^\infty x^{(1+x)(1-p)}\left(1-2\varphi(x)\right)^{p-1} f^p(x)\,dx < +\infty \quad \text{and}$$
$$0 < \int_0^\infty y^{(1+y)(1-q)}\left(1-2\varphi(y)\right)^{q-1} g^q(y)\,dy < +\infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{ax^{1+x}+by^{1+y}}dxdy \leq \frac{\mu\pi}{2\sin\frac{\pi}{p}}\left\{\int_0^\infty x^{(1+x)(1-p)}\left(1-2\varphi(x)\right)^{p-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{(1+y)(1-q)}\left(1-2\varphi(y)\right)^{q-1} g^q(y)\,dy\right\}^{\frac{1}{q}}, \tag{2.10}$$

where $\mu = (1/a)^{1/q}(1/b)^{1/p}$ and the constant factor $\frac{\mu\pi}{2\sin\frac{\pi}{p}}$ is best possible.

In particular, for case $p=2$, some extensions on (1.2) are obtained. According to Theorem 2.1, we get the following results.

Corollary 2.3. *If*

$$0 < \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x) \right) f^2(x)\, dx < +\infty \quad \text{and}$$

$$0 < \int_0^\infty y^{-(1+y)} \left(\frac{1}{2} - \varphi(y) \right) g^2(y)\, dy < +\infty,$$

where $\varphi(x)$ is a function defined by (1.4), then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{\sqrt{ab}} \left\{ \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x) \right) f^2(x)\, dx \right\}^{\frac{1}{2}} \times \left\{ \int_0^\infty y^{-(1-y)} \left(\frac{1}{2} - \varphi(y) \right) g^2(y)\, dy \right\}^{\frac{1}{2}}, \tag{2.11}$$

where the constant factor $\frac{\pi}{\sqrt{ab}}$ is best possible.

Corollary 2.4. *Let $\varphi(x)$ be a function defined by (1.4). If*

$$0 < \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x) \right) f^2(x)\, dx < +\infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, f(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{\sqrt{ab}} \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x) \right) f^2(x)\, dx, \tag{2.12}$$

where the constant factor $\frac{\pi}{\sqrt{ab}}$ is best possible.

A equivalent proposition of Corollary 2.3 is:

Corollary 2.5. *Let $\varphi(x)$ be a function defined by (1.4),*

$$0 < \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx < +\infty \quad \textit{and}$$
$$0 < \int_0^\infty y^{-(1+y)} (1 - 2\varphi(y)) g^2(y)\, dy < +\infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{ax^{1+x} + by^{1+y}} dxdy \le \frac{\pi}{2\sqrt{ab}} \left\{ \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx \right\}^{\frac{1}{2}} \times \left\{ \int_0^\infty y^{-(1+y)} (1 - 2\varphi(y)) g^2(y)\, dy \right\}^{\frac{1}{2}}, \tag{2.13}$$

where the constant factor $\frac{\pi}{2\sqrt{ab}}$ is best possible.

Similarly, an equivalent proposition to Corollary 2.4 is:

Corollary 2.6. *Let $\varphi(x)$ be a function defined by (1.4). If*

$$0 < \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x)\, dx + \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x) f(y)}{ax^{1+x} + by^{1+y}} dxdy \leq \frac{\pi}{2\sqrt{ab}} \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx, \tag{2.14}$$

where the constant factor $\frac{\pi}{2\sqrt{ab}}$ *is best possible.*

3. Application

In this section, we will give various extensions of Hardy-Littlewood's integral inequality.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)\,dx, \quad n = 0, 1, 2, \dots$$

then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)\,dx, \tag{3.1}$$

where π is the best constant that keeps (3.1) valid. In our previous paper [5], the inequality (3.1) was extended and the following inequality established:

$$\int_0^{\infty} f^2(x)\,dx < \pi \int_0^1 h^2(x)\,dx, \tag{3.2}$$

where $f(x) = \int_0^1 t^x h(x)\,dx$, $x \in [0, +\infty)$.

Afterwards the inequality (3.2) was refined into the form in the paper [6]:

$$\int_0^{\infty} f^2(x)\,dx \leq \pi \int_0^1 t h^2(t)\,dt. \tag{3.3}$$

We will further extend the inequality (3.3), some new results can be obtained by further extending inequality (3.3).

Theorem 3.1. *Let $h(t) \in L^2(0,1)$, $h(t) \neq 0$. Define a function by*

$$f(x) = \int_0^1 t^{u(x)} |h(t)|\, dt, \quad x \in [0, +\infty),$$

where $u(x) = x^{1+x}$. Also, let $\varphi(x)$ be a weight function defined by (1.4), $(r = p, q)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq q > 1$. If

$$0 < \int_0^\infty x^{(1+x)(1-r)} \left(\frac{1}{2} - \varphi(x)\right)^{r-1} f^r(x)\, dx < +\infty,$$

then

$$\left(\int_0^\infty f^2(x)\, dx\right)^2 < \frac{\mu\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty x^{(1+x)(1-p)} \left(\frac{1}{2} - \varphi(x)\right)^{p-1} f^p(x)\, dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{(1+y)(1-q)} \left(\frac{1}{2} - \varphi(y)\right) f^q(y)\, dy\right)^{\frac{1}{q}} \int_0^1 th^2(t)\, dt, \tag{3.4}$$

where the constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ in (3.4) is best possible, and $\mu = (1/a)^{1/q}(1/b)^{1/p}$.

Proof. Let us write $f^2(x)$ in the form:

$$f^2(x) = \int_0^1 f(x)\, t^{u(x)} |h(t)|\, dt.$$

We apply, in turn, Schwarz's inequality and Theorem 2.1 to obtain

$$
\begin{aligned}
(3.5)\quad & \left(\int_0^\infty f^2(x)\,dx\right)^2 \\
&= \left\{\int_0^\infty \left(\int_0^1 f(x)\,t^{u(x)}\,|h(t)|\,dt\right)dx\right\}^2 \\
&= \left\{\int_0^1 \left(\int_0^\infty f(x)\,t^{u(x)-1/2}dx\right)t^{1/2}\,|h(t)|\,dt\right\}^2 \\
&\le \int_0^1 \left(\int_0^\infty f(x)\,t^{u(x)-1/2}dx\right)^2 dt \int_0^1 th^2(t)\,dt \\
&= \int_0^1 \left(\int_0^\infty f(x)\,t^{u(x)-1/2}dx\right) \\
&\qquad \times \left(\int_0^\infty f(y)\,t^{u(y)-1/2}dy\right)dt \int_0^1 th^2(t)\,dt \\
&= \int_0^1 \left(\int_0^\infty \int_0^\infty f(x)\,f(y)\,t^{u(x)+u(y)-1}dxdy\right)dt \int_0^1 th^2(t)\,dt \\
&= \left(\int_0^\infty \int_0^\infty \frac{f(x)\,f(y)}{u(x)+u(y)}dxdy\right)\int_0^1 th^2(t)\,dt \\
&\le \frac{\mu\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty x^{(1+x)(1-p)}\left(\frac{1}{2}-\varphi(x)\right)^{p-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\qquad \times \left\{\int_0^\infty y^{(1+y)(1-q)}\left(\frac{1}{2}-\varphi(y)\right)^{q-1} f^q(y)\,dy\right\}^{\frac{1}{q}} \int_0^1 th^2(t)\,dt.
\end{aligned}
$$

Since $h(t) \neq 0$, $f^2(x) \neq 0$. It is impossible to take equality in (3.5). We therefore complete the proof of the theorem. □

An equivalent proposition to Theorem 3.1 is:

Theorem 3.2. *Let the functions $h(t)$, $f(x)$ and $u(x)$ satisfy the assumptions of Theorem 3.1, and assume that*

$$0 < \int_0^\infty x^{(1+x)(1-r)} (1-2\varphi(x))^{r-1} f^r(x)\,dx < +\infty \quad (r = p, q).$$

Then

$$\left(\int_0^\infty f^2(x)\,dx\right)^2 < \frac{\mu\pi}{2\sin\frac{\pi}{p}} \left(\int_0^\infty x^{(1+x)(1-p)} (1-2\varphi(x))^{p-1} f^p(x)\,dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{(1+y)(1-q)} (1-2\varphi(y))^{q-1} f^q(y)\,dy\right)^{\frac{1}{q}} \int_0^1 th^2(t)\,dt, \tag{3.6}$$

and the constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ in (3.6) is best possible, where $\mu = (1/a)^{1/q}(1/b)^{1/p}$.

In particular, when $p = q = 2$, we have the following result.

Corollary 3.3. *Let the functions $h(t)$, $f(x)$ and $u(x)$ satisfy the assumptions of Theorem 3.1, and assume that*

$$0 < \int_0^\infty x^{-(1+x)} \left(\frac{1}{2} - \varphi(x)\right) f^2(x)\,dx < +\infty,$$

where $\varphi(x)$ is a function defined by (1.4). Then

$$\left(\int_0^\infty f^2(x)\,dx\right)^2 < \frac{\pi}{\sqrt{ab}}\left(\int_0^\infty x^{-(1+x)}\left(\frac{1}{2}-\varphi(x)\right)f^2(x)\,dx\right)\int_0^1 th^2(t)\,dt, \tag{3.7}$$

and the constant factor $\frac{\pi}{\sqrt{ab}}$ in (3.7) is best possible.

A result equivalent to Corollary 3.3 is:

Corollary 3.4. *Let the functions $h(t)$, $f(x)$ and $u(x)$ satisfy the assumptions of Theorem 3.1, and assume that*

$$0 < \int_0^\infty x^{-(1+x)}(1-2\varphi(x))\,f^2(x)\,dx < +\infty,$$

where $\varphi(x)$ is a function defined by (1.4). Then

$$\left(\int_0^\infty f^2(x)\,dx\right)^2 < \frac{\pi}{2\sqrt{ab}}\left(\int_0^\infty x^{-(1+x)}(1-2\varphi(x))\,f^2(x)\,dx\right)\int_0^1 th^2(t)\,dt, \tag{3.8}$$

and the constant factor $\frac{\pi}{2\sqrt{ab}}$ in (3.8) is best possible.

The inequalities (3.4), (3.6), (3.7) and (3.8) are extensions of (3.3).

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