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A SAWYER DUALITY PRINCIPLE FOR RADIALLY MONOTONE FUNCTIONS IN $\mathbb{R}^n$

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ABSTRACT. Let f be a non-negative function on $\mathbb{R}^n$, which is radially monotone $(0 < f \downarrow r)$. For $1 < p < \infty$, $g \geq 0$ and v a weight function, an equivalent expression for

$$\sup_{f\downarrow r} \frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}}$$

is proved as a generalization of the usual Sawyer duality principle. Some applications involving boundedness of certain integral operators are also given.

Key words and phrases: Duality theorems, radially monotone functions, weighted inequalities.

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1. INTRODUCTION

An explicit duality principle for positive decreasing functions of one variable was proved by E. Sawyer in [8], and also some applications are well-known. Here we also refer to the useful proof and ideas concerning this principle presented by V. Stepanov in [9]. See also [6] and the proof and comments given there. Moreover, it is natural to look for extensions to functions of

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several variables. Such generalizations were recently obtained in [1], [2] and [3]. To be able to describe some of these generalizations we require some notations: We write

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : i = 1, 2, \ldots, n\}$$

and $\mathbb{R}^1 = \mathbb{R}$. If $f : \mathbb{R}^n \to \mathbb{R}$ is decreasing (increasing) separately in each variable we write $0 \le f \downarrow (0 \le f \uparrow)$. A set $D \subset \mathbb{R}^n$ is said to be decreasing if its characteristic function χ_D is decreasing, and clearly if $0 \le h \downarrow$ and $t > 0$, then the set $D_{h,t} = \{x \in \mathbb{R}^n : h(x) > t\}$ is decreasing. For $0 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, it was shown in [3] that

$$\sup_{0 \le f \downarrow} \frac{\left(\int_{\mathbb{R}^n} f^q u\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} \approx \frac{\left(\int_{\mathbb{R}^n} u\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n} v\right)^{\frac{1}{p}}} + \sup_{0 \le h \downarrow} \left(\int_0^\infty \left(\int_{D_{h,t}} u \right)^{\frac{r}{q}} d \left(\int_{D_{h,t}} v \right)^{-\frac{r}{p}} \right)^{\frac{1}{p}}, \tag{1.1}$$

where u and v are weights, i.e. positive and locally integrable functions on $\mathbb{R}^n$. For the case $0 < p \le q < \infty$ c.f. [2].

If $q = 1 < p < \infty$ and $u = g \ge 0$, then (1.1) with $n = 1$ is a variant of the duality theorem given in [8] (c.f. also [9]).

An explicit form of (1.1) for $n \in \mathbb{Z}_+$ was given in [1] for $g = u$, but only when v is of product type, that is weights of the form $v(x) = v(x_1, x_2, \ldots, x_n) = v_1(x_1) v_2(x_2) \cdots v_n(x_n)$, $v_i \ge 0$, $i = 1, 2, \ldots, n$.

We say that $f : \mathbb{R}^n \to \mathbb{R}$ is a radially decreasing (increasing) function if $f(x) = f(y)$ when $|x| = |y|$ and $f(x) \ge f(y)$ $(f(x) \le f(y))$ when $|x| < |y|$ and we write $f \downarrow r$ and $f \uparrow r$, respectively (see also [7] for further explanations and applications of these notions).

In this paper we prove a duality formula of the type (1.1) for radially decreasing functions and with general weights (see Theorem 2.1). We also state the corresponding result for radially increasing functions (see Theorem 3.1). In particular, these results imply that we can describe mapping properties of operators defined on the cone of such monotone functions between weighted Lebesgue spaces. Moreover, we point out that these results can also be used to describe mapping properties between some corresponding general weighted multidimensional Lebesgue spaces (see Theorem 3.5 and c.f. also [9] for the case $n = 1$). We illustrate this useful technique for the identity operator (see Corollary 4.3) and for the Hardy integral operator over the n-dimensional sphere (see Corollary 4.8).

The paper is organized as follows: In Section 2 we present some known and easily derived results required in the proofs of our statements. The announced duality theorems are stated and proved in Section 3. Finally, in Section 4 the afore mentioned applications are given and also some further results and remarks.

Notations and conventions: Throughout this paper all functions are assumed to be measurable. Constants, denoted a, b, c, are always positive and may be different at different places. Moreover $n \in \mathbb{Z}_+$, $1 \le p < \infty$, $p' = \frac{p}{p-1}$ ($p' = \infty$ if $p = 1$), $v(x)$ and $u(x)$ are weights (positive and measurable functions on $\mathbb{R}^n$) and

$$\begin{aligned} L_v^p &= L_v^p(\mathbb{R}^n) \\ &= \left\{ f : \mathbb{R}^n \to \mathbb{R}, \text{measurable s.t. } \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

Inequalities such as (2.1) are interpreted to mean that if the right hand side is finite, then so is the left hand side and the inequality holds. The symbol $\approx$ (c.f. (1.1)) means that the quotient of the right and left hand sides is bounded from above and below by positive constants, while expressions such as $0 \cdot \infty = \infty \cdot 0$ are taken as zero. Other notations will be introduced when required.

2. PRELIMINARY RESULTS

Consider the Hardy integral operator H of the form

$$(Hf)(x) = \int_{B(x)} f(y)dy,$$

where $B(x)$ is the ball in $\mathbb{R}^n$ centered at the origin and with radius $|x|$. We need the following well-known n-dimensional form of Hardy's inequality (see [5]):

Theorem 2.1. *Let W and U be weights on $\mathbb{R}^n$ and $1 < p \le q < \infty$.*

(i) *The inequality*

$$\left(\int_{\mathbb{R}^n} W(x) \left(\int_{B(x)} f(y)dy\right)^q dx\right)^{\frac{1}{q}} \le c \left(\int_{\mathbb{R}^n} U(x) f^p(x)dx\right)^{\frac{1}{p}} \tag{2.1}$$

holds for $f \ge 0$ if and only if

$$a := \sup_{\alpha>0} \left(\int_{|x|\ge\alpha} W(x)dx\right)^{\frac{1}{q}} \left(\int_{|x|\le\alpha} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty\,.$$

Moreover, if c is the smallest constant for which (2.1) holds, then

$$a \le c \le a p'^{\frac{1}{p'}} p^{\frac{1}{q}}.$$

(ii) *The inequality*

$$\left(\int_{\mathbb{R}^n} W(x) \left(\int_{\mathbb{R}^n\setminus B(x)} f(y)dy\right)^q dx\right)^{\frac{1}{q}} \le c \left(\int_{\mathbb{R}^n} U(x) f^p(x)dx\right)^{\frac{1}{p}} \tag{2.2}$$

holds if and only if

$$b := \sup_{\alpha>0} \left(\int_{|x|\le\alpha} W(x)dx\right)^{\frac{1}{q}} \left(\int_{|x|\ge\alpha} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty\,.$$

Moreover, if c is the smallest constant for which (2.2) holds, then

$$b \le c \le b p'^{\frac{1}{p'}} p^{\frac{1}{q}}.$$

In particular we need the following special case of Theorem 2.1 (ii):

Lemma 2.2. *Let v be a weight function, $V(x) = \int_{B(x)} v(y)dy$ and $1 < p < \infty$. Then for $f \ge 0$*

$$\int_{\mathbb{R}^n} v(x) \left(\int_{\mathbb{R}^n\setminus B(x)} f(y)dy\right)^p dx \le p \int_{\mathbb{R}^n} f^p(x) V^p(x)\, v^{1-p}(x)dx \tag{2.3}$$

is satisfied.

Proof. Apply Theorem 2.1 (ii) with $W(x) = v(x)$, $U(x) = v^{1-p}(x) \left(\int_{B(x)} v(y)dy\right)^p$ and $q = p$. Denote

$$v_n(s) = \int_{\Sigma_{n-1}} s^{n-1} v(s\sigma)d\sigma, \tag{2.4}$$

where Σ_{n-1} as usual denotes the unit sphere in $\mathbb{R}^n$, $s \in \mathbb{R}$. We note that

$$\begin{aligned}
b &= \sup_{\alpha>0} \left(\int_{|x|\le\alpha} v(x)dx \right)^{\frac{1}{p}} \left(\int_{|x|\ge\alpha} \left(\int_{B(x)} v(y)dy \right)^{-p'} v(x)dx \right)^{\frac{1}{p'}} \\
&= \sup_{\alpha>0} V^{\frac{1}{p}}(\alpha) \left(\int_{|x|\ge\alpha} \left(\int_0^{|x|} v_n(s)\,ds \right)^{-p'} v(x)\,dx \right)^{\frac{1}{p'}} \\
&= \sup_{\alpha>0} V^{\frac{1}{p}}(\alpha) \left(\int_\alpha^\infty \int_{\Sigma_{n-1}} t^{n-1} \left(\int_0^t v_n(s)\,ds \right)^{-p'} v(t\delta)\,dt d\delta \right)^{\frac{1}{p'}} \\
&= \sup_{\alpha>0} V^{\frac{1}{p}}(\alpha) \left(\int_\alpha^\infty \left(\int_0^t v_n(s)\,ds \right)^{-p'} \left(\int_{\Sigma_{n-1}} t^{n-1} v(t\sigma)\,d\sigma \right) dt \right)^{\frac{1}{p'}} \\
&= \sup_{\alpha>0} V^{\frac{1}{p}}(\alpha) \left(\int_\alpha^\infty \left(\int_0^t v_n(s)\,ds \right)^{-p'} v_n(t)\,dt \right)^{\frac{1}{p'}} \\
&= \sup_{\alpha>0} V^{\frac{1}{p}}(\alpha) \left(\int_\alpha^\infty \left(\frac{d}{dt} \left(\int_0^t v_n(s)\,ds \right)^{-p'+1} \right) \frac{1}{1-p'} dt \right)^{\frac{1}{p'}} \\
&\le \sup_{\alpha>0} \frac{1}{(p'-1)^{\frac{1}{p'}}} V^{\frac{1}{p}}(\alpha) \left(\int_0^\alpha v_n(t)\,dt \right)^{\frac{-p'+1}{p'}} \\
&= \sup_{\alpha>0} \frac{1}{(p'-1)^{\frac{1}{p'}}} V^{\frac{1}{p}}(\alpha)\, V^{\frac{1}{p'}-1}(\alpha) = \frac{1}{(p'-1)^{\frac{1}{p'}}} < \infty.
\end{aligned}$$

Therefore, by Theorem 2.1 (ii), (2.3) holds with a constant $c \le \frac{1}{(p'-1)^{\frac{1}{p'}}} (p')^{\frac{1}{p'}} p^{\frac{1}{p}} = p$ and the proof is complete. $\square$

3. The Duality Principle for radially monotone functions

In the sequel we sometimes delete the integration variable and write e.g. $\int_{\mathbb{R}^n} fg$ instead of $\int_{\mathbb{R}^n} f(x)\,g(x)\,dx$ when it cannot be misinterpreted. Moreover, as usual, $\|g\|_1 = \|g\|_{L_1} = \int_{\mathbb{R}^n} |g(x)|\,dx$. Our main result in this section reads:

Theorem 3.1. *Suppose that v is a weight on $\mathbb{R}^n$ and $1 < p < \infty$. If f is a positive radially decreasing function on $\mathbb{R}^n$ and g a positive measurable function on $\mathbb{R}^n$, then*

$$C(g) := \sup_{f\downarrow r} \frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} \approx I_1 + I_2, \tag{3.1}$$

where

$$I_1 = \|v\|_1^{\frac{-1}{p}} \|g\|_1$$

and

$$I_2 = \left(\int_{\mathbb{R}^n} G(t)^{p'} V(t)^{-p'} v(t)\,dt \right)^{\frac{1}{p'}}$$

with $V(t) = \int_{B(t)} v(x)\,dx$ and $G(t) = \int_{B(t)} g(x)\,dx$.

Remark 3.2. Theorem 2.1 for the case $n = 1$ is simply Theorem 1 in [8]. However, our proof below is based on the technique in [9] and our investigation in Section 3.

Proof. If $f \equiv c > 0$, then

$$C(g) \geq \sup_{f=c} \frac{\int_{\mathbb{R}^n_+} fg}{\left(\int_{\mathbb{R}^n_+} f^p v\right)^{\frac{1}{p}}} = \frac{\|g\|_1}{\|v\|_1^{\frac{1}{p}}} = I_1.$$

Moreover, we use the test function $f(x) = \int_{|t|>|x|} h(t)dt$, where $h(t) \geq 0$ (note that f is radially decreasing), Lemma 2.2 and the usual duality in L^p_v-spaces to find that

$$\begin{aligned}
C(g) &= \sup_{0\leq f\downarrow} \frac{\int_{\mathbb{R}^n} f(x)\, g(x)\, dx}{\left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx\right)^{\frac{1}{p}}} \\
&\geq \sup_{h\geq 0} \frac{\int_{\mathbb{R}^n} \left(\int_{|x|<|t|} h(t)\, dt\right) g(x)\, dx}{\left(\int_{\mathbb{R}^n} \left(\int_{|x|<|t|} h(t)\, dt\right)^p v(x)\, dx\right)^{\frac{1}{p}}} \\
&= \sup_{h\geq 0} \frac{\int_{\mathbb{R}^n} h(t) \int_{|x|<|t|} g(x)\, dxdt}{\left(\int_{\mathbb{R}^n} \left(\int_{|x|<|t|} h(t)\, dt\right)^p v(x)\, dx\right)^{\frac{1}{p}}} \\
&\geq \frac{1}{p} \sup_{h\geq 0} \frac{\int_{\mathbb{R}^n} h \int_{|x|<|t|} g}{\left(\int_{\mathbb{R}^n} h^p V^p v^{1-p}\right)^{\frac{1}{p}}} \\
&= \frac{1}{p} \sup_{h\geq 0} \frac{\int_{\mathbb{R}^n} \left(\frac{hV}{v}\right) \frac{G}{V} v}{\left(\int_{\mathbb{R}^n} \left(\frac{hV}{v}\right)^p v\right)^{\frac{1}{p}}} \\
&= \frac{1}{p} \left(\int_{\mathbb{R}^n} \left(\frac{G}{V}\right)^{p'} v\right)^{\frac{1}{p'}} = I_2.
\end{aligned}$$

To prove the upper bound of $C(g)$ we use the monotonicity of f. Since f is radially decreasing we have

$$\begin{aligned}
\int_{\mathbb{R}^n} gf &= \int_{\mathbb{R}^n} gfV \frac{1}{V} \\
&= \int_0^\infty \int_{\Sigma_{n-1}} f(x)\, g(x) \frac{1}{V(x)} \left(\int_{B(x)} v(t)dt\right) dx \\
&= \int_{\mathbb{R}^n} v(t) \left(\int_{|x|>|t|} f(x)\, g(x) \frac{1}{V(x)} dx\right) dt \\
&\leq \int_{\mathbb{R}^n} v(t)\, f(t) \left(\int_{|x|>|t|} g(x) \frac{1}{V(x)} dx\right) dt. \qquad (3.2)
\end{aligned}$$

To estimate the inner integral we define $g_n(s)$ and $v_n(s)$ analogously to (2.4), note that $V(x) = V(|x|)$ and find that

$$\begin{aligned}
\int_{|x|>|t|} g(x) \frac{1}{V(x)} dx &= \int_{|t|}^\infty s^{n-1} \int_{\Sigma_{n-1}} g(s\sigma) \frac{1}{V(s\sigma)} ds d\sigma \\
&= \int_{|t|}^\infty \left(\int_{\Sigma_{n-1}} s^{n-1} g(s\sigma)\, d\sigma\right) \frac{1}{V(s)} ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_{|t|}^{\infty} g_n(s)\, \frac{1}{V(s)} ds \\
&= \left[\frac{1}{V(s)} \int_0^s g_n(z)\, dz \right]_{|t|}^{\infty} \\
&\qquad + \int_{|t|}^{\infty} \frac{1}{V^2(s)} \left(\int_{\Sigma_{n-1}} s^{n-1} v(s\sigma)\, d\sigma \right) \left(\int_0^s g_n(z)\, dz \right) ds \\
&\le \underbrace{\frac{1}{V(\infty)} \int_0^{\infty} g_n(z)\, dz}_{K_1} + \underbrace{\int_{|t|}^{\infty} \frac{1}{V^2(s)} v_n(s) \left(\int_0^s g_n(z)\, dz \right) ds}_{K_2}.
\end{aligned}
$$

Hence the inner integral can be estimated by $K_1 + K_2$ and by substituting this into (3.2) and applying Hölder's and Minkowski's inequalities we get

$$
\begin{aligned}
\int_{\mathbb{R}^n} fg &\le \int_{\mathbb{R}^n} fv\,(K_1 + K_2) \\
&\le \left(\int_{\mathbb{R}^n} f^p v \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} (K_1 + K_2)^{p'}\, v \right)^{\frac{1}{p'}} \\
&\le \left(\int_{\mathbb{R}^n} f^p v \right)^{\frac{1}{p}} \left(\left(\int_{\mathbb{R}^n} K_1^{p'} v \right)^{\frac{1}{p'}} + \left(\int_{\mathbb{R}^n} K_2^{p'} v \right)^{\frac{1}{p'}} \right).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} K_1^{p'} v \right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{R}^n} \left(\frac{1}{V(\infty)} \int_0^{\infty} g_n(z)\, dz \right)^{p'} v(x)\, dx \right)^{\frac{1}{p'}} \\
&= \frac{1}{V(\infty)} \left(\int_0^{\infty} \int_{\Sigma_{n-1}} s^{n-1} g(s\sigma)\, d\sigma ds \right) \left(\int_{\mathbb{R}^n} v(x)\, dx \right)^{\frac{1}{p'}} \\
&= \|v\|_1^{-1+\frac{1}{p'}} \|g\|_1 = \|v\|_1^{-\frac{1}{p}} \|g\|_1 = I_1,
\end{aligned}
$$

and, according to Lemma 2.2,

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} K_2^{p'} v \right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{R}^n} \left(\int_{|t|}^{\infty} \frac{1}{V^2(s)} v_n(s) \left(\int_0^s g_n(z)\, dz \right) ds \right)^{p'} v(t)\, dt \right)^{\frac{1}{p'}} \\
&= \left(\int_{\mathbb{R}^n} \left(\int_{|t|}^{\infty} \int_{\Sigma_{n-1}} \frac{s^{n-1} v(s\sigma)}{V^2(s)} \left(\int_0^s \int_{\Sigma_{n-1}} z^{n-1} g(z\delta)\, dz d\delta \right) d\sigma ds \right)^{p'} v \right)^{\frac{1}{p'}} \\
&= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus B(t)} \frac{v(x)}{V^2(x)} \int_{B(x)} g(y)\, dy \right)^{p'} v(t)\, dt \right)^{\frac{1}{p'}} \\
&\le p' \left(\int_{\mathbb{R}^n} V(t)^{-p'} \left(\int_{B(|x|)} g \right) v(t)\, dt \right)^{\frac{1}{p'}} = p' I_2.
\end{aligned}
$$

The upper bound of $C(g)$ follows by combining the last estimates and the proof is complete. $\square$

Remark 3.3. According to our proof we see that the duality constant $C(g)$ in (3.1) can in fact be estimated in the following more precise way:

$$\max(I_1, I_2) \leq C(g) \leq I_1 + p' I_2.$$

In particular we have the following useful information:

Corollary 3.4. *Let the assumptions in Theorem 3.1 be satisfied and $\int_{\mathbb{R}^n} v = \infty$. Then*

$$I_2 \leq \sup_{f \downarrow r} \frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} \leq p' I_2. \tag{3.3}$$

The proof above is self-contained and does not depend directly on the one-dimensional result (only on our investigations in Section 2 and similar arguments as V.D. Stepanov used when he proved the case $n = 1$). Here we give another shorter proof where we directly use the (Sawyer) one-dimensional result.

Proof. Make the following changes of variables

$$t = s\sigma \text{ and } x = y\tau, \tag{3.4}$$

where $s, y \in (0, \infty)$ and $\sigma, \tau \in \sum_{n-1}$. By using the fact that $f(s\sigma) = f(s)$ since f is radial, we get:

$$\begin{aligned}
\frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} &= \frac{\int_0^\infty \int_{\sum_{n-1}} f(s\sigma) g(s\sigma) s^{n-1} d\sigma ds}{\left(\int_0^\infty \int_{\sum_{n-1}} f^p(s\sigma) v(s\sigma) s^{n-1} d\sigma ds\right)^{\frac{1}{p}}} \\
&= \frac{\int_0^\infty f(s) \widetilde{G}(s) ds}{\left(\int_0^\infty f^p(s) \widetilde{V}(s) ds\right)^{\frac{1}{p}}},
\end{aligned}$$

and hence, by using the (Sawyer) one-dimensional result we find that

$$\begin{aligned}
\frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} &\approx \left(\int_0^\infty \widetilde{V}(s) ds\right)^{-\frac{1}{p}} \left(\int_0^\infty \widetilde{G}(s) ds\right) \\
&\quad + \left(\int_0^\infty \frac{\left(\int_0^s \widetilde{G}(y) dy\right)^{p'}}{\left(\int_0^s \widetilde{V}(y) dy\right)^{p'}} \widetilde{V}(s) ds\right)^{\frac{1}{p'}} := I.
\end{aligned} \tag{3.5}$$

Moreover,

$$\begin{aligned}
I &= \left(\int_0^\infty \int_{\sum_{n-1}} v(s\sigma) s^{n-1} d\sigma\, ds\right)^{-\frac{1}{p}} \left(\int_0^\infty \int_{\sum_{n-1}} g(s\sigma) s^{n-1} d\sigma ds\right) \\
&\quad + \left(\int_0^\infty \frac{\left(\int_0^s \int_{\sum_{n-1}} g(y\tau) y^{n-1} d\tau dy\right)^{p'}}{\left(\int_0^s \int_{\sum_{n-1}} v(y\tau) y^{n-1} d\tau dy\right)^{p'}} \int_{\sum_{n-1}} v(s\sigma) s^{n-1} d\sigma ds\right)^{\frac{1}{p'}}
\end{aligned}$$

$$= \left(\int_{\mathbb{R}^n} v(t) dt\right)^{-\frac{1}{p}} \left(\int_{\mathbb{R}^n} g(t) dt\right) + \left(\int_{\mathbb{R}^n} \frac{\left(\int_{B(t)} g(x) dx\right)^{p'}}{\left(\int_{B(t)} v(x) dx\right)^{p'}} v(t) dt\right)^{\frac{1}{p'}}. \tag{3.6}$$

The proof follows by combining (3.5) and (3.6). $\square$

For completeness and later use in our applications we also state the corresponding result for radially increasing functions in $\mathbb{R}^n$:

Theorem 3.5. *Suppose that v is a weight on $\mathbb{R}^n$ and $1 < p < \infty$. If f is a positive radially increasing function on $\mathbb{R}^n$ and g a positive measurable function on $\mathbb{R}^n$, then*

$$D(g) := \sup_{f\uparrow r} \frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} \approx I_1 + I_3,$$

where

$$I_1 = \|v\|_1^{\frac{-1}{p}} \|g\|_1,$$

and

$$I_3 = \left(\int_{\mathbb{R}^n} G_1(t)^{p'} V_1(t)^{-p'} v(t)\, dt\right)^{\frac{1}{p'}},$$

with $G_1(t) = \int_{\mathbb{R}^n \setminus B(t)} g(x)\, dx$ and $V_1(t) = \int_{\mathbb{R}^n \setminus B(t)} v(x)\, dx$.

Proof. We now use Theorem 2.1 (i) (instead of (ii) as in the proof of Lemma 2.2) and obtain as in the proof of (2.3):

$$\int_{\mathbb{R}^n} v(x) \left(\int_{B(x)} f(y) dy\right)^p dx \le p \int_{\mathbb{R}^n} f^p(x) V_1^p(x)\, v^{1-p}(x) dx.$$

By using this estimate the proof follows similarly as the proof of Theorem 3.1 so we leave out the details. □

Remark 3.6. In fact, similar to Remark 3.3 and Corollary 3.4, we find that

$$\max(I_1, I_3) \le D(g) \le I_1 + p' I_3$$

and if, in addition to the assumptions in Theorem 3.5, $\int_{\mathbb{R}^n} v = \infty$, then

$$I_3 \le \sup_{f\uparrow r} \frac{\int_{\mathbb{R}^n} fg}{\left(\int_{\mathbb{R}^n} f^p v\right)^{\frac{1}{p}}} \le p' I_3.$$

4. Further Results and Applications

Let T be an integral operator defined on the cone of functions $f : \mathbb{R}^n \to \mathbb{R}$, which are radially decreasing $(0 < f \downarrow r)$ and let T^* be the adjoint operator. Then our results imply the following useful duality result:

Theorem 4.1. *Let $1 < p, q < \infty$, u, v be weights on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} v(x)\, dx = \infty$. Then the inequality*

$$\left(\int_{\mathbb{R}^n} (Tf(x))^q u(x)\, dx\right)^{\frac{1}{q}} \le c \left(\int_{\mathbb{R}^n} f^p(x) v(x)\, dx\right)^{\frac{1}{p}} \tag{4.1}$$

holds for all $f \downarrow r$ if and only if

$$\left(\int_{\mathbb{R}^n} \left(\int_{B(x)} T^* g(y)\, dy\right)^{p'} V^{-p'}(x)\, v(x)\, dx\right)^{\frac{1}{p'}} \le c \left(\int_{\mathbb{R}^n} g^{q'}(x) u^{1-q'}(x)\, dx\right)^{\frac{1}{q'}} \tag{4.2}$$

holds for every positive measurable function g.

Proof. Assume first that (4.1) holds for all $0 < f \downarrow r$. Then, by using Corollary 3.4, duality and Hölder's inequality, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\int_{B(x)} T^* g(y)\, dy \right)^{p'} V^{-p'}(x)\, v(x)\, dx \\
&\quad \le \sup_{f \downarrow r} \frac{\int_{\mathbb{R}^n} f(x)\, T^* g(x)\, dx}{\left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}}} \\
&\quad = \sup_{f \downarrow r} \frac{\int_{\mathbb{R}^n} Tf(x)\, g(x)\, dx}{\left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}}} \\
&\quad \le \sup_{f \downarrow r} \frac{\left(\int_{\mathbb{R}^n} (Tf(x))^q\, u(x)\, dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} g^{q'}(x)\, u^{\frac{-q'}{q}}(x)\, dx \right)^{\frac{1}{q'}}}{\left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}}} \\
&\quad = c \left(\int_{\mathbb{R}^n} g^{q'}(x)\, u^{1-q'}(x)\, dx \right)^{\frac{1}{q'}}.
\end{aligned}$$

On the contrary assume that (4.2) holds for all $g \ge 0$. Then, by using Corollary 3.4 again, we have

$$\begin{aligned}
p'c \left(\int_{\mathbb{R}^n} (g(x))^{q'} (u(x))^{1-q'}\, dx \right)^{\frac{1}{p'}} &\ge p' \left(\int_{\mathbb{R}^n} \left(\int_{B(x)} T^* g(t)\, dt \right)^{p'} V^{-p'}(x)\, v(x)\, dx \right)^{\frac{1}{p'}} \\
&\ge \frac{\int_{\mathbb{R}^n} f(x)\, T^* g(x)\, dx}{\left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}}} = \frac{\int_{\mathbb{R}^n} Tf(x)\, g(x)\, dx}{\left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}}}
\end{aligned}$$

for each fixed $0 < f \downarrow r$. Therefore we have

$$\int_{\mathbb{R}^n} h(x)\, Tf(x)\, u^{\frac{1}{q}}(x)\, dx \le p'c \left(\int_{\mathbb{R}^n} f^p(x)\, v(x)\, dx \right)^{\frac{1}{p}}, \tag{4.3}$$

where

$$h(x) = \frac{g(x)\, u^{-\frac{1}{q}}(x)}{\left(\int_{\mathbb{R}^n} \left(g(x)\, u^{-\frac{1}{q}}(x) \right)^{q'} dx \right)^{\frac{1}{q'}}}.$$

Since $\|h\|_{L^{q'}} = 1$ we obtain (4.1) by taking the supremum in (4.3) and usual duality in L^p-spaces. □

Remark 4.2. By modifying the proof above we see that a similar duality result also holds for positive radially increasing functions. In fact, in this case we just need to replace $\int_{B(x)}$by $\int_{\mathbb{R}^n \setminus B(x)}$ and $V(x)$ by $V_1(x) = \int_{\mathbb{R}^n \setminus B(x)} v(x)\, dx$ in (4.2).

For example when T is the identity operator we obtain the following:

Corollary 4.3. *Let $1 < p \le q < \infty$ and suppose that u, v are weights on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} v = \infty$ and $V(x) = \int_{B(x)} v(y)\, dy$.*

a) *The following conditions are equivalent:*

i) *The inequality*

$$\left(\int_{\mathbb{R}^n} f^q u \right)^{\frac{1}{q}} \le c \left(\int_{\mathbb{R}^n} f^p v \right)^{\frac{1}{p}} \tag{4.4}$$

is satisfied for all $0 \leq f \downarrow r$.

ii) *The inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(x)} g\left(y\right)dy\right)^{p'} V^{-p'}\left(x\right)v\left(x\right)dx\right)^{\frac{1}{p'}} \leq c\left(\int_{\mathbb{R}^n} g^{q'}(x)u^{1-q'}\left(x\right)dx\right)^{\frac{1}{q'}} \tag{4.5}$$

holds for all $g \geq 0$.

iii)

$$\sup_{\alpha>0}\left(\int_{B(\alpha)} v(x)dx\right)^{-\frac{1}{p}}\left(\int_{B(\alpha)} u(x)dx\right)^{\frac{1}{q}} < \infty\,.$$

b) *If* $V_1\left(x\right)=\int_{\mathbb{R}^n\setminus B(x)} v\left(t\right)dt$, *then for* $0 \leq f \uparrow r$ *(4.4) is equivalent to*

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B(x)} g\left(y\right)dy\right)^{p'} V_1^{-p'}\left(x\right)v\left(x\right)dx\right)^{\frac{1}{p'}} \leq c\left(\int_{\mathbb{R}^n} g^{q'}(x)u^{1-q'}\left(x\right)dx\right)^{\frac{1}{q'}}$$

which in turn is equivalent to

$$\sup_{\alpha>0}\left(\int_{\mathbb{R}^n\setminus B(\alpha)} v(x)dx\right)^{-\frac{1}{p}}\left(\int_{\mathbb{R}^n\setminus B(\alpha)} u(x)dx\right)^{\frac{1}{q}} < \infty.$$

Proof. a) The equivalence of i) and ii) is just a special case of Theorem 4.1. Moreover, the fact that ii) and iii) are equivalent follows from Theorem 2.1 (i) with f replaced by g, q by p', p by q', W by $vV^{-p'}$ and U by $u^{1-q'}$. In fact, then (4.5) is equivalent to (note that $(1-q)(1-q')=1$)

$$\sup_{\alpha>0}\left(\int_{|x|>\alpha} vV^{-p'}\right)^{\frac{1}{p'}}\left(\int_{|x|<\alpha} u\right)^{\frac{1}{q}} = \sup_{\alpha>0}\frac{\left(\int_{|x|<\alpha} v\right)^{-\frac{1}{p}}}{(p'-1)^{\frac{1}{p'}}}\left(\int_{|x|<\alpha} u\right)^{\frac{1}{q}} < \infty.$$

The proof of b) follows similarly by just using Remark 4.2 and Theorem 2.1 (ii). □

Remark 4.4. The equivalence of i) and iii) can also be proved using the technique from [2]. For the case $q < p$ cf. also [3].

The next result concerns the multidimensional Hardy operator, defined on the cone of radially decreasing functions in $\mathbb{R}^n$.

Proposition 4.5. *Let* $1 < p \leq q < \infty$ *and suppose that* u *and* v *are weights on* $\mathbb{R}^n$ *with* $\int_{\mathbb{R}^n} v = \infty$ *and* $V\left(x\right)=\int_{B(x)} v\left(x\right)dx$. *If* $0 \leq f$ *is a radially decreasing function in* $\mathbb{R}^n$, *then*

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(x)} f\left(y\right)dy\right)^{q} u\left(x\right)dx\right)^{\frac{1}{q}} \leq c\left(\int_{\mathbb{R}^n} f^p(x)v\left(x\right)dx\right)^{\frac{1}{p}} \tag{4.6}$$

is satisfied if and only if the following conditions hold:

$$\sup_{\alpha>0}\left(\int_{B(\alpha)} v(x)dx\right)^{-\frac{1}{p}}\left(\int_{B(\alpha)} u(x)\left|B\left(x\right)\right|^{q}dx\right)^{\frac{1}{q}} < \infty \tag{4.7}$$

and

$$\sup_{\alpha>0}\left(\int_{B(\alpha)} \left|B\left(x\right)\right|^{p'} V^{-p'}\left(x\right)v\left(x\right)dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n\setminus B(\alpha)} u(x)dx\right)^{\frac{1}{q}} < \infty. \tag{4.8}$$

Here, as usual, $\left|B\left(x\right)\right|$ denotes the Lebesgue measure of the ball with center at 0 and with radius $|x|$.

Remark 4.6. For the case $n = 1$ this result is due to V. Stepanov (see [9], Theorem 2).

Proposition 4.5 can be proved by using the method of reduction to the one-dimensional case but here we present an independent proof:

Proof. Since $Tf(x) = \int_{B(x)} f(t)\,dt$ its conjugate T^* is defined by $T^*g(x) = \int_{\mathbb{R}^n \setminus B(x)} g(t)\,dt$. Assume first that (4.7) and (4.8) hold. We note that, according to Theorem 4.1, (4.6) for $0 < f \downarrow r$ is equivalent to (4.2) for arbitrary $g \geq 0$. Moreover, to be able to characterize weights for which (4.2) is satisfied, we first compute:

$$\begin{aligned}
\int_{B(x)} T^*g &= \int_{B(x)} \left(\int_{|y|>|z|} g(y)\,dy \right) dz \\
&= \int_{B(x)} \left(\int_{|z|}^{\infty} \int_{\Sigma_{n-1}} t^{n-1} g(t\delta)\,d\delta dt \right) dz \\
&= \int_{B(x)} \left(\int_{|z|}^{\infty} g_n(t)\,dt \right) dz \\
&= \int_0^{|x|} \int_{\Sigma_{n-1}} s^{n-1} \left(\int_s^{\infty} g_n(t)\,dt \right) d\sigma ds \\
&= |\Sigma_{n-1}| \left(\int_0^{|x|} s^{n-1} \left(\int_s^{|x|} g_n(t)\,dt \right) ds + \int_0^{|x|} \int_{|x|}^{\infty} g_n(t)\,dt ds \right) \\
&= |\Sigma_{n-1}| \int_0^{|x|} \left(\int_0^t s^{n-1} ds \right) g_n(t)\,dt + |\Sigma_{n-1}| \int_0^{|x|} ds \int_{|x|}^{\infty} g_n(t)\,dt \\
&= \int_0^{|x|} \left(\int_{\Sigma_{n-1}} d\sigma \int_0^t s^{n-1} ds \int_{\Sigma_{n-1}} t^{n-1} g(t\delta)\,d\delta \right) dt \\
&\quad + \left(\int_0^{|x|} \int_{\Sigma_{n-1}} d\sigma ds \right) \int_{|x|}^{\infty} \int_{\Sigma_{n-1}} t^{n-1} g(t\delta)\,d\delta dt \\
&= \underbrace{\int_{B(x)} |B(y)|\, g(y)\,dy}_{I_1} + \underbrace{|B(x)| \int_{\mathbb{R}^n \setminus |B(x)|} g(y)\,dy}_{I_2} \\
&= I_1(x) + I_2(x).
\end{aligned}$$

This means that (4.6) holds if and only if

$$\left(\int_{\mathbb{R}^n} (I_1(x) + I_2(x))^{p'} V^{-p'}(x)\, v(x)\,dx \right)^{\frac{1}{p'}} \leq c \left(\int_{\mathbb{R}^n} g^{q'}(x)\, U^{1-q'}(x)\,dx \right)^{\frac{1}{q'}}. \tag{4.9}$$

Moreover, by Theorem 2.1 (i) with q replaced by p' and p replaced by q', we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} (I_1(x))^{p'} V^{-p'}(x)\, v(x)\,dx \right)^{\frac{1}{p'}} \\
&\quad = \left(\int_{\mathbb{R}^n} \left(\int_{B(x)} |B(y)|\, g(y)\,dy \right)^{p'} V^{-p'}(x)\, v(x)\,dx \right)^{\frac{1}{p'}} \\
&\quad \leq c \left(\int_{\mathbb{R}^n} (|B(x)|\, g(x))^{q'} \frac{u(x)^{1-q'}}{|B(x)|^{q'}} dx \right)^{\frac{1}{q}} = c \left(\int_{\mathbb{R}^n} g(x)^{q'}\, u(x)^{1-q'}\,dx \right)^{\frac{1}{q'}},
\end{aligned} \tag{4.10}$$

which holds because, according to (4.7),

$$\sup_{\alpha>0}\left(\int_{\mathbb{R}^n\setminus B(\alpha)} v(x)\left(\int_{B(x)} v(y)\,dy\right)^{-p'} dx\right)^{\frac{1}{p'}} \cdot \left(\int_{B(\alpha)}\left(\frac{u(y)^{1-q'}}{|B(y)|^{q'}}\right)^{1-q} dy\right)^{\frac{1}{q}}$$
$$\leq \sup_{\alpha>0}\frac{\left(\int_{B(\alpha)} v(y)\,dy\right)^{-\frac{1}{p}}}{(p'-1)^{\frac{1}{p'}}}\left(\int_{B(\alpha)} u(y)\,|B(y)|^{q}\,dy\right)^{\frac{1}{q}} < \infty.$$

Similarly, according to Theorem 2.1 (ii),

$$\left(\int_{\mathbb{R}^n} (I_2(x))^{p'} V^{-p'}(x)\,v(x)\,dx\right)^{\frac{1}{p'}} \tag{4.11}$$
$$= \left(\int_{\mathbb{R}^n\setminus B(\alpha)}\left(|B(x)|\int_{\mathbb{R}^n\setminus B(x)} g(y)\,dy\right)^{-p'} V^{-p'}(x)\,v(x)\,dx\right)^{\frac{1}{p'}}$$
$$\leq c\left(\int_{\mathbb{R}^n} g^{q'}(x)\,u^{1-q'}(x)\,dx\right)^{\frac{1}{q'}}$$

because

$$\sup_{\alpha>0}\left(\int_{B(\alpha)} |B(x)|^{p'} V^{-p'}(x)\,v(x)\,dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n\setminus B(\alpha)} u(x)\,dx\right)^{\frac{1}{q}} < \infty$$

which holds by (4.8). Thus by using (4.9), Minkowski's inequality and (4.10) – (4.11) we see that (4.6) holds.

Now assume that (4.6) holds, i.e., that (4.9) holds. Then, in particular,

$$\left(\int_{\mathbb{R}^n} (I_1(x))^{p'} V^{-p'}(x)\,v(x)\,dx\right)^{\frac{1}{p'}} \leq c\left(\int_{\mathbb{R}^n} g^{q'}(x)\,u^{1-q'}(x)\,dx\right)^{\frac{1}{q'}} \tag{4.12}$$

and by using Theorem 2.1 (i) and arguing as above, we find that (4.7) holds. Moreover, (4.12) holds also with I_1 replaced by I_2 so that, by using Theorem 2.1 (ii) and again arguing as in the sufficiency part, we see that (4.8) holds. The proof is complete. □

Remark 4.7. For the case when the weights are also radially decreasing or increasing some of our results can be written in a more suitable form. Here we only state the following consequence of Proposition 4.5:

Corollary 4.8. *Let $1 < p \leq q < \infty$ and let $f(x)$ be positive and radially decreasing in $\mathbb{R}^n$. Then the Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(x)|}\int_{B(x)} f(y)\,dy\right)^{q} |B(x)|^{b}\,dx\right)^{\frac{1}{q}} \leq c\left(\int_{\mathbb{R}^n} f^{p}(x)\,|B(x)|^{a}\,dx\right)^{\frac{1}{p}}$$

holds if and only if $-1 < a < p-1$, $-1 < b < q-1$ and

$$\frac{a+1}{p} = \frac{b+1}{q}. \tag{4.13}$$

Proof. Apply Proposition 4.5 with $v(x) = |B(x)|^{a}$ and $u(x) = |B(x)|^{b}$. We note that some straightforward calculations give

$$\left(\int_{B(\alpha)} v(x)\,dx\right)^{-\frac{1}{p}}\left(\int_{B(\alpha)} u(x)\,dx\right)^{\frac{1}{q}} \approx \alpha^{\frac{-(a+1)n}{p}+\frac{(b+1)n}{q}} \tag{4.14}$$

whenever $a > -1$, $b > -1$ and

$$\left(\int_{B(\alpha)} |B(x)|^{p'} V^{-p'}(x)\, v(x)\, dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n \setminus B(\alpha)} u(x)dx\right)^{\frac{1}{q}} \approx \alpha^{\frac{an-anp'+n}{p'}+\frac{bn-nq+n}{q}} = \alpha^{n\left(-\frac{a+1}{p}+\frac{b+1}{q}\right)} \tag{4.15}$$

whenever $a < p-1$ and $b < q-1$.

Moreover, according to the estimates (4.14) and (4.15) the condition (4.13) (and only this) gives a finite supremum. The proof is complete. □

Remark 4.9. It is easy to see that Theorem 2.1 also holds if $\mathbb{R}^n$ is replaced by $\mathbb{R}^n_+$ or even some more general cone in $\mathbb{R}^n$. Therefore, by modifying our proofs, we see that all our results in this chapter indeed hold also when $\mathbb{R}^n$ is replaced by $\mathbb{R}^n_+$ or even general cones in $\mathbb{R}^n$ as defined in [4].

REFERENCES

[1] S. BARZA, H. HEINIG AND L-E. PERSSON, Duality theorems over the cone of monotone functions and sequences in higher dimensions, *J. Inequal. Appl.*, **7**(1) (2002), 79–108.

[2] S. BARZA, L-E. PERSSON AND J. SORIA, Sharp weighted multidimensional integral inequalities for monotone functions, *Math. Nachr.*, **210** (2000), 43–58.

[3] S. BARZA, L-E. PERSSON AND V.D. STEPANOV, On weighted multidimensional embeddings for monotone functions, *Math. Scand.*, **88**(2) (2001), 303–319.

[4] A. ČIŽMEŠIJA, L-E. PERSSON AND A. WEDESTIG, Weighted integral inequalities for Hardy and geometric mean operators with kernels over cones in R^n, to appear in *Italian J. Pure & Appl. Math.*

[5] P. DRÁBEK, H.P. HEINIG AND A. KUFNER, Higher dimensional Hardy inequality, *General Inequalities*, 7 (Oberwolfach, 1995), 3–16, *Internat. Ser. Numer. Math.*, 123, Birkhäuser, Basel, 1997.

[6] A. KUFNER AND L-E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co., Singapore/ New Jersey/ London/ Hong Kong, 2003.

[7] E.H. LIEB AND M. LOSS, Analysis, *Graduate Studies in Mathematics*, 14, AMS. (1997).

[8] E. SAWYER, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.*, **96**(2) (1990), 145–158.

[9] V.D. STEPANOV, Integral operators on the cone of monotone functions, *J. London Math. Soc.*, **48**(3) (1993), 465–487.