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ON THE VALUE DISTRIBUTION OF $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$

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Abstract

In this paper, the value distribution of $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)$ is studied, where $f(z)$ is a transcendental meromorphic function, $\varphi(z)(\not\equiv 0)$ is a function such that $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, n and k are positive integers such that $n = 1$ or $n \geq k + 3$. This generalizes a result of Hiong.

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Contents

1. Introduction and the Main Result

Throughout this paper, we use the notations $[f(z)]^n$ or $[f]^n$ to denote the n-power of a meromorphic function f. Similarly, $f^{(k)}(z)$ or $f^{(k)}$ are used to denote the k-order derivative of f.

In 1940, Milloux [5] showed that

Theorem A. *Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Further, let*

$$\phi(z) = \sum_{i=0}^{k} a_i(z) f^{(i)}(z),$$

where $a_i(z)(i = 0, 1, \dots, k)$ are small functions of $f(z)$. Then we have

$$m\left(r, \frac{\phi}{f}\right) = S(r, f)$$

and

$$T(r, \phi) \leq (k+1)T(r, f) + S(r, f)$$

as $r \to +\infty$.

From this, it is easy for us to derive the following inequality which states a relationship between $T(r, f)$ and the 1-point of the derivatives of f. For the proof, please see [4], [7] or [8],

Theorem B. *Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Then*

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f)$$

as $r \to +\infty$.

In fact, the above estimate involves the consideration of the zeros and poles of $f(z)$. Then a natural question is: Is it possible to use only the counting functions of the zeros of $f(z)$ and an a-point of $f^{(k)}(z)$ to estimate the function $T(r,f)$? Hiong proved that the answer to this question is yes. Actually, Hiong [6] obtained the following inequality

Theorem C. *Let $f(z)$ be a non-constant meromorphic function. Further, let a, b and c be three finite complex numbers such that $b \ne 0$, $c \ne 0$ and $b \ne c$. Then*

$$T(r,f) < N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f)$$

as $r \to +\infty$.

Following this idea, a natural question to Theorem C is: Can we extend the three complex numbers to small functions of $f(z)$? In [9], by studying the zeros of the function $f(z)f'(z) - c(z)$, where $c(z)$ is a small function of $f(z)$, the author generalized the above inequality under an extra condition on the derivatives of $f^{(k)}(z)$. In fact, we have

Theorem D. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z)(\not\equiv 0)$ is a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$. Then for any finite non-zero distinct complex numbers b and c and any positive integer k such that $\varphi(z)f^{(k)}(z) \not\equiv$ constant, we have*

$$T(r,f) < N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\varphi f^{(k)} - b}\right) + N\left(r,\frac{1}{\varphi f^{(k)} - c}\right) - N(r,f) - N\left(r,\frac{1}{(\varphi f^{(k)})'}\right) + S(r,f)$$

as $r \to +\infty$.

In this paper, we are going to show that Theorem D is still valid for all positive integers k. As a result, this generalizes Theorem C to small functions completely. More generally, we show that:

Theorem 1.1. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z)(\not\equiv 0)$ is a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$. Suppose further that b and c are any finite non-zero distinct complex numbers, and k and n are positive integers. If $n = 1$ or $n \geq k+3$, then we have*

(1.1) $$T(r,f)$$

$$< N\left(r,\frac{1}{f}\right)+\frac{1}{n}\left[N\left(r,\frac{1}{\varphi[f]^{n-1}f^{(k)}-b}\right)+N\left(r,\frac{1}{\varphi[f]^{n-1}f^{(k)}-c}\right)\right] - \frac{1}{n}\left[N(r,f)+N\left(r,\frac{1}{(\varphi[f]^{n-1}f^{(k)})'}\right)\right]+S(r,f)$$

as $r\to+\infty$.
If $f(z)$ *is entire, then (2.1) is true for all positive integers* $n(\neq 2)$.

As an immedicate application of our theorem, we have

Corollary 1.2. *If we take* $n=1$ *in the theorem, then we have Theorem D.*

Corollary 1.3. *If we take* $n=1$, $\varphi(z)\equiv 1$ *and* $f(z)=g(z)-a$, *where* a *is any complex number, then we obtain Theorem C.*

Remark 1.1. *We shall remark that our main theorem and corollaries are also valid if* $f(z)$ *is rational since* $\varphi(z)\equiv constant$ *and* $\varphi(z)[f(z)]^{n-1}f^{(k)}(z)\not\equiv constant$ *in this case.*

Here, we assume that the readers are familiar with the basic concepts of the Nevanlinna value distribution theory and the notations $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $T(r,f)$, $S(r,f)$, etc., see e.g. [1].

2. Lemmae

For the proof of the main result, we need the following three lemmae.

Lemma 2.1. *[3] If $F(z)$ is a transcendental meromorphic function and $K > 1$, then there exists a set $M(K)$ of upper logarithmic density at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K-1)) \exp(e(1-K))\}$$

such that for every positive integer q,

$$\overline{\lim_{r\to\infty, r\notin M(K)}} \frac{T(r,F)}{T(r,F^{(q)})} \leq 3eK. \tag{2.1}$$

If $F(z)$ is entire, then we can replace $3eK$ by $2eK$ in (2.1).

Lemma 2.2. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\varphi(z)(\not\equiv 0)$ is a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$. Suppose further that k and n are positive integers. If $n = 1$ or $n \geq k+3$, then $\varphi(z)[f(z)]^{n-1}f^{(k)}(z) \not\equiv constant$.*

Proof. Without loss of generality, we suppose that the constant is 1. If $n = 1$, then $\varphi f^{(k)} \equiv 1$. Hence, $T(r,\varphi) = T(r, f^{(k)}) + O(1)$ as $r \to +\infty$ and this implies that

$$\overline{\lim_{r\to\infty, r\notin M(K)}} \frac{T(r,f)}{T(r,f^{(k)})} = \infty.$$

This contradicts Lemma (2.1).

If $n \geq k+3$, then $T(r, \varphi f^{(k)}) = (n-1)T(r,f)$ as $r \to +\infty$ and

$$(n-1)T(r,f) \leq T(r,f^{(k)}) + S(r,f) \tag{2.2}$$

as $r \to +\infty$. On the other hand,

$$T(r, f^{(k)}) \leq (k+1)T(r,f) + S(r,f) \tag{2.3}$$

as $r \to +\infty$. By (2.2) and (2.3), we have $n \leq k+2$, a contradiction.

Hence, we have $\varphi[f]^{n-1}f^{(k)} \not\equiv constant$ in both cases and the lemma is proven. □

Lemma 2.3. *If $f(z)$ is entire, then $\varphi(z)[f(z)]^{n-1}f^{(k)}(z) \not\equiv constant$ for all positive integers $n (\neq 2)$ and k.*

Proof. For the case $n = 1$, we still have $T(r,\varphi) = T(r, f^{(k)}) + O(1)$ as $r \to +\infty$, so a contradiction to Lemma (2.1) again.

For $n \geq 3$, instead of (2.3), we have

$$T(r, f^{(k)}) \leq T(r,f) + S(r,f) \tag{2.4}$$

as $r \to +\infty$.

So by (2.2) and (2.4), we have $n \leq 2$, a contradiction. □

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3. Proof of the Main Result

Proof. First of all, by the given conditions and Lemma 2.2, we know that $\varphi[f]^{n-1}f^{(k)} \not\equiv constant$ for $n \geq 1$. Therefore, we have

$$m\left(r, \frac{1}{\varphi[f]^n}\right) \leq m\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1). \tag{3.1}$$

From

$$m\left(r, \frac{1}{\varphi[f]^n}\right) = T(r, \varphi[f]^n) - N\left(r, \frac{1}{\varphi[f]^n}\right) + O(1),$$
$$m\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) = T(r, \varphi[f]^{n-1}f^{(k)}) - N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + O(1),$$

and (3.1), we have

$$T(r, \varphi[f]^n) \leq N\left(r, \frac{1}{\varphi[f]^n}\right) + T(r, \varphi[f]^{n-1}f^{(k)}) - N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1). \tag{3.2}$$

Since $\varphi(z)[f(z)]^{n-1}f^{(k)} \not\equiv constant$, from the second fundamental theorem,

$$T(r, \varphi[f]^{n-1}f^{(k)}) < N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)}}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - b}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - c}\right) - N_1(r) + S(r, \varphi f^{(k)}) \tag{3.3}$$

as $r \to +\infty$, where b and c are two non-zero distinct complex numbers and, as usual, $N_1(r)$ is defined as

$$N_1(r) = 2N(r, \varphi[f]^{n-1}f^{(k)}) - N(r, (\varphi[f]^{n-1}f^{(k)})') + N\left(r, \frac{1}{(\varphi[f]^{n-1}f^{(k)})'}\right).$$

Let z_0 be a pole of order $p \geq 1$ of f. Then $[f]^{n-1}f^{(k)}$ and $([f]^{n-1}f^{(k)})'$ have a pole of order $k + np$ and $k + np + 1$ at z_0 respectively. Thus $2(k + np) - (k + np + 1) = k + np - 1 \geq p$ and

$$N_1(r) \geq N(r, f) + N\left(r, \frac{1}{(\varphi[f]^{n-1}f^{(k)})'}\right) + S(r, f). \tag{3.4}$$

It is clear that $S(r, f^{(k)}) = S(r, f)$ and $m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. Thus by (3.2), (3.3) and (3.4),

$$\begin{aligned} &T(r, \varphi[f]^n) \\ &< N\left(r, \frac{1}{\varphi[f]^n}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - b}\right) + N\left(r, \frac{1}{\varphi[f]^{n-1}f^{(k)} - c}\right) \\ &\qquad - N(r, f) - N\left(r, \frac{1}{(\varphi[f]^{n-1}f^{(k)})'}\right) + S(r, f) \end{aligned}$$

as $r \to +\infty$. Since $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$, we have the desired result. □

If f is entire, then by Lemma (2.3), we still have $\varphi[f]^{n-1}f^{(k)} \not\equiv constant$ for all positive integers $n (\neq 2)$, (3.3) and (3.4). Thus the same argument can be applied and the same result is obtained.

4. Concluding Remarks and a Conjecture

Remark 4.1. *We expect that our theorem is also valid for the case* $n = 2$ *if* $f(z)$ *is entire.*

Remark 4.2. *In [10], Zhang studied the value distribution of* $\varphi(z)f(z)f'(z)$ *and he obtained the following result: If* $f(z)$ *is a non-constant meromorphic function and* $\varphi(z)$ *is a non-zero meromorphic function such that* $T(r,\varphi) = S(r,f)$ *as* $r \to +\infty$*, then*

$$T(r,f) < \frac{9}{2}\overline{N}(r,f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi f f' - 1}\right) + S(r,f)$$

as $r \to +\infty$*.*

Hence, by this remark, we expect the following conjecture would be true.

Conjecture 4.1. *Let* n *and* k *be positive integers. If* $n = 1$ *or* $n \geq k+3$*,* $f(z)$ *is a non-constant meromorphic function and* $\varphi(z)$ *is a non-zero meromorphic function such that* $T(r,\varphi) = S(r,f)$ *as* $r \to +\infty$*, then*

$$T(r,f) < \frac{9}{2}\overline{N}(r,f) + \frac{9}{2}\overline{N}\left(r, \frac{1}{\varphi [f]^{n-1} f^{(k)} - 1}\right) + S(r,f)$$

as $r \to +\infty$*.*

References

[1] W.K. HAYMAN, *Meromorphic Functions*, Oxford, Clarendon Press, 1964.

[2] W.K. HAYMAN, Picard values of meromorphic functions and their derivatives, *Ann. Math.*, **70** (1959), 9–42.

[3] W.K. HAYMAN AND J. MILES, On the growth of a meromorphic function and its derivatives, *Complex Variables*, **12** (1989), 245–260.

[4] H. MILLOUX, Extension d'un théorème de M. R. Nevanlinna et applications, *Act. Scient. et Ind.*, no.888, 1940.

[5] H. MILLOUX, *Les fonctions méromorphes et leurs dérivées*, Paris, 1940.

[6] K.L. HIONG, Sur la limitation de $T(r, f)$ sans intervention des pôles, *Bull. Sci. Math.*, **80** (1956), 175–190.

[7] L. YANG, *Value distribution theory and its new researches* (Chinese), Beijing, 1982.

[8] H.X. YI and C.C. YANG, *On the uniqueness theory of meromorphic functions* (Chinese), Science Press, China, 1996.

[9] K.W. YU, A note on the product of a meromorphic function and its derivative, to appear in *Kodai Math.* J.

[10] Q.D. ZHANG, On the value distribution of $\varphi(z)f(z)f'(z)$ (Chinese), *Acta Math. Sinica*, **37** (1994), 91–97.