

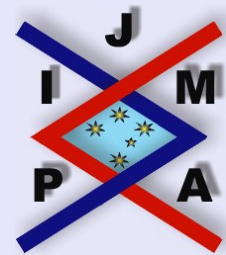
Journal of Inequalities in Pure and Applied Mathematics

ON THE VALUE DISTRIBUTION OF $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

KIT-WING YU

Rm 205, Kwai Shun Hse.,
Kwai Fong Est., Hong Kong,
The People's Republic of China
EMail: maykw00@alumni.ust.hk

©2000 Victoria University
ISSN (electronic): 1443-5756
037-01



volume 3, issue 1, article 8,
2002.

*Received 01 May, 2001;
accepted 04 October, 2001.*

Communicated by: H.M. Srivastava

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper, the value distribution of $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$ is studied, where $f(z)$ is a transcendental meromorphic function, $\varphi(z)(\neq 0)$ is a function such that $T(r, \varphi) = o(T(r, f))$ as $r \rightarrow +\infty$, n and k are positive integers such that $n = 1$ or $n \geq k + 3$. This generalizes a result of Hiong.

2000 Mathematics Subject Classification: 30D35, 30A10.

Key words: Derivatives, Inequality, Meromorphic Functions, Small Functions, Value Distribution.

The author would like to express his sincere appreciation to the referee for the thorough and helpful comments that have aided significantly in improving the paper.

Contents

1	Introduction and the Main Result	3
2	Lemmas	7
3	Proof of the Main Result	9
4	Concluding Remarks and a Conjecture	11
	References	



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 2 of 12

1. Introduction and the Main Result

Throughout this paper, we use the notations $[f(z)]^n$ or $[f]^n$ to denote the n -power of a meromorphic function f . Similarly, $f^{(k)}(z)$ or $f^{(k)}$ are used to denote the k -order derivative of f .

In 1940, Milloux [5] showed that

Theorem A. *Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Further, let*

$$\phi(z) = \sum_{i=0}^k a_i(z) f^{(i)}(z),$$

where $a_i(z)$ ($i = 0, 1, \dots, k$) are small functions of $f(z)$. Then we have

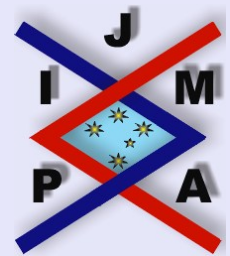
$$m\left(r, \frac{\phi}{f}\right) = S(r, f)$$

and

$$T(r, \phi) \leq (k + 1)T(r, f) + S(r, f)$$

as $r \rightarrow +\infty$.

From this, it is easy for us to derive the following inequality which states a relationship between $T(r, f)$ and the 1-point of the derivatives of f . For the proof, please see [4], [7] or [8],



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 3 of 12

Theorem B. Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Then

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f)$$

as $r \rightarrow +\infty$.

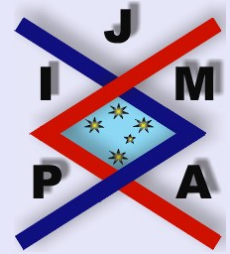
In fact, the above estimate involves the consideration of the zeros and poles of $f(z)$. Then a natural question is: Is it possible to use only the counting functions of the zeros of $f(z)$ and an a -point of $f^{(k)}(z)$ to estimate the function $T(r, f)$? Hiong proved that the answer to this question is yes. Actually, Hiong [6] obtained the following inequality

Theorem C. Let $f(z)$ be a non-constant meromorphic function. Further, let a , b and c be three finite complex numbers such that $b \neq 0$, $c \neq 0$ and $b \neq c$. Then

$$T(r, f) < N\left(r, \frac{1}{f - a}\right) + N\left(r, \frac{1}{f^{(k)} - b}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f)$$

as $r \rightarrow +\infty$.

Following this idea, a natural question to Theorem C is: Can we extend the three complex numbers to small functions of $f(z)$? In [9], by studying



On the Value Distribution of $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 4 of 12

the zeros of the function $f(z)f'(z) - c(z)$, where $c(z)$ is a small function of $f(z)$, the author generalized the above inequality under an extra condition on the derivatives of $f^{(k)}(z)$. In fact, we have

Theorem D. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z) (\not\equiv 0)$ is a meromorphic function such that $T(r, \varphi) = o(T(r, f))$ as $r \rightarrow +\infty$. Then for any finite non-zero distinct complex numbers b and c and any positive integer k such that $\varphi(z)f^{(k)}(z) \not\equiv \text{constant}$, we have*

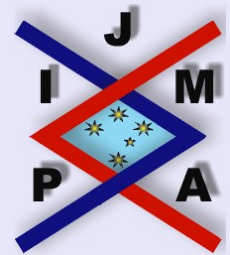
$$T(r, f) < N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\varphi f^{(k)} - b}\right) + N\left(r, \frac{1}{\varphi f^{(k)} - c}\right) - N(r, f) - N\left(r, \frac{1}{(\varphi f^{(k)})'}\right) + S(r, f)$$

as $r \rightarrow +\infty$.

In this paper, we are going to show that Theorem D is still valid for all positive integers k . As a result, this generalizes Theorem C to small functions completely. More generally, we show that:

Theorem 1.1. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z) (\not\equiv 0)$ is a meromorphic function such that $T(r, \varphi) = o(T(r, f))$ as $r \rightarrow +\infty$. Suppose further that b and c are any finite non-zero distinct complex numbers, and k and n are positive integers. If $n = 1$ or $n \geq k + 3$, then we have*

$$(1.1) \quad T(r, f)$$



On the Value Distribution of $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 5 of 12

$$\begin{aligned}
&< N\left(r, \frac{1}{f}\right) + \frac{1}{n} \left[N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - b}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - c}\right) \right] \\
&\quad - \frac{1}{n} \left[N(r, f) + N\left(r, \frac{1}{(\varphi[f]^{n-1}f^{(k)})'}\right) \right] + S(r, f)
\end{aligned}$$

as $r \rightarrow +\infty$.

If $f(z)$ is entire, then (2.1) is true for all positive integers $n(\neq 2)$.

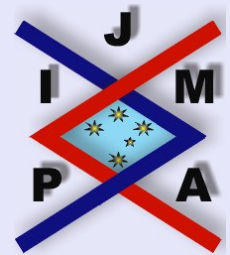
As an immediate application of our theorem, we have

Corollary 1.2. *If we take $n = 1$ in the theorem, then we have Theorem D.*

Corollary 1.3. *If we take $n = 1$, $\varphi(z) \equiv 1$ and $f(z) = g(z) - a$, where a is any complex number, then we obtain Theorem C.*

Remark 1.1. *We shall remark that our main theorem and corollaries are also valid if $f(z)$ is rational since $\varphi(z) \equiv \text{constant}$ and $\varphi(z)[f(z)]^{n-1}f^{(k)}(z) \not\equiv \text{constant}$ in this case.*

Here, we assume that the readers are familiar with the basic concepts of the Nevanlinna value distribution theory and the notations $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$, $S(r, f)$, etc., see e.g. [1].



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 6 of 12

2. Lemmae

For the proof of the main result, we need the following three lemmae.

Lemma 2.1. [3] *If $F(z)$ is a transcendental meromorphic function and $K > 1$, then there exists a set $M(K)$ of upper logarithmic density at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K - 1)) \exp(e(1 - K))\}$$

such that for every positive integer q ,

$$(2.1) \quad \overline{\lim}_{r \rightarrow \infty, r \notin M(K)} \frac{T(r, F)}{T(r, F^{(q)})} \leq 3eK.$$

If $F(z)$ is entire, then we can replace $3eK$ by $2eK$ in (2.1).

Lemma 2.2. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z) (\not\equiv 0)$ is a meromorphic function such that $T(r, \varphi) = o(T(r, f))$ as $r \rightarrow +\infty$. Suppose further that k and n are positive integers. If $n = 1$ or $n \geq k + 3$, then $\varphi(z)[f(z)]^{n-1} f^{(k)}(z) \not\equiv \text{constant}$.*

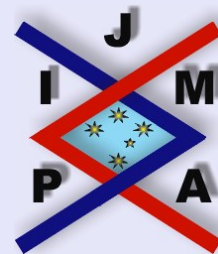
Proof. Without loss of generality, we suppose that the constant is 1. If $n = 1$, then $\varphi f^{(k)} \equiv 1$. Hence, $T(r, \varphi) = T(r, f^{(k)}) + O(1)$ as $r \rightarrow +\infty$ and this implies that

$$\overline{\lim}_{r \rightarrow \infty, r \notin M(K)} \frac{T(r, f)}{T(r, f^{(k)})} = \infty.$$

This contradicts Lemma (2.1).

If $n \geq k + 3$, then $T(r, \varphi f^{(k)}) = (n - 1)T(r, f)$ as $r \rightarrow +\infty$ and

$$(2.2) \quad (n - 1)T(r, f) \leq T(r, f^{(k)}) + S(r, f)$$



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1} f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 7 of 12

as $r \rightarrow +\infty$. On the other hand,

$$(2.3) \quad T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f)$$

as $r \rightarrow +\infty$. By (2.2) and (2.3), we have $n \leq k+2$, a contradiction.

Hence, we have $\varphi[f]^{n-1}f^{(k)} \not\equiv \text{constant}$ in both cases and the lemma is proven. \square

Lemma 2.3. *If $f(z)$ is entire, then $\varphi(z)[f(z)]^{n-1}f^{(k)}(z) \not\equiv \text{constant}$ for all positive integers $n(\neq 2)$ and k .*

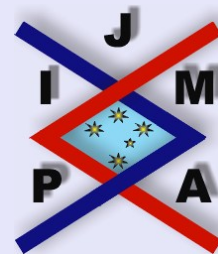
Proof. For the case $n = 1$, we still have $T(r, \varphi) = T(r, f^{(k)}) + O(1)$ as $r \rightarrow +\infty$, so a contradiction to Lemma (2.1) again.

For $n \geq 3$, instead of (2.3), we have

$$(2.4) \quad T(r, f^{(k)}) \leq T(r, f) + S(r, f)$$

as $r \rightarrow +\infty$.

So by (2.2) and (2.4), we have $n \leq 2$, a contradiction. \square



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 8 of 12

3. Proof of the Main Result

Proof. First of all, by the given conditions and Lemma 2.2, we know that $\varphi[f]^{n-1}f^{(k)} \not\equiv \text{constant}$ for $n \geq 1$. Therefore, we have

$$(3.1) \quad m\left(r, \frac{1}{\varphi[f]^n}\right) \leq m\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1).$$

From

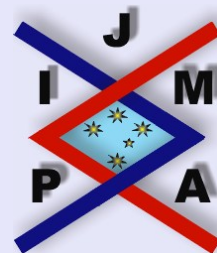
$$\begin{aligned} m\left(r, \frac{1}{\varphi[f]^n}\right) &= T(r, \varphi[f]^n) - N\left(r, \frac{1}{\varphi[f]^n}\right) + O(1), \\ m\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) &= T(r, \varphi[f]^{n-1}f^{(k)}) - N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + O(1), \end{aligned}$$

and (3.1), we have

$$(3.2) \quad \begin{aligned} T(r, \varphi[f]^n) &\leq N\left(r, \frac{1}{\varphi[f]^n}\right) + T(r, \varphi[f]^{n-1}f^{(k)}) \\ &\quad - N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1). \end{aligned}$$

Since $\varphi(z)[f(z)]^{n-1}f^{(k)} \not\equiv \text{constant}$, from the second fundamental theorem,

$$(3.3) \quad \begin{aligned} T(r, \varphi[f]^{n-1}f^{(k)}) &< N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - b}\right) \\ &\quad + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - c}\right) - N_1(r) + S(r, \varphi f^{(k)}) \end{aligned}$$



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 9 of 12

as $r \rightarrow +\infty$, where b and c are two non-zero distinct complex numbers and, as usual, $N_1(r)$ is defined as

$$N_1(r) = 2N(r, \varphi[f]^{n-1} f^{(k)}) - N(r, (\varphi[f]^{n-1} f^{(k)})') + N\left(r, \frac{1}{(\varphi[f]^{n-1} f^{(k)})'}\right).$$

Let z_0 be a pole of order $p \geq 1$ of f . Then $[f]^{n-1} f^{(k)}$ and $([f]^{n-1} f^{(k)})'$ have a pole of order $k + np$ and $k + np + 1$ at z_0 respectively. Thus $2(k + np) - (k + np + 1) = k + np - 1 \geq p$ and

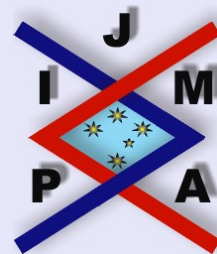
$$(3.4) \quad N_1(r) \geq N(r, f) + N\left(r, \frac{1}{(\varphi[f]^{n-1} f^{(k)})'}\right) + S(r, f).$$

It is clear that $S(r, f^{(k)}) = S(r, f)$ and $m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. Thus by (3.2), (3.3) and (3.4),

$$\begin{aligned} T(r, \varphi[f]^n) &< N\left(r, \frac{1}{\varphi[f]^n}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1} f^{(k)} - c}\right) \\ &\quad - N(r, f) - N\left(r, \frac{1}{(\varphi[f]^{n-1} f^{(k)})'}\right) + S(r, f) \end{aligned}$$

as $r \rightarrow +\infty$. Since $T(r, \varphi) = o(T(r, f))$ as $r \rightarrow +\infty$, we have the desired result. \square

If f is entire, then by Lemma (2.3), we still have $\varphi[f]^{n-1} f^{(k)} \neq \text{constant}$ for all positive integers $n (\neq 2)$, (3.3) and (3.4). Thus the same argument can be applied and the same result is obtained.



On the Value Distribution of $(\varphi(z)[f(z)]^{n-1} f^{(k)}(z))$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 10 of 12

4. Concluding Remarks and a Conjecture

Remark 4.1. We expect that our theorem is also valid for the case $n = 2$ if $f(z)$ is entire.

Remark 4.2. In [10], Zhang studied the value distribution of $\varphi(z)f(z)f'(z)$ and he obtained the following result: If $f(z)$ is a non-constant meromorphic function and $\varphi(z)$ is a non-zero meromorphic function such that $T(r, \varphi) = S(r, f)$ as $r \rightarrow +\infty$, then

$$T(r, f) < \frac{9}{2}\overline{N}(r, f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi f f' - 1}\right) + S(r, f)$$

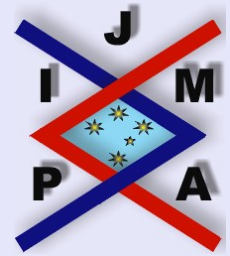
as $r \rightarrow +\infty$.

Hence, by this remark, we expect the following conjecture would be true.

Conjecture 4.1. Let n and k be positive integers. If $n = 1$ or $n \geq k + 3$, $f(z)$ is a non-constant meromorphic function and $\varphi(z)$ is a non-zero meromorphic function such that $T(r, \varphi) = S(r, f)$ as $r \rightarrow +\infty$, then

$$T(r, f) < \frac{9}{2}\overline{N}(r, f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - 1}\right) + S(r, f)$$

as $r \rightarrow +\infty$.



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 11 of 12

References

- [1] W.K. HAYMAN, *Meromorphic Functions*, Oxford, Clarendon Press, 1964.
- [2] W.K. HAYMAN, Picard values of meromorphic functions and their derivatives, *Ann. Math.*, **70** (1959), 9–42.
- [3] W.K. HAYMAN AND J. MILES, On the growth of a meromorphic function and its derivatives, *Complex Variables*, **12** (1989), 245–260.
- [4] H. MILLOUX, Extension d'un théorème de M. R. Nevanlinna et applications, *Act. Scient. et Ind.*, no.888, 1940.
- [5] H. MILLOUX, *Les fonctions méromorphes et leurs dérivées*, Paris, 1940.
- [6] K.L. HIONG, Sur la limitation de $T(r, f)$ sans intervention des pôles, *Bull. Sci. Math.*, **80** (1956), 175–190.
- [7] L. YANG, *Value distribution theory and its new researches* (Chinese), Beijing, 1982.
- [8] H.X. YI and C.C. YANG, *On the uniqueness theory of meromorphic functions* (Chinese), Science Press, China, 1996.
- [9] K.W. YU, A note on the product of a meromorphic function and its derivative, to appear in *Kodai Math. J.*
- [10] Q.D. ZHANG, On the value distribution of $\varphi(z)f(z)f'(z)$ (Chinese), *Acta Math. Sinica*, **37** (1994), 91–97.



On the Value Distribution of
 $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

Kit-Wing Yu

Title Page

Contents



Go Back

Close

Quit

Page 12 of 12