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$W^{2,2}$ ESTIMATES FOR SOLUTIONS TO NON-UNIFORMLY ELLIPTIC PDE'S WITH MEASURABLE COEFFICIENTS

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ABSTRACT. We propose to extend Talenti's estimates on the L^2 norm of the second order derivatives of the solutions of a uniformly elliptic PDE with measurable coefficients satisfying the Cordes condition to the non-uniformly elliptic case.

Key words and phrases: Cordes conditions, Elliptic partial differential equations.

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1. INTRODUCTION

The Cordes conditions first were used by H. O. Cordes [1] and later by G. Talenti [5] to prove C^α, $C^{1,\alpha}$ and $W^{2,2}$ estimates for the solutions of second order linear and elliptic partial differential equations in non-divergence form

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u,$$

where $A = (a_{ij}) \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ is a symmetric matrix function. As an introductory remark about the Cordes condition we can say that by using the normalization (see [5])

$$\sum_{i=1}^{n} a_{ii}(x) = 1$$

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or strictly positive lower and upper bounds (see [1])

$$0 < p \le \sum_{i=1}^{n} a_{ii}(x) \le P$$

we get a condition equivalent to the uniform ellipticity condition in $\mathbb{R}^2$ and stronger than it in $\mathbb{R}^n,\ n \ge 3$. At the same time it seems to be the weakest condition which implies that $\mathcal{A}$ is an isomorphism between the spaces $W_0^{2,2}(\Omega)$ and $L^2(\Omega)$ and implicitly gives existence and uniqueness for boundary value problems with measurable coefficients [4]. As an application it was used to prove the second order differentiability of $p-$harmonic functions [3].

If we assume that the Cordes condition is satisfied, then it is possible to give an optimal upper bound of the L^2 norm of the second order derivatives to the solution $u \in W_0^{2,2}(\Omega)$ of the problem

$$\mathcal{A}u = f, \quad f \in L^2(\Omega)$$

in terms of a constant times the L^2 norm of f. An interesting method, that connects linear algebra to PDE's, has been developed in [5]. In this paper we will extend this method to not necessarily uniformly elliptic problems and as an application we will also show a change in Talenti's constant. More exactly, estimate (1.2) below holds in the case of operators with constant coefficients, but needs a change to cover the general case.

Let us consider the bounded domain $\Omega \in \mathbb{R}^n$ with a sufficiently regular boundary and the Sobolev space

$$W^{2,2}(\Omega) = \left\{ u \in L^2(\Omega) : D_{ij}u \in L^2(\Omega)\,,\ \forall\, i,j \in \{1,\dots,n\} \right\}$$

endowed with the inner-product

$$(u,v)_{W^{2,2}} = \int_\Omega \left(u(x)v(x) + \sum_{i,j=1}^{n} D_{ij}u(x)\cdot D_{ij}v(x) \right) dx.$$

Let $W_0^{2,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{2,2}(\Omega)$ and denote by D^2u the matrix of the second order derivatives.

We state now Talenti's result using our setting.

Theorem 1.1 ([5]). *Let us suppose that for a fixed* $0 < \varepsilon < 1$ *and almost every* $x \in \Omega$ *the following conditions hold:*

$$\sum_{i=1}^{n} a_{ii}(x) = 1 \quad \text{and} \quad \sum_{i,j=1}^{n} \Big(a_{ij}(x)\Big)^2 \le \frac{1}{n-1+\varepsilon}. \tag{1.1}$$

Then, for all $u \in W_0^{2,2}(\Omega)$ *we have*

$$||D^2u||_{L^2(\Omega)} \le \frac{\sqrt{n-1+\varepsilon}}{\varepsilon}\Big(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)}\Big)||\mathcal{A}u||_{L^2(\Omega)}\,. \tag{1.2}$$

2. MAIN RESULT

Consider the matrix valued mapping $A : \Omega \to \mathcal{M}_n(\mathbb{R})$, where $A(x) = (a_{ij}(x))$ with $a_{ij} \in L^\infty(\Omega)$, and let

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}(u). \tag{2.1}$$

We use the notations $||a|| = \sqrt{a_1^2 + \cdots + a_n^2}$ for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\operatorname{trace} A = \sum_{i=1}^n a_{ii}$ for the trace of an $n \times n$ matrix $A = (a_{ij})$. Also, we denote by $\langle A, B \rangle = \sum_{i,j=1}^n a_{ij} b_{ij}$ the inner product and by $||A|| = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ the Euclidean norm in $\mathbb{R}^{n \times n}$.

Definition 2.1 (Cordes condition K_ε). We say that A satisfies the Cordes condition K_ε if there exists $\varepsilon \in (0, 1]$ such that

$$0 < ||A(x)||^2 \leq \frac{1}{n - 1 + \varepsilon} \Big(\operatorname{trace} A(x) \Big)^2, \tag{2.2}$$

for almost every $x \in \Omega$ and

$$\frac{1}{\operatorname{trace} A} \in L^2_{\mathrm{loc}}(\Omega).$$

Remark 2.1. We observe that inequality (2.2) implies that for

$$\sigma(x) = \frac{\sqrt{n}}{\operatorname{trace} A(x)}$$

we have

$$0 < \frac{1}{\sigma^2(x)} \leq ||A(x)||^2 \leq \frac{1}{n - 1 + \varepsilon} \Big(\operatorname{trace} A(x) \Big)^2 \tag{2.3}$$

with $\sigma(\cdot) \in L^2_{\mathrm{loc}}(\Omega)$. Therefore without a strictly positive lower bound for $\operatorname{trace} A(x)$, the Cordes condition K_ε does not imply uniform ellipticity. As an example we can mention

$$A(x, y) = \begin{bmatrix} y & \sqrt{\frac{xy}{2}} \\ \sqrt{\frac{xy}{2}} & x \end{bmatrix}$$

defined on

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ 0 < x^2 + y^2 < 1,\ 1 < \frac{y}{x} < 2 \right\}.$$

In this case inequality (2.2) looks like

$$x^2 + xy + y^2 < \frac{1}{1 + \varepsilon}(x + y)^2.$$

Considering the lines $y = mx$ we see that

$$\varepsilon = \inf \left\{ \frac{m}{m^2 + m + 1} : 1 < m < 2 \right\} = \frac{2}{7}$$

and

$$\sigma(x) = \frac{\sqrt{2}}{x + y}.$$

Remark 2.2. In the case when we want to have a strictly positive lower bound for $\operatorname{trace} A$ we should use a Cordes condition $K_{\varepsilon,\gamma}$ that asks for the existence of a number $\gamma > 0$ such that

$$0 < \frac{1}{\gamma} \leq \frac{1}{\sigma^2(x)} \leq ||A(x)||^2 \leq \frac{1}{n - 1 + \varepsilon} \Big(\operatorname{trace} A(x) \Big)^2 \tag{2.4}$$

for almost every $x \in \Omega$. In this way the normalized condition (1.1) corresponds to the $K_{\varepsilon,n}$, since $\sum_{i=1}^n a_{ii} = 1$ implies that $\gamma = n$.

We recall the following lemma from [5].

Lemma 2.3. *Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Suppose that*

$$(a_1 + \cdots + a_n)^2 > (n-1)||a||^2. \tag{2.5}$$

If for $\alpha > 1$ and $\beta > 0$ the condition

$$(a_1 + \cdots + a_n)^2 \geq \left(n - 1 + \frac{1}{\alpha}\right)||a||^2 + \frac{1}{\beta}\left(n - 1 + \frac{1}{\alpha}\right)(\alpha - 1), \tag{2.6}$$

holds, then we have

$$||k||^2 + 2\alpha \sum_{i<j} k_i k_j \leq \beta(a_1 k_1 + \cdots + a_n k_n)^2 \tag{2.7}$$

for all $k = (k_1, \dots, k_n) \in \mathbb{R}^n$.

The next lemma is the nonsymmetric version of the original one in Talenti's paper [5]. By nonsymmetric version we mean that we drop the assumption that A is symmetric. On the other hand, it is easy to see that Lemma 2.4 below will not hold for arbitrary nonsymmetric matrices P, even in the case when A is diagonal. For the completeness of our paper we include the proof, which can be considered as a natural extension of the original one.

Lemma 2.4. *Let $A = (a_{ij})$ be an $n \times n$ real matrix. Suppose that*

$$(\operatorname{trace} A)^2 > (n-1)||A||^2. \tag{2.8}$$

If for $\alpha > 1$ and $\beta > 0$ the condition

$$(\operatorname{trace} A)^2 \geq \left(n - 1 + \frac{1}{\alpha}\right)||A||^2 + \frac{1}{\beta}\left(n - 1 + \frac{1}{\alpha}\right)(\alpha - 1) \tag{2.9}$$

holds, then we have

$$||P||^2 + \alpha \sum_{i,j=1}^{n} \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ij} & p_{jj} \end{vmatrix} \leq \beta \langle A, P \rangle^2 \tag{2.10}$$

for all real and symmetric $n \times n$ matrices $P = (p_{ij})$.

Proof. Consider an arbitrary but fixed real and symmetric matrix P. It follows that there exists a real orthogonal matrix C and a diagonal matrix

$$D = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix}$$

such that $P = C^{-1}DC$. Observe that

$$-\frac{1}{2} \sum_{i,j=1}^{n} \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ij} & p_{jj} \end{vmatrix}$$

is the coefficient of λ^{n-2} in the characteristic polynomial of P, therefore

$$\frac{1}{2} \sum_{i,j=1}^{n} \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ij} & p_{jj} \end{vmatrix} = \sum_{i<j} k_i k_j.$$

Moreover,

$$\sum_{i,j=1}^{n} p_{ij}^2 = \operatorname{trace}(P^2) = \sum_{i=1}^{n} k_i^2. \tag{2.11}$$

Hence, inequality (2.10) can be rewritten as

$$|k|^2 + 2\alpha \sum_{i<j} k_i k_j \le \beta \left(\sum_{i,j=1}^{n} a_{ij} p_{ij} \right)^2 . \tag{2.12}$$

Let $B = CAC^{-1}$. Then $\operatorname{trace} B = \operatorname{trace} A$ and

$$\begin{aligned} \langle A, P \rangle &= \operatorname{trace}(AP) \\ &= \operatorname{trace}(CAPC^{-1}) \\ &= \operatorname{trace}(CAC^{-1}CPC^{-1}) \\ &= \operatorname{trace}(BD) \\ &= \sum_{i=1}^{n} b_{ii} k_i. \end{aligned} \tag{2.13}$$

Also, because B and A are unitary equivalent, we have

$$\sum_{i=1}^{n} b_{ii}^2 \le \sum_{ij}^{n} b_{ij}^2 = \sum_{i,j=1}^{n} a_{ij}^2 .$$

Therefore, $b = (b_{11}, \dots, b_{nn})$, α and β satisfy the condition (2.6) from Lemma 2.3, and hence

$$\sum_{i=1}^{n} k_1^2 + 2\alpha \sum_{i<j} k_i k_j \le \beta (b_{11} k_1 + \cdots + b_{nn} k_n)^2 = \beta \langle A, P \rangle^2 .$$

Using (2.11) – (2.13) we get (2.10). $\square$

Theorem 2.5. *Suppose that A satisfies the Cordes condition K_ε. Then for all $u \in C_0^\infty(\Omega)$ we have*

$$||D^2 u||_{L^2(\Omega)} \le \frac{1}{\varepsilon} \left(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)} \right) ||\sigma \mathcal{A} u||_{L^2(\Omega)} . \tag{2.14}$$

Proof. Fix $x \in \Omega$ such that (2.3) holds and consider an arbitrary $\alpha > 1/\varepsilon$. Then

$$\left(\sum_{i=1}^{n} a_{ii}(x) \right)^2 > \left(n - 1 + \frac{1}{\alpha} \right) ||A(x)||^2 .$$

In order to choose $\beta(x) > 0$ such that

$$\left(\sum_{i=1}^{n} a_{ii}(x) \right)^2 \ge \left(n - 1 + \frac{1}{\alpha} \right) ||A(x)||^2 + \frac{1}{\beta(x)} \left(n - 1 + \frac{1}{\alpha} \right) (\alpha - 1), \tag{2.15}$$

observe that condition K_ε is equivalent to

$$\left(\sum_{i=1}^{n} a_{ii}(x) \right)^2 \ge \left(n - 1 + \frac{1}{\alpha} \right) ||A(x)||^2 + \left(\varepsilon - \frac{1}{\alpha} \right) ||A(x)||^2 .$$

Therefore we have to ask $\beta(x)$ to satisfy

$$\left(\varepsilon - \frac{1}{\alpha} \right) ||A(x)||^2 \ge \frac{1}{\beta(x)} \left(n - 1 + \frac{1}{\alpha} \right) (\alpha - 1),$$

and hence

$$\beta(x) \ge \sigma^2(x) \, \frac{(n-1)\alpha^2 + (2-n)\alpha - 1}{\varepsilon\alpha - 1} . \tag{2.16}$$

Considering the function $f : (1/\varepsilon, +\infty) \to \mathbb{R}$ defined by

$$f(\alpha) = \frac{(n-1)\alpha^2 + (2-n)\alpha - 1}{\varepsilon\alpha - 1},$$

we get that its minimum point is

$$\alpha = \frac{n-1+\sqrt{(n-1)(1-\varepsilon)(n-1+\varepsilon)}}{(n-1)\varepsilon}.$$

Therefore, the minimum value of $\sigma^2(x)\, f(\alpha)$, which is coincidentally the best choice of $\beta(x)$, is

$$\begin{aligned}\beta(x) &= \sigma^2(x)\frac{2\varepsilon - \varepsilon n + 2n - 2 + \sqrt{(n-1)(1-\varepsilon)(n-1+\varepsilon)}}{\varepsilon^2} \\ &= \frac{\sigma^2(x)}{\varepsilon^2}\left(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)}\right)^2.\end{aligned}$$

Applying Lemma 2.4 in the case of $u \in C_0^\infty(\Omega)$ and $p_{ij} = D_{ij}u(x)$, we get

$$\int_\Omega \sum_{i,j=1}^n (D_{ij}u(x))^2 dx + \alpha \sum_{i\neq j} \int_\Omega \begin{vmatrix} D_{ii}u(x) & D_{ij}u(x) \\ D_{ij}u(x) & D_{jj}u(x) \end{vmatrix} dx \leq \int_\Omega \beta(x)(\mathcal{A}u(x))^2 dx. \tag{2.17}$$

But, integrating by parts two times we get

$$\int_\Omega D_{ii}u(x)D_{jj}u(x)dx = \int_\Omega D_{ij}u(x)D_{ij}u(x)dx, \tag{2.18}$$

and hence

$$\int_\Omega \begin{vmatrix} D_{ii}u(x) & D_{ij}(x) \\ D_{ij}u(x) & D_{jj}u(x) \end{vmatrix} dx = 0. \tag{2.19}$$

Therefore, for all $u \in C_0^\infty(\Omega)$ we have

$$||D^2u||_{L^2(\Omega)} \leq \frac{1}{\varepsilon}\left(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)}\right)||\sigma\mathcal{A}u||_{L^2(\Omega)}.$$

□

Theorem 2.5 clearly implies the following result.

Corollary 2.6. *Suppose that A satisfies Cordes condition $K_{\varepsilon,\gamma}$. Then for all $u \in W_0^{2,2}(\Omega)$ we have*

$$||D^2u||_{L^2(\Omega)} \leq \frac{\sqrt{\gamma}}{\varepsilon}\left(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)}\right)||\mathcal{A}u||_{L^2(\Omega)}. \tag{2.20}$$

Remark 2.7. In the case of $\operatorname{trace} A = 1$ we get that

$$||D^2u||_{L^2(\Omega)} \leq \frac{\sqrt{n}}{\varepsilon}\left(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)}\right)||\mathcal{A}u||_{L^2(\Omega)}. \tag{2.21}$$

If we compare estimate (1.2) with ours from (2.21) we realize that our constant on the right hand side is larger. The interesting fact is that the two constants in (1.2) and (2.21) coincide in the case when $A = \frac{1}{n}I$ and $\varepsilon = 1$, and give (see [2])

$$||D^2u||_{L^2(\Omega)} \leq ||\Delta u||_{L^2(\Omega)}, \quad \text{for all } \ u \in W_0^{2,2}(\Omega)\,.$$

Looking at Talenti's paper [5] we realize that the way in which the constant B is chosen on page 303 leads to

$$||A(x)||^2 \geq \frac{1}{n-1+\varepsilon}. \tag{2.22}$$

Comparing this inequality to (1.1) which gives

$$\frac{1}{n} \le ||A(x)||^2 \le \frac{1}{n-1+\varepsilon}$$

and therefore

$$||A(x)||^2 = \frac{1}{n-1+\varepsilon},$$

we conclude that (2.22) (and hence (1.2)) holds for constant matrices A but may fail for a nonconstant $A(x)$ on a subset of Ω with positive Lebesgue measure. Therefore, the estimate (2.21) is the right one for nonconstant matrix functions $A(x)$ satisfying (1.1).

Remark 2.8. Another interesting fact is found when applying our method to the case of convex functions u. In this case we can further generalize the Cordes condition in the following way: We say that A satisfies the condition $K_{\varepsilon(x)}$ if

$$\frac{1}{\operatorname{trace} A} \in L^2_{\text{loc}}(\Omega)$$

and there exists a measurable function $\varepsilon : \Omega \to \mathbb{R}$ such that $0 < \varepsilon(x) \le 1$ for a.e. $x \in \Omega$ and $\frac{1}{\varepsilon} \in L^2(\Omega)$, and the following inequalities hold:

$$0 < \frac{1}{\sigma^2(x)} = \frac{\Big(\operatorname{trace} A(x)\Big)^2}{n} \le ||A(x)||^2 \le \frac{\Big(\operatorname{trace} A(x)\Big)^2}{n-1+\varepsilon(x)}. \tag{2.23}$$

Inequality (2.17) in this case looks like

$$\int_\Omega \sum_{i,j=1}^n (D_{ij}u(x))^2\,dx + \sum_{i\neq j}\int_\Omega \alpha(x) \begin{vmatrix} D_{ii}u(x) & D_{ij}u(x) \\ D_{ij}u(x) & D_{jj}u(x) \end{vmatrix} dx \le \int_\Omega \beta(x)(\mathcal{A}u(x))^2 dx\,.$$

Observe that the convexity of u implies that $D^2u(x)$ is positive definite, which makes the determinants

$$\begin{vmatrix} D_{ii}u(x) & D_{ij}u(x) \\ D_{ij}u(x) & D_{jj}u(x) \end{vmatrix}$$

positive. We conclude in this way that under the Cordes condition $K_{\varepsilon(x)}$ for all convex functions $u \in W^{2,2}(\Omega)$ we still have

$$||D^2u||_{L^2(\Omega)} \le \left\| \frac{1}{\varepsilon}\, \left(\sqrt{n-1+\varepsilon} + \sqrt{(1-\varepsilon)(n-1)}\right) \sigma \mathcal{A}u \right\|_{L^2(\Omega)}.$$

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