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UPPER ESTIMATES FOR GÂTEAUX DIFFERENTIABILITY OF BUMP FUNCTIONS IN ORLICZ-LORENTZ SPACES

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ABSTRACT. Upper estimates for the order of Gâteaux smoothness of bump functions in Orlicz–Lorentz spaces $d(w, M, \mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, are obtained. The best possible order of Gâteaux differentiability in the class of all equivalent norms in $d(w, M, \mathbf{\Gamma})$ is found.

Key words and phrases: Orlicz function, Orlicz–Lorentz space, Smooth bump functions.

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1. INTRODUCTION

The existence of higher order Fréchet smooth norms and bump functions and its impact on the geometrical properties of a Banach space have been subject to many investigations beginning with the classical result for L_p–spaces in [1] and [6]. An extensive study and bibliography may be found in [2]. As any negative result on the existence of Gâteaux smooth bump functions immediately applies to the problem of existence of Fréchet smooth bump functions and norms, the question arises of estimating the best possible order of Gâteaux smoothness of bump functions in a given Banach space. A variational technique (the Ekeland variational principle) was applied in [2] to show that in $\ell_1(\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, there is no continuous Gâteaux differentiable bump function. Following the same idea and using Stegall's variational principle, an extension of this result to Banach spaces with uncountable unconditional basis was given in [4] and to Banach spaces with uncountable symmetric basis in [9]. As an application in [4] it was shown that in $\ell_p(\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, there is no continuous p–times Gâteaux differentiable bump function when p is odd and there is no continuous $([p]+1)$–times Gâteaux differentiable bump function in the case $p \notin \mathbb{N}$. This is essentially different from the case $\ell_p(\mathbb{N})$, p-odd, where equivalent p–times Gâteaux differentiable and even uniformly Gâteaux differentiable norms are constructed (see [10] and [8] respectively). As examples of the main result in [9], Orlicz $\ell_M(\mathbf{\Gamma})$

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and Lorentz $d(w,p,\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable are considered and estimates for the order of Gâteaux smoothness of bump functions are obtained. Recently a deep result on embedding of ℓ_p spaces in Orlicz–Lorentz sequence spaces $d_0(p,M)$ have been found in [5]. It is shown there that $\ell_p \hookrightarrow d_0(w,M)$ iff $\ell_p \hookrightarrow h_M$ iff $p \in [\alpha_M, \beta_M]$. From this result naturally arises the question of finding upper estimates for the order of Gâteaux smoothness of bump functions in Orlicz–Lorentz spaces.

It is worthwhile to mention that results about differentiability of bump functions in $\ell_p(\mathbf{\Gamma})$ cannot be used directly for $\ell_M(\mathbf{\Gamma})$ and $d(w,p,\mathbf{\Gamma})$. Indeed, in [3] it is proved that $\ell_p(\boldsymbol{A})$ is isomorphic to a subspace of $d(w,p,\mathbf{\Gamma})$ iff $\boldsymbol{A}$ is countable. On the other hand $\ell_M(\mathbf{\Gamma})$ for $M \equiv t^p(1+|\log t|)^q$ at zero, $p \geq 1$, $q \neq 0$, contains an isomorphic copy of $\ell_p(\boldsymbol{A})$ iff $\boldsymbol{A}$ is countable. The problem of embedding $\ell_p(\boldsymbol{A})$ or $\ell_M(\boldsymbol{A})$ into $d(w,M,\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable is open.

In this note we give one new application of the main result of [9] in Orlicz–Lorentz spaces $d(w,M,\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable for finding upper estimates for the order of Gâteaux smoothness of bump functions.

Let U be an open set in a Banach space X and let $f : U \to \mathbb{R}$ be continuous. Following [4] we shall say that f is $G^0_{\omega,k}$–smooth, $k \in \mathbb{N}$ in U for some $\omega : (0,1] \to \mathbb{R}^+$, $\lim_{t\to 0} t^{-k}\omega(t) = 0$ if for any $x \in U$, $y \in X$ the representation holds

$$f(x+ty) = f(x) + \sum_{i=1}^{k} \frac{t^i}{i!} f^{(i)}(x)(y^i) + R^k_f(x,y,t),$$

where $f^{(i)}$, $i = 1,2,\dots,k$ are i–linear bounded symmetric forms on X and $\lim\limits_{t\to 0} \frac{|R^k_f(x,y,t)|}{\omega(|t|)} = 0$.

If $U = X$ we use the notation $G^0_{\omega,[p]}$ instead of $G^0_{\omega,[p]}(X)$ and G_k, $k \in \mathbb{N}$ for the set of all continuous k–times Gâteaux differentiable functions on X, for which $\lim_{t\to 0} |R^k_f(x,y,t)|/|t|^k = 0$. We say that the norm $\|\cdot\|$ in X is k–times Gâteaux differentiable if it is from the class $G_k(X\backslash\{0\})$.

2. PRELIMINARIES

We use the standard Banach space terminology from [7]. Let us recall that an Orlicz function M is an even, continuous, non-decreasing convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is a non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$.

A weight sequence $w = \{w_n\}_{n=1}^{\infty}$ is a positive decreasing sequence such that $w_1 = 1$ and $\lim_{n\to\infty} W(n) = \infty$, where $W(n) = \sum_{j=1}^{n} w_j$, for any $n \in \mathbb{N}$.

The Orlicz–Lorentz space $d(w,M,\mathbf{\Gamma})$ is the space of all real functions $x = x(\alpha)$ defined on the set $\mathbf{\Gamma}$, for which

$$I(\lambda x) = \sup \left\{ \sum_{i=1}^{\infty} w_i M(\lambda x(\alpha_i)) \right\} < \infty$$

for some $\lambda > 0$, where the supremum is taken over all sequences $\{\alpha_i\}_{i=1}^{\infty}$ of different elements on $\mathbf{\Gamma}$. There exists a sequence $\{\alpha_i^*\}_{i=1}^{\infty}$, such that $|x(\alpha_1^*)| \geq |x(\alpha_2^*)| \geq \cdots \geq |x(\alpha_i^*)| \geq \cdots$, $\lim_{i\to\infty} x(\alpha_i^*) = 0$, $|x(\alpha^*)| = 0$ if $\alpha \neq \alpha_i^*$ for $i \in \mathbb{N}$ and $I(\lambda x) = \sum_{i=1}^{\infty} w_i M(\lambda x(\alpha_i^*))$. The space $d(w,M,\mathbf{\Gamma})$, equipped with the Luxemburg norm:

$$\|x\| = \inf \left\{ \lambda > 0 : I\left(\frac{x}{\lambda}\right) \leq 1 \right\}$$

is a Banach space.

By $\operatorname{supp} x$ we denote the set $\{\alpha \in \mathbf{\Gamma} : x(\alpha) \neq 0\}$.

The symbol e_γ, $\gamma \in \mathbf{\Gamma}$ will stand for the unit vectors.

If $M(u) = u^p$, $1 \le p < \infty$ then $d(w, M, \mathbf{\Gamma})$ is the Lorentz space $d(w, p, \mathbf{\Gamma})$. If $w_i = 1$ for every $i \in \mathbb{N}$ then $d(w, M, \mathbf{\Gamma})$ is the Orlicz space $\ell_M(\mathbf{\Gamma})$. In this case we use the notation $I(x) = \widetilde{M}(x)$

To every Orlicz function M the following numbers are associated:

$$\alpha_M = \sup\left\{p > 0 : \sup_{0<u,v\le 1} \frac{M(uv)}{u^p M(v)} < \infty\right\},$$

$$\beta_M = \inf\left\{q > 0 : \inf_{0<u,v\le 1} \frac{M(uv)}{u^q M(v)} > 0\right\}.$$

We consider only spaces generated by an Orlicz function M satisfying the Δ_2–condition at zero, i.e., $\beta_M < \infty$, which implies of course that

$$M(uv) \ge u^q M(v), \quad u, v \in [0,1] \tag{2.1}$$

for some $q > \beta_M$ (see [7]).

Finally we mention that the unit vectors $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ form a symmetric basis of $d(w, M, \mathbf{\Gamma})$ with symmetric constant 1, which is boundedly complete [5], [7].

For a function $g : (0,1] \to \mathbb{R}^+$ denote:

$$d_M(g) = \sup\left\{\frac{M(uv)}{g(u)M(v)} : u, v \in (0,1]\right\}.$$

Let us recall a well known definition. Let X have symmetric basis $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ with a symmetric constant 1 and let $z \in X$, $z \ne 0$, $z = \sum_{i=1}^{\infty} u_i e_{\gamma_i}$, $\gamma_i \ne \gamma_j$ for $i \ne j$. A sequence $\{z_k\}_{k=1}^{\infty}$, $z_k = \sum_{i=1}^{\infty} u_i e_{\alpha_{i,k}}$, $\alpha_{i,k} \ne \alpha_{j,l}$ for $(i,k) \ne (j,l)$, $\alpha_{i,k} \in \mathbf{\Gamma}$ is called a block basis generated by the vector z.

We will apply a general result for upper estimates for the order of Gâteaux smoothness of bump functions in a Banach space with a symmetric, boundedly complete basis with a symmetric constant 1, obtained in [9].

Theorem 2.1. [9] *Let X be a Banach space, let $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$, $\sharp\mathbf{\Gamma} > \aleph_0$ be a symmetric, boundedly complete basis in X with a symmetric constant 1 and let:*

$$\lim_{n\to\infty} \left\|\sum_{j=1}^{n} z_j\right\| n^{-\frac{1}{k}} = 0$$

for every $z \in X$.

Let $\omega : [0,1] \to \mathbb{R}^+$ be such that for every $x \in X$ there exist $y \in X$, $\operatorname{supp} y \cap \operatorname{supp} x = \emptyset$ and a sequence $t_n \searrow 0$, which satisfy the inequality

$$\|x + t_n y\| - \|x\| \ge \omega(t_n), \quad n \in \mathbb{N}.$$

Then in X there is no continuous:

(i) *$G^0_{\omega,k}$–smooth bump when $\omega(t) = o(t^k)$;*
(ii) *$G^0_{\omega,k+1}$–smooth bump when $\omega(t) = o(t^{k+1})$, k–even;*
(iii) *k–times Gâteaux differentiable bump if $\omega(t) = t^k$;*
(iv) *$(k+1)$–times Gâteaux differentiable bump if $\omega(t) = t^{k+1}$, k–even.*

3. MAIN RESULT

Theorem 3.1. *Let M be an Orlicz function. If f is a continuous k–times Gâteaux differentiable bump function in $d(w,M,\mathbf{\Gamma})$, then*

$$k \leq E_M = \begin{cases} [\alpha_M], & d_M(t^{[\alpha_M]}) < \infty \\ \alpha_M - 1, & \alpha_M \in \mathbb{N}, d_M(t^{\alpha_M}) = \infty. \end{cases}$$

4. AUXILIARY LEMMAS

To apply Theorem 2.1 for $d(w,M,\mathbf{\Gamma})$ we need the following lemmas.

Lemma 4.1. *Let $p \geq 1$ and let M be an Orlicz function satisfying the conditions $\lim_{t\to 0}\frac{M(t)}{t^p} = 0$, $d_M(t^p) = c < \infty$. Then every block basis $\{z_j\}_{j=1}^{\infty}$ of the unit vector basis $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ in $d(w,M,\mathbf{\Gamma})$, generated by one vector, satisfies*

$$\lim_{n\to\infty}\left\|\sum_{j=1}^{n} z_j\right\| n^{-\frac{1}{p}} = 0.$$

Proof. Let $z = \sum_{i=1}^{\infty} u_i e_{\gamma_i} \in d(w,M,\mathbf{\Gamma})$. Let $\{e_{j,i}\}_{i=1}^{\infty}$, $j \in \mathbb{N}$ be disjoint subsets of $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$. Then we define $z_j = \sum_{i=1}^{\infty} u_i e_{j,i}$. Let $\mu(t) = \frac{M(t)}{t^p}$. It follows that $\lim_{t\to 0}\mu(t) = 0$ and $\mu(t_1) \leq c\mu(t_2)$ for every $0 < t_1 < t_2 \leq 1$. Let $\lambda_n(z) = \sum_{j=1}^{n} z_j$. Then

$$I(\lambda_n(z)) = \sum_{i=1}^{\infty}\sum_{j=n(i-1)+1}^{ni} w_j M(u_i^*).$$

For every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\sum_{i=m+1}^{\infty}\sum_{j=n(i-1)+1}^{ni} w_j M(u_i^*) < \frac{\varepsilon}{2c}.$$

By the definition of the function μ it follows that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=n(i-1)+1}^{ni} w_j |u_i^*|^p \mu\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) &= \|\lambda_n(z)\|^p \sum_{i=1}^{\infty}\sum_{j=n(i-1)+1}^{ni} w_j M\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) \\ &= \|\lambda_n(z)\|^p.\end{aligned}$$

Using the inequality

$$1 = I\left(\frac{\lambda_n(z)}{\|\lambda_n(z)\|}\right) = \sum_{i=1}^{\infty}\sum_{j=n(i-1)+1}^{ni} w_j M\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) \geq \sum_{j=1}^{n} w_j M\left(\frac{u_1^*}{\|\lambda_n(z)\|}\right)$$

we get that $\lim_{n\to\infty}\|\lambda_n(z)\|^{-1} = 0$.

For every $m \in \mathbb{N}$ we have

$$\sum_{i=1}^{m}\sum_{j=n(i-1)+1}^{ni} \frac{w_j |u_i^*|^p}{n}\mu\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) \leq \frac{w_1 + w_2 \dots w_n}{n}\sum_{i=1}^{m} |u_i^*|^p \mu\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right).$$

Because $\lim_{j\to\infty} w_j = 0$ it follows that for every $\varepsilon > 0$ and every $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$ holds

$$\sum_{i=1}^{m}\sum_{j=n(i-1)+1}^{ni} \frac{w_j |u_i^*|^p}{n}\mu\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) \leq \frac{\varepsilon}{2}.$$

On the other hand for all $n \in \mathbb{N}$ such that $\|\lambda_n(z)\|^{-1} \leq 1$ we can write the chain of inequalities

$$\begin{aligned}\sum_{i=m+1}^{\infty} \sum_{j=n(i-1)+1}^{ni} \frac{w_j |u_i^*|^p}{n} \mu\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) &\leq c \sum_{i=m+1}^{\infty} \sum_{j=n(i-1)+1}^{ni} \frac{w_j |u_i^*|^p}{n} \mu(u_i^*) \\ &\leq c \sum_{i=m+1}^{\infty} \sum_{j=n(i-1)+1}^{ni} \frac{w_j}{n} M(u_i^*) \\ &\leq \frac{\varepsilon}{2n}.\end{aligned}$$

Therefore for every $\varepsilon > 0$ and $n \geq N$ we have

$$\frac{\|\lambda_n(z)\|^p}{n} = \sum_{i=1}^{\infty} \sum_{j=n(i-1)+1}^{ni} \frac{w_j |u_i^*|^p}{n} \mu\left(\frac{u_i^*}{\|\lambda_n(z)\|}\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2n} < \varepsilon$$

and thus

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} z_j \right\| n^{-\frac{1}{p}} = 0$$

□

Lemma 4.2. *Let $d_M(\omega) = \infty$ then for any $x \in d(w, M, \mathbf{\Gamma})$ there exist $y \in d(w, M, \mathbf{\Gamma})$ with $\operatorname{supp} y \cap \operatorname{supp} x = \emptyset$ and a sequence $t_n \searrow 0$ such that*

$$\|x + t_n y\| \geq \|x\| + c\omega(t_n) \tag{4.1}$$

for some constant $c > 0$ and any $n \in \mathbb{N}$.

Proof. We note first that

$$\liminf_{t\to 0} \frac{\omega(t)}{t} = 0.$$

If $x = 0$, choose sequence $t_n \searrow 0$ such that $\lim_{n\to\infty} \omega(t_n)/t_n = 0$. Then (4.1) holds trivially for any $y \neq 0$ with $c = \|y\| > 0$.

WLOG suppose that $M(1) = 1$.

Fix an arbitrary $x = \sum_{n=1}^{\infty} x_n e_{\gamma_n} \in d(w, M, \mathbf{\Gamma})$ and $\|x\| = 1$. Just for simplicity of notation we will assume that $|x_1| \geq |x_2| \geq \cdots \geq |x_n| \geq \cdots$.

We will choose sequences $t_n \searrow 0$ and $v_n \searrow 0$ inductively:

(1) $t_1 = v_1 = u_1 = 1$, $k_0 = 0$, $k_1 = 1$.

(2) Find $k_2 > k_1$, $k_2 \in \mathbb{N}$ such that

$$\frac{1}{2^1 \sum_{j=k_1+1}^{k_2} w_j} < M(v_1) \quad \text{and} \quad M(x_i) < \frac{M(t_1 v_1)}{2}$$

for $i \geq k_2$.

Find $t_2 < t_1$, $v_2 < v_1$ such that

$$\frac{M(t_2 v_2)}{\omega(t_2) M(v_2)} > 2^2 \quad \text{and} \quad M(v_2) < \frac{1}{2^2 \sum_{j=k_1+1}^{k_2} w_j}.$$

(3) Find $k_3 > k_2$, $k_3 \in \mathbb{N}$ such that

$$\frac{1}{2^2 \sum_{j=k_2+1}^{k_3} w_j} < M(v_2) \quad \text{and} \quad M(x_i) < \frac{M(t_2 v_2)}{2}$$

for $i \geq k_3$.

Find $t_3 < t_2$, $v_3 < v_2$ such that

$$\frac{M(t_3v_3)}{\omega(t_3)M(v_3)} > 2^3 \quad \text{and} \quad M(v_3) < \frac{1}{2^3\sum_{j=k_2+1}^{k_3} w_j}.$$

If we have chosen t_{n-1}, v_{n-1} and k_{n-1} then

(4) Find $k_n > k_{n-1}$, $k_n \in \mathbb{N}$ such that

$$\frac{1}{2^{n-1}\sum_{j=k_{n-1}+1}^{k_n} w_j} < M(v_{n-1}) \quad \text{and} \quad M(x_i) < \frac{M(t_{n-1}v_{n-1})}{2}$$

for $i \geq k_n$.

Find $t_n < t_{n-1}$, $v_n < v_{n-1}$ such that

$$\frac{M(t_nv_n)}{\omega(t_n)M(v_n)} > 2^n \quad \text{and} \quad M(v_n) < \frac{1}{2^n\sum_{j=k_{n-1}+1}^{k_n} w_j}.$$

For a sequence $\{A_n\}_{n=1}^{\infty}$ of finite disjoint subsets of $\mathbf{\Gamma}$, such that $A_n \cap \operatorname{supp} x = \emptyset$, $\sharp A_n = k_n - k_{n-1}$, put

$$y_n = v_n \sum_{\gamma\in A_n} e_\gamma \quad \text{and} \quad y = \sum_{n=1}^{\infty} y_n.$$

Obviously

$$\begin{aligned} I(y) &= \sum_{n=1}^{\infty} \sum_{j=k_{n-1}+1}^{k_n} w_j M(v_n) \\ &= w_1 M(v_1) + \sum_{n=2}^{\infty} M(v_n) \sum_{j=k_{n-1}+1}^{k_n} w_j \\ &\leq 1 + \sum_{n=2}^{\infty} \frac{1}{2^n} < \infty, \end{aligned}$$

which ensures $y \in d(w, M, \mathbf{\Gamma})$. We have $\operatorname{supp}(x + t_n y) = \operatorname{supp} x \cup (\cup_{n=1}^{\infty} A_n)$ for any $t \neq 0$ and therefore

$$\begin{aligned} I(x+t_ny) - I(x) &\geq I(x+t_ny_n) - I(x) \\ &\geq \sum_{j=1}^{k_{n+1}} w_j M(x_j) + \sum_{j=k_{n+1}+1}^{k_{n+2}} w_j M(t_nv_n) \\ &\qquad + \sum_{j=k_{n+2}+1}^{\infty} w_j M(x_{j+k_{n+1}-k_{n+2}}) - \sum_{j=1}^{\infty} w_j M(x_j) \\ &= M(t_nv_n) \sum_{j=k_{n+1}+1}^{k_{n+2}} w_j - \sum_{j=k_{n+1}+1}^{k_{n+2}} w_j M(x_j) \\ &\qquad + \sum_{j=k_{n+2}+1}^{\infty} w_j (M(x_{j+k_{n+1}-k_{n+2}}) - M(x_j)) \end{aligned} \tag{4.2}$$

$$\geq \frac{1}{2} M(t_n v_n) \sum_{j=k_{n+1}+1}^{k_{n+2}} w_j \geq \frac{1}{2} 2^n \omega(t_n) M(v_n) \sum_{j=k_{n+1}+1}^{k_{n+2}} w_j$$
$$\geq 2^{n-1}\omega(t_n) \frac{M(v_n)}{2^{n+1} M(v_{n+1})} \geq \frac{\omega(t_n)}{4}.$$

Remove as many elements of the sequence $\{t_n\}_{n=1}^{\infty}$ as necessary to obtain

$$0 < d_n = \|x + t_n y\| - 1 \leq 1$$

and keep the same notation for the remaining sequence. Now (2.1) implies

$$\begin{aligned} I(x+t_n y) - I(x) &= I\left(\|x + t_n y\| \frac{x + t_n y}{\|x + t_n y\|} \right) - 1 \\ &\leq \|x + t_n y\|^q - 1 \\ &= (1 + d_n)^q - 1 \leq q 2^{q-1} d_n, \end{aligned} \tag{4.3}$$

for some $q > \beta_M$.

Combining (4.2) and (4.3), we obtain

$$\|x + t_n y\| - 1 \geq c\omega(t_n),$$

where $c = \frac{1}{q 2^{q+1}}$.

Now let $x \neq 0$ be arbitrary. Find $\overline{y}$ such that $\operatorname{supp} \overline{y} \cap \operatorname{supp} x = \emptyset$ and $\left\| \frac{x}{\|x\|} - t_n \overline{y} \right\| - 1 \geq c\omega(t_n)$. Obviously for $y = \|x\| \overline{y}$ we have

$$\|x + t_n y\| - \|x\| \geq c\|x\|\omega(t_n).$$

□

5. GÂTEAUX DIFFERENTIABILITY OF BUMPS IN $d(w, M, \mathbf{\Gamma})$ AND $d(w, p, \mathbf{\Gamma})$

Theorem 5.1. *Let M be an Orlicz function and $\omega : (0, 1] \to \mathbb{R}^+$, $d_M(\omega) = \infty$.*

(i) *If $\alpha_M \notin \mathbb{N}$ then there is no continuous $G^0_{\omega,[\alpha_M]}$–smooth bump function in $d(w, M, \mathbf{\Gamma})$;*

(ii) *If $\alpha_M \in \mathbb{N}$ then there is no continuous G^0_{ω,α_M}–smooth bump function, provided $d_M(t^{\alpha_M}) < \infty$ in $d(w, M, \mathbf{\Gamma})$ and there is no continuous G^0_{ω,α_M-1}–smooth bump function, provided $d_M(t^{\alpha_M}) = \infty$ in $d(w, M, \mathbf{\Gamma})$.*

Proof. The proof in all cases is straightforward, applying Lemma 4.1 for appropriate p, Lemma 4.2 and Theorem 2.1. □

Proof of Theorem 3.1. The proof in the two cases is straightforward, applying Theorem 5.1. □

It is well known that in a Banach space X a norm of some order of smoothness generates a bump function with the same order of smoothness (see e.g. [2]), therefore the next corollary is a direct consequence of Theorem 3.1

Corollary 5.2. *Let M be an Orlicz function. If $|\cdot|$ is an equivalent norm in $d(w, M, \mathbf{\Gamma})$, which is k–times Gâteaux differentiable then $k \leq E_M$.*

As a consequence of Theorem 5.1 and Theorem 3.1 we get for $M(t) = t^p$, $p \geq 1$ the results from [9].

Corollary 5.3 ([9, Theorem 3])**.** *Let $p \geq 1$, $w_n \searrow 0$, $\sum_{n=1}^{\infty} w_n = \infty$ and $\omega : (0, 1] \to \mathbb{R}^+$ be such that $\omega(t) = o(t^p)$. Then there is no continuous $G^0_{\omega,[p]}$–smooth bump function in $d(w, p, \mathbf{\Gamma})$.*

Proof. Indeed in this case $\alpha_M = p$ and $d_{t^p}(\omega) = \infty$. If $p \notin \mathbb{N}$ then by Theorem 5.1 i), it follows that there is no continuous $G^0_{\omega,[p]}$–smooth bump in $d(w, M, \mathbf{\Gamma})$. If $p \in \mathbb{N}$ then $d_{t^p}(t^p) = 1 < \infty$ and by Theorem 5.1 ii), there is no continuous $G^0_{\omega,p}$–smooth bump in $d(w, M, \mathbf{\Gamma})$. □

Corollary 5.4 ([9, Corollary 2])**.** *Let* $p \geq 1$, $w_n \searrow 0$, $\sum_{n=1}^{\infty} w_n = \infty$. *If* f *is a continuous* k*–times Gâteaux differentiable bump function in* $d(w, p, \mathbf{\Gamma})$*, then* $k \leq [p]$.

Proof. In this case it is obvious that $d_{t^p}(t^{[p]}) < \infty$ and $d_{t^p}(t^p) < \infty$. Therefore by Theorem 3.1 it follows that $k \leq [p]$. □

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