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# CONVEX FUNCTIONS IN A HALF-PLANE

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## Abstract

The class of convex hydrodynamically normalized functions in a half-plane was introduced by J. Stankiewicz. In this paper we introduce the general class of convex functions in the upper half-plane $D$ (not necessarily hydrodynamically normalized) and we obtain necessary and sufficient conditions for an analytic function in $D$, to be convex univalent in $D$.



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# Contents






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# 1. Introduction

We denote by $D$ the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, by $\mathcal{H}$ the class of analytic functions in $D$, and by $\mathcal{H}_1$ the class of functions $f \in \mathcal{H}$ satisfying:

$$\lim_{D \ni z \to \infty} [f(z) - z] = 0. \tag{1.1}$$

The normalization (1.1) is known in the literature as hydrodynamic normalization, being related to fluid flows in Mechanics.

The notion of convexity for functions belonging to the class $\mathcal{H}_1$ was introduced by J. Stankiewicz and Z. Stankiewicz ([4], [5]) as follows:

**Definition 1.1.** *The function $f \in \mathcal{H}_1$ is said to be convex if $f$ is univalent in $D$ and $f(D)$ is a convex domain.*

We denote by $C_{\mathcal{H}_1}(D)$ the class of convex functions satisfying the hydrodynamic normalization (1.1).

J. Stankiewicz and Z. Stankiewicz obtained ([4], [5]) the following sufficient conditions for a function $f \in \mathcal{H}_1$ to be a convex function:

**Theorem 1.1.** *If the function $f \in \mathcal{H}_1$ satisfies:*

$$f'(z) \neq 0, \text{ for all } z \in D$$

*and*

$$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0, \text{ for all } z \in D, \tag{1.2}$$

*then $f$ is a convex function.*

The class of analytic univalent functions in a half-plane has been studied by F.G. Avhadiev [1] starting from the 1970's. He examined the class of convex and univalent functions in a half plane that are not hydrodynamically normalized, obtaining the following theorem:

**Theorem 1.2.** *([1]) The function $f : D \to \mathbb{C}$, analytic in $D$, is convex and univalent in $D$ if and only if $f'(i) \neq 0$ and for any $z \in D$ the following inequality holds:*

$$\operatorname{Im}\left(2z + \left(z^2 + 1\right)\frac{f''(z)}{f'(z)}\right) > 0.$$

Another result that characterizes the convexity property for univalent functions in the half plane that are not hydrodynamically normalized was obtained by the second author in [2].

After 1974, the year when Avhadiev's paper was published, the only classes of univalent functions in the half-plane that had been studied were the univalent functions hydrodynamically normalized. We make the remark that the analytic representation of a geometric property (in this case the convexity property) is not unique.

## 2. Main Results

The function $\varphi : U \to D$ given by

$$\varphi(u) = i\frac{1-u}{1+u}$$

is a conformal mapping of the unit disk $U$ onto the upper half-plane $D$.

For $0 < r < 1$, the image of the disk $U_r = \{z \in \mathbb{C} : |z| < r\}$ under $\varphi$ is the disk $D_r = \{z \in \mathbb{C} : |z - z_r| < R_r\}$, where:

$$\begin{cases} z_r = i\dfrac{1+r^2}{1-r^2}; \\ R_r = \dfrac{2r}{1-r^2} \end{cases} . \tag{2.1}$$

To see this, note that in polar coordinates $u = re^{it}$, using the identity:

$$\left|1 + re^{-it}\right| = \left|1 + re^{it}\right|$$

we obtain:

$$\left| i\frac{1-re^{it}}{1+re^{it}} - i\frac{1+r^2}{1-r^2} \right| = \left| \frac{2r\left(1+re^{-it}\right)}{\left(1+re^{it}\right)\left(1-r^2\right)} \right| = \frac{2r}{1-r^2},$$

for any $r \in (0,1)$ and any $t \in [0, 2\pi)$, which shows that the image under $\varphi$ of the boundary of the disk $U_r$ is the boundary of the disk $D_r$. Since $\varphi(0) = i \in D_r$, it follows that $\varphi(U_r) = D_r$.

**Lemma 2.1.** *([3])The family of domains* $\{D_r\}_{r\in(0,1)}$ *has the following properties:*

i) *for any positive real numbers* $0 < r < s < 1$ *we have* $D_r \subset D_s$*;*

ii) *for any complex number* $z \in D$ *there exists* $r_z \in (0,1)$ *such that* $z \in D_r$*, for any* $r \in (r_z, 1)$*;*

iii) *for any* $z \in D$ *and* $r \in (r_z, 1)$ *arbitrarily fixed, there exists* $u_r \in U$ *such that*

$$z = z_r + R_r u_r.$$

*Moreover, we have the following equalities:*

$$\begin{cases} \lim\limits_{r\to 1} u_r = -i \\ \lim\limits_{r\to 1} R_r \left(1 - |u_r|\right) = \operatorname{Im} z \end{cases}.$$

*Proof.*

i) For any $0 < r < s < 1$ we have:

$$D_r = \varphi\left(U_r\right) \subset \varphi\left(U_s\right) = D_s.$$

ii) For $z \in D$ we have $\varphi^{-1}(z) \in U$, hence considering $r_z = \left|\varphi^{-1}\left(z\right)\right|$ we have $r_z \in (0,1)$, and for any $r \in (r_z, 1)$ we obtain $z \in \varphi\left(U_r\right) = D_r$.

iii) If $z = X + iY$ is an arbitrarily fixed point in $D_r$, $r \in (r_z, 1)$, then the complex number $u_r = x_r + iy_r$ given by:

$$u_r = \frac{z - z_r}{R_r}$$

has the property that $|u_r| < 1$. Using the relations (2.1) we get:

$$X + iY = \frac{2r}{1-r^2} x_r + i\left(\frac{1+r^2}{1-r^2} + \frac{2r}{1-r^2} y_r\right)$$

and therefore:

$$\begin{cases} x_r = \dfrac{1-r^2}{2r} X, \\ y_r = \dfrac{(1-r^2)\,Y - (1+r^2)}{2r}, \end{cases}$$

hence it follows:

$$\begin{aligned} \lim_{r\to 1} u_r &= \lim_{r\to 1} \frac{(1-r^2)}{2r} X + i \frac{(1-r^2)\,Y - (1+r^2)}{2r} \\ &= -i, \end{aligned}$$

and

$$\begin{aligned} &\lim_{r\to 1} R_r \left(1 - |u_r|^2\right) \\ &\quad = \lim_{r\to 1} \frac{2r}{1-r^2} \cdot \frac{4r^2 - |z|^2 \left(1-r^2\right)^2 + 2(1-r^4)Y - \left(1+r^2\right)^2}{4r^2} \\ &\quad = \lim_{r\to 1} - \frac{|z|^2 \left(1-r^2\right)}{2r} + Y\left(1+r^2\right) - \frac{1-r^2}{2r} \\ &\quad = 2Y = 2\operatorname{Im} z. \end{aligned}$$

As $\lim_{r\to 1} 1 + |u_r| = 2$, follows from the previous inequality, the final result follows from the second part of iii), completing the proof. □

The next theorem is obtained as a consequence of "the second coefficient inequality" for univalent functions in the unit disk, due to Bieberbach:

**Theorem 2.2.** *If $g : U \to \mathbb{C}$ is analytic and univalent in $U$, then for any $z \in U$ the following inequality holds:*

$$\left| -2|z|^2 + \left(1 - |z|^2\right) \frac{zg''(z)}{g'(z)} \right| \leq 4|z|.$$

Using Lemma 2.1 we obtain the following result, which corresponds to the previous theorem in the case of univalent functions in the half-plane:

**Theorem 2.3.** *([3]) If the function $f : D \to \mathbb{C}$ is analytic and univalent in the half-plane $D$, then for any $z \in D$ we have the inequality*

$$\left| i - \operatorname{Im}(z) \frac{f''(z)}{f'(z)} \right| \leq 2. \tag{2.2}$$

The equality is satisfied for the function given by

$$f(z) = z^2$$

at the point $z = i$.

We make the observation that a simple function such as $f : D \to \mathbb{C}$ defined by

$$f(z) = \sqrt{z},$$

(where we consider a fixed branch of the logarithm for the square root) is univalent in the domain $D$, $f(D)$ is a convex domain, yet the function $f$ is not considered to be convex in the sense of Definition 1.1 since it does not belong to the class $\mathcal{H}_1$ ($f$ does not satisfy the hydrodynamic normalization (1.1)).

This observation suggested the idea that it is necessary to give up the hydrodynamic normalization condition, a much too restrictive normalization. In this sense we propose a new definition of convexity for analytic functions in $D$, to include a larger class of analytic functions in $D$, not necessarily hydrodynamically normalized:

**Definition 2.1.** *A function $f \in \mathcal{H}$ is said to be convex in $D$ if $f$ is univalent in $D$ and $f(D)$ is a convex domain.*

We will denote by $C(D)$ the class of convex functions (in the sense of Definition 2.1). The next theorem gives necessary and sufficient conditions for a function $f \in \mathcal{H}$ to belong to the class $C(D)$:

**Theorem 2.4.** *For an analytic function $f : D \to \mathbb{C}$, the following are equivalent:*

*i)* $f \in C(D)$;

*ii)* $f'(iy) \neq 0$ *for any* $y > 1$, *and for any* $r \in (0,1)$ *and* $z \in D_r$ *the following inequality holds:*

$$\operatorname{Re} \frac{(z - z_r) f''(z)}{f'(z)} + 1 > 0, \tag{2.3}$$

*where $D_r$ is the disk $\{z \in \mathbb{C} : |z - z_r| < R_r\}$ and*

$$\begin{cases} z_r = i\dfrac{1+r^2}{1-r^2}, \\ \\ R_r = \dfrac{2r}{1-r^2}. \end{cases} \tag{2.4}$$

*Proof.* Given the function $f \in C(D)$, denote by $\Delta$ the convex domain $f(D)$. The function $\varphi : U \to D$ given by

$$\varphi(u) = i\frac{1-u}{1+u}$$

represents conformally the disk $U$ to the half-plane $D$, and for any $r \in (0,1)$ we have $\varphi(U_r) = D_r$.

The function $f \circ \varphi : U \to \mathbb{C}$ represents conformally the unit disk $U$ onto $\Delta = f(D)$. Since the domain $\Delta$ is convex, it follows that the function $f \circ \varphi$ is convex and univalent in the unit disk $U$, and hence represents conformally any disk $U_r$ $(0 < r < 1)$, onto a convex domain. Since $\varphi(U_r) = D_r$, it follows that for any $r \in (0,1)$ the domain $\Delta_r = f(D_r)$ is convex. For $r \in (0,1)$ arbitrarily fixed, the function $g_r : U \to \mathbb{C}$ given by

$$g_r(u) = f(z_r + R_r u), \tag{2.5}$$

where $z_r, R_r$ are given by (2.4), represents conformally the disk $U$ onto the convex domain $\Delta_r$. Using the results for convex and univalent functions in the unit disk, it follows that the domain $\Delta_r$ is convex if and only if

$$g_r'(0) = R_r f'(z_r) \neq 0 \tag{2.6}$$

and for any $u \in U$ the following inequality holds:

$$\text{Re}\,\frac{zg_r''(u)}{g_r'(u)} + 1 = \text{Re}\,\frac{zR_r f''(z_r + R_r u)}{f'(z_r + R_r u)} + 1 > 0. \tag{2.7}$$

Denoting $z = z_r + R_r u$, and observing that $u \in U$ if and only if $z \in D_r$, the previous inequality can be written as

$$\text{Re}\,\frac{(z - z_r) f''(z)}{f'(z)} + 1 > 0, \tag{2.8}$$

for any $z \in D_r$, proving the necessity for condition (2.3).

Since $z_r = i\frac{1+r^2}{1-r^2}$, for $r \in (0,1)$ we have:

$$|z_r| = \frac{1+r^2}{1-r^2} > 1$$

for any $r \in (0,1)$ and thus the condition (2.6) is equivalent to $f'(iy) \neq 0$ for any $y > 1$.

Conversely, if ii) holds, then for any arbitrarily fixed $r \in (0,1)$ the function $g_r(u) = f(z_r + R_r u)$ is convex and univalent in the disk $U$. It follows that for any $r \in (0,1)$ the domain $\Delta_r = g_r(U)$ is convex, and since $\Delta_r = f(D_r)$, it follows that the function $f$ is convex and univalent in the domain $D_r$, for any $r \in (0,1)$. Since $\bigcup_{r\in(0,1)} D_r = D$, it follows that the function $f$ is convex and univalent in the half-plane $D$, completing the proof. □

In the previous proof we obtained the following result:

**Corollary 2.5.** *If the function $f : D \to \mathbb{C}$ is convex and univalent in $D$, then $\Delta_r = f(D_r)$ is a convex domain for any $r \in (0, 1)$.*

**Remark 2.1.** *In [2] the second author introduced the subclass $C_1(D)$ of the class of convex univalent functions as follows:*

**Definition 2.2.** *([2]) We say that the analytic function $f : D \to \mathbb{C}$ belongs to the class $C_1(D)$ if for any $z \in D$ we have:*

$$f'(z) \neq 0 \tag{2.9}$$

*and*

$$\begin{cases} \operatorname{Re} \dfrac{zf''(z)}{f'(z)} + 1 > 0, \\ \\ \operatorname{Im} \dfrac{f''(z)}{f'(z)} > 0. \end{cases} \tag{2.10}$$

It is known that if the function $g : U \to \mathbb{C}$ is convex and univalent in the unit disk $U$, with the Taylor series expansion:

$$g(z) = z + a_2 z^2 + \cdots ,$$

then $|a_2| \leq 1$. The class of convex and univalent functions in the unit disk, normalized by $f(0) = f'(0) - 1 = 0$ is denoted by $C$.

The above property for functions belonging to the class $C$ has the following important consequence:

**Theorem 2.6.** *If the function $g : U \to \mathbb{C}$ belongs to the class $C$, then for any $z \in U$ the following inequality holds:*

$$\left| -2|z|^2 + \left(1 - |z|^2\right) \frac{zg''(z)}{g'(z)} \right| \le 2|z|.$$

Using this result we obtain a differential characterization of the class $C(D)$ of convex and univalent functions in the half-plane $D$:

**Theorem 2.7.** *If the function $f : D \to \mathbb{C}$ belongs to the class $C(D)$, then for any $z \in D$ we have the inequality:*

$$\left| i - \operatorname{Im}(z) \frac{f''(z)}{f'(z)} \right| \le 1. \tag{2.11}$$

*Proof.* If the function $f$ belongs to $C(D)$, by Corollary 2.5 it follows that $\Delta_r = f(D_r)$ is a convex domain for any $r \in (0,1)$.

The function $g_r$ given by formula (2.5) represents conformally the unit disk $U$ onto $g_r(U) = \Delta_r$, and since $\Delta_r$ is a convex domain, it follows that the function $g_r$ is convex and univalent. The function

$$\frac{g_r(u) - g_r(0)}{g_r'(0)} = \frac{f(z_r + R_r u) - f(z_r)}{R_r f'(z_r)}$$

is therefore convex and univalent in $u \in U$, normalized by $g_r(0) = g_r'(0) - 1 = 0$ for any $r \in (0,1)$. By Theorem 2.6 it follows that for any $r \in (0,1)$ and any $u \in U$ the following inequality holds:

$$\left| -2|u|^2 + \left(1 - |u|^2\right) \frac{uR_r f''(z_r + R_r u)}{f'(z_r + R_r u)} \right| \le 2|u|. \tag{2.12}$$

Given $z \in D$, by Lemma 2.1 there exists $r_z \in (0,1)$ such that for any fixed $r \in (r_z, 1)$, there is $u_r \in U$ such that $z = z_r + R_r u_r \in D_r$ and

$$\begin{cases} \lim_{r\to 1} u_r = -i, \\ \lim_{r\to 1} (1 - |u_r|) R_r = \operatorname{Im} z. \end{cases}$$

Considering $u = u_r$ in the inequality (2.12) and passing to the limit with $r \to 1$, we obtain:

$$\left| -2 + 2\operatorname{Im}(z) \frac{-if''(z)}{f'(z)} \right| \le 2.$$

Since $z \in D$ was arbitrarily chosen, we have shown that for any $z \in D$ the following inequality holds:

$$\left| i - \operatorname{Im}(z) \frac{f''(z)}{f'(z)} \right| \le 1,$$

and the theorem is proved. □

The next result is an important consequence of Theorem 2.7:

**Corollary 2.8.** *If the function $f : D \to \mathbb{C}$ is convex and univalent in the half-plane $D$, then for any $z \in D$ we have the inequality:*

(2.13) $$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0.$$

*Proof.* If the function $f$ is convex and univalent in the half-plane $D$, by the inequality (2.11) given by Theorem 2.7, it follows that for any $z \in D$, the point $w = \operatorname{Im}(z) \frac{f''(z)}{f'(z)}$ belongs to the disk centered at $i$ with radius 1. Since this disk belongs to the upper half-plane, it follows that for any $z \in D$ the inequality (2.13) holds. □

**Remark 2.2.** *The result in the previous corollary was obtained, using different methods, by F.G. Avhadiev [1].*

**Example 2.1.** *The function $f : D \to \mathbb{C}$ given by*

$$f(z) = z^a,$$

*is convex and univalent for any $a \in [-1, 0) \cup (0, 1]$, since the function $f$ is analytic and univalent in $D$, and the domains: $f(D) = \{z \in \mathbb{C} : \arg(z) \in (0, a\pi)\}$, for $a \in (0, 1)$, and $f(D) = \{z \in \mathbb{C} : \arg(z) \in (a\pi, 0)\}$, for $a \in (-1, 0)$, are convex.*

The following inequalities hold:

$$\begin{aligned} \operatorname{Re} \frac{(z - z_r) f''(z)}{f'(z)} + 1 &= (a-1) \operatorname{Re} \frac{z - z_r}{z} + 1 \\ &= \frac{a |z^2| - |z_r| (a-1) \operatorname{Im} z}{|z|^2} \\ &= \frac{a}{|z|^2} \left[ (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 - \frac{a-1}{a} |z_r| \operatorname{Im} z \right]. \end{aligned}$$

Let us observe that if $a \in [-1, 0)$, then for any $r \in (0, 1)$, we have the following inequality:

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 - \frac{a-1}{a} |z_r| \operatorname{Im} z < 0,$$

for any $z$ in the disk centered at $\frac{i(a-1)|z_r|}{2a}$ with radius $\frac{(a-1)|z_r|}{2a}$. Since

$$\frac{a-1}{2a} |z_r| \geq |z_r| ,$$

this disk is contained in the disk $D_r$, and hence by Theorem 2.4 it follows that for $a \in [-1, 0)$ we have $f \in C(D)$.

For $a \in (0, 1]$ we have:

$$\operatorname{Re} \frac{(z - z_r) f''(z)}{f'(z)} + 1 = a - (a - 1) |z_r| \frac{\operatorname{Im} z}{|z^2|} > 0$$

for any $r \in (0, 1)$ and for any $z \in D$, and therefore by Theorem 2.4 the function $f$ belongs to the class $C(D)$ for $a \in (0, 1]$ as well.

Applying Theorem 1.2 to the same function $f$, we obtain:

$$\begin{aligned} \operatorname{Im} \left[ 2z + \frac{(z^2 + 1) f''(z)}{f'(z)} \right] &= 2y + \operatorname{Im} \frac{(z^2 + 1)(a - 1)}{z} \\ &= |z|^{-2} y \left[ (a + 1) |z|^2 - (a - 1) \right] > 0 \end{aligned}$$

for any $z \in D$, if and only if $a \in [-1, 1]$. The condition $f'(i)$ is satisfied for $a \neq 0$, hence it follows that $f \in C(D)$ for any $a \in [-1, 0) \cup (0, 1]$.

Trying to apply the result due to J. Stankiewicz, we can see that $f \notin C_{\mathcal{H}_1}(D)$ for any value of $a \in [-1,0) \cup (0,1]$ since the considered function $f$ satisfies the hydrodynamic normalization just for $a = 1$, but in this case

$$\operatorname{Im} \frac{f''(z)}{f'(z)} = 0,$$

and the condition obtained by J. Stankiewicz is not satisfied. We therefore have the inclusion $C_{\mathcal{H}_1}(D) \subsetneq C(D)$.



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