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ON SOME SUBCLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and w is a fixed point in $\mathbb{U}$. The aim of this paper is to continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in $\mathbb{U}$.

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1. INTRODUCTION

Let $\mathcal{H}(\mathbb{U})$ be the set of functions which are regular in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$,

$$A = \{f \in \mathcal{H}(\mathbb{U}) : f(0) = f'(0) - 1 = 0\} \quad \text{and} \quad S = \{f \in A : \ f \text{ is univalent in } \mathbb{U}\}.$$

We recall here the definitions of the well-known classes of starlike, convex, close-to-convex and α-convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{U} \right\},$$

$$S^c = \left\{ f \in A : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{U} \right\},$$

$$CC = \left\{ f \in A : \exists g \in S^*, \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{U} \right\},$$

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$$M_\alpha = \left\{ f \in A : \frac{f(z)f'(z)}{z} \neq 0,\ \operatorname{Re} J(\alpha, f : z) > 0,\ z \in \mathbb{U} \right\},$$

where

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Let w be a fixed point in $\mathbb{U}$ and $A(w) = \{f \in \mathcal{H}(\mathbb{U}) : f(w) = f'(w) - 1 = 0\}$.

In [3], S. Kanas and F. Ronning introduced the following classes:

$$S(w) = \{f \in A(w) : f \text{ is univalent in } \mathbb{U}\}$$

$$ST(w) = S^*(w) = \left\{ f \in S(w) : \operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0,\ z \in \mathbb{U} \right\}$$

$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re}\left(\frac{(z-w)f''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{U} \right\}.$$

The class $S^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $S^c(w)$ is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones for uniformly starlike and convex functions introduced by A. W. Goodman in [1] and [2], except that in this case the point w is fixed.

It is obvious that there exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \text{ if and only if } f(z) = (z-w)g'(z) \in S^*(w).$$

Let $\mathcal{P}(w)$ denote the class of all functions

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n (z-w)^n$$

that are regular in U and satisfy $p(w) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{U}$.

The purpose of this note is to define the classes of close to convex and α-convex functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in $\mathbb{U}$, and to obtain some results concerning these classes.

2. PRELIMINARY RESULTS

It is easy to see that a function $f \in A(w)$ has the series expansion:

$$f(z) = (z-w) + a_2(z-w)^2 + \cdots.$$

In [7], J.K. Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$ as follows.

Theorem 2.1. *If $p \in \mathcal{P}(w)$,*

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n (z-w)^n,$$

then

$$|B_n| \leq \frac{2}{(1+d)(1-d)^n}, \tag{2.1}$$

where $d = |w|$ and $n \geq 1$.

Using the above result, S. Kanas and F. Ronning [3] obtain the following:

Theorem 2.2. *Let $f \in S^*(w)$ and $f(z) = (z-w) + a_2(z-w)^2 + \cdots$. Then*

$$|a_2| \leq \frac{2}{1-d^2}, \quad |a_3| \leq \frac{3+d}{(1-d^2)^2}, \tag{2.2}$$

$$|a_4| \leq \frac{2}{3}\frac{(2+d)(3+d)}{(1-d^2)^3}, \quad |a_5| \leq \frac{1}{6}\frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4} \tag{2.3}$$

where $d = |w|$.

Remark 2.3. It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.

The next theorem is the result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [4], [5], [6]).

Theorem 2.4. *Let h be convex in $\mathbb{U}$ and $\mathrm{Re}[\beta h(z)+\gamma] > 0$, $z \in \mathbb{U}$. If $p \in \mathcal{H}(\mathbb{U})$ with $p(0) = h(0)$ and p satisfies the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z), \quad z \in \mathbb{U},$$

then $p(z) \prec h(z),\ z \in \mathbb{U}$.

3. MAIN RESULTS

Let us consider the integral operator $L_a : A(w) \to A(w)$ defined by

$$f(z) = L_a F(z) = \frac{1+a}{(z-w)^a}\int_w^z F(t)(t-w)^{a-1}dt, \quad a \in \mathbb{R},\ a \geq 0. \tag{3.1}$$

We denote by

$$D(w) = \left\{ z \in \mathbb{U} : \mathrm{Re}\left(\frac{w}{z}\right) < 1 \quad \text{and} \quad \mathrm{Re}\left(\frac{z(1+z)}{(z-w)(1-z)}\right) > 0 \right\},$$

with $D(0) = \mathbb{U}$, and

$$s(w) = \{f : D(w) \to \mathbb{C}\} \cap S(w),$$

where w is a fixed point in $\mathbb{U}$. Denoting $s^*(w) = S^*(w) \cap s(w)$, where w is a fixed point in $\mathbb{U}$, we obtain

Theorem 3.1. *Let w be a fixed point in $\mathbb{U}$ and $F(z) \in s^*(w)$. Then $f(z) = L_a F(z) \in S^*(w)$, where the integral operator L_a is defined by (3.1).*

Proof. By differentiating (3.1), we obtain

$$(1+a)F(z) = af(z) + (z-w)f'(z). \tag{3.2}$$

From (3.2), we also have

$$(1+a)F'(z) = (1+a)f'(z) + (z-w)f''(z). \tag{3.3}$$

Using (3.2) and (3.3), we obtain

$$\frac{(z-w)F'(z)}{F(z)} = \frac{(1+a)(z-w)\frac{f'(z)}{f(z)} + (z-w)^2\frac{f''(z)}{f(z)}}{a + (z-w)\frac{f'(z)}{f(z)}}. \tag{3.4}$$

Letting

$$p(z) = \frac{(z-w)f'(z)}{f(z)},$$

where $p \in \mathcal{H}(\mathbb{U})$ and $p(0) = 1$, we have

$$(z-w)p'(z) = p(z) + (z-w)^2 \cdot \frac{f''(z)}{f(z)} - [p(z)]^2$$

and thus

$$(z-w)^2 \frac{f''(z)}{f(z)} = (z-w)p'(z) - p(z)[1-p(z)]. \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\frac{(z-w)F'(z)}{F(z)} = p(z) + \frac{(z-w)p'(z)}{a+p(z)}. \tag{3.6}$$

Since $F \in s^*(w)$, from (3.6), we have

$$p(z) + \frac{z-w}{a+p(z)} p'(z) \prec \frac{1+z}{1-z} \equiv h(z)$$

or

$$p(z) + \frac{1-\frac{w}{z}}{a+p(z)} zp'(z) \prec \frac{1+z}{1-z}.$$

From the hypothesis, we have

$$\mathrm{Re}\left(\frac{1}{1-\frac{w}{z}} h(z) + \frac{a}{1-\frac{w}{z}}\right) > 0$$

and thus from Theorem 2.4, we obtain

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}$$

or

$$\mathrm{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}.$$

This means that $f \in S^*(w)$. □

Definition 3.1. Let $f \in S(w)$ where w is a fixed point in $\mathbb{U}$. We say that f is w-close-to-convex if there exists a function $g \in S^*(w)$ such that

$$\mathrm{Re}\left(\frac{(z-w)f'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{U}.$$

We denote this class by $CC(w)$.

Remark 3.2. If we consider $f = g$, $g \in S^*(w)$, then we have $S^*(w) \subset CC(w)$. If we take $w = 0$, then we obtain the well-known close-to-convex functions.

Theorem 3.3. *Let w be a fixed point in $\mathbb{U}$ and $f \in CC(w)$, where*

$$f(z) = (z-w) + \sum_{n=2}^{\infty} b_n (z-w)^n,$$

with respect to the function $g \in S^*(w)$, where

$$g(z) = (z-w) + \sum_{n=2}^{\infty} a_n (z-w)^n.$$

Then

$$|b_n| \le \frac{1}{n}\left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot \frac{2}{(1+d)(1-d)^{n-k}}\right],$$

where $d = |w|$, $n \geq 2$ *and* $a_1 = 1$.

Proof. Let $f \in CC(w)$ with respect to the function $g \in S^*(w)$. Then there exists a function $p \in \mathcal{P}(w)$ such that

$$\frac{(z-w)f'(z)}{g(z)} = p(z),$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n (z-w)^n.$$

Using the hypothesis through identification of $(z-w)^n$ coefficients, we obtain

$$nb_n = a_n + \sum_{k=1}^{n-1} a_k B_{n-k}, \tag{3.7}$$

where $a_1 = 1$ and $n \geq 2$. From (3.7), we have

$$|b_n| \leq \frac{1}{n} \left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot |B_{n-k}| \right], \quad a_1 = 1,\ n \geq 2.$$

Applying the above and the estimates (2.1), we obtain the result. □

Remark 3.4. If we use the estimates (2.2), we obtain the same estimates for the coefficients b_n, $n = 2, 3, 4, 5$.

Definition 3.2. Let $\alpha \in \mathbb{R}$ and w be a fixed point in $\mathbb{U}$. For $f \in S(w)$, we define

$$J(\alpha, f, w; z) = (1-\alpha)\frac{(z-w)f'(z)}{f(z)} + \alpha \left[1 + \frac{(z-w)f''(z)}{f'(z)} \right].$$

We say that f is $w-\alpha-$convex function if

$$\frac{f(z)f'(z)}{z-w} \neq 0, \quad z \in \mathbb{U}$$

and $\operatorname{Re}\ J(\alpha, f, w; z) > 0,\ z \in \mathbb{U}$. We denote this class by $M_\alpha(w)$.

Remark 3.5. It is easy to observe that $M_\alpha(0)$ is the well-known class of α-convex functions.

Theorem 3.6. *Let* w *be a fixed point in* $\mathbb{U}$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$ *and* $m_\alpha(w) = M_\alpha(w) \cap s(w)$. *Then we have*

(1) *If* $f \in m_\alpha(w)$ *then* $f \in S^*(w)$. *This means* $m_\alpha(w) \subset S^*(w)$.
(2) *If* $\alpha, \beta \in \mathbb{R}$, *with* $0 \leq \beta/\alpha < 1$, *then* $m_\alpha(w) \subset m_\beta(w)$.

Proof. From $f \in m_\alpha(w)$, we have $\operatorname{Re} J(\alpha, f, w; z) > 0$, $z \in \mathbb{U}$. Putting

$$p(z) = \frac{(z-w)f'(z)}{f(z)},$$

with $p \in \mathcal{H}(\mathbb{U})$ and $p(0) = 1$, we obtain

$$\operatorname{Re}\ J(\alpha, f, w; z) = \operatorname{Re}\left[p(z) + \alpha \frac{(z-w)p'(z)}{p(z)} \right] > 0, \quad z \in \mathbb{U}$$

or

$$p(z) + \frac{\alpha\left(1 - \frac{w}{z}\right)}{p(z)} zp'(z) \prec \frac{1+z}{1-z} \equiv h(z).$$

In particular, for $\alpha = 0$, we have

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$

Using the hypothesis, we have for $\alpha > 0$,

$$\operatorname{Re}\left(\frac{1}{\alpha\left(1-\frac{w}{z}\right)}h(z)\right) > 0, \quad z \in \mathbb{U}$$

and from Theorem 2.4, we obtain

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$

This means that

$$\operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}$$

for $\alpha \geq 0$ or $f \in S^*(w)$.

If we denote by $A = \operatorname{Re} p(z)$ and $B = \operatorname{Re}\left((z-w)p'(z)/p(z)\right)$, then we have $A > 0$ and $A + B\alpha > 0$, where $\alpha \geq 0$. Using the geometric interpretation of the equation $y(x) = A + Bx$, $x \in [0, \alpha]$, we obtain

$$y(\beta) = A + B\beta > 0 \quad \text{for every} \;\; \beta \in [0, \alpha].$$

This means that

$$\operatorname{Re}\left(p(z) + \beta\frac{(z-w)p'(z)}{p(z)}\right) > 0, \quad z \in \mathbb{U}$$

or $f \in m_\beta(w)$. □

Remark 3.7. From Theorem 3.6, we have

$$m_1(w) \subseteq s^c(w) \subseteq m_\alpha(w) \subseteq s^*(w),$$

where $0 \leq \alpha \leq 1$ and $s^c(w) = S^c(w) \cap s(w)$.

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