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GOOD LOWER AND UPPER BOUNDS ON BINOMIAL COEFFICIENTS

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Abstract

We provide good bounds on binomial coefficients, generalizing known ones, using some results of H. Robbins and of Sasvári.

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Contents

1. Motivation

Analytic techniques can be often used to obtain asymptotics for simply-indexed sequences. Asymptotic estimates for doubly(multiply)-indexed sequences are considerably more difficult to obtain (cf. [4], p. 204). Very little is known about how to obtain asymptotic estimates of these sequences. The estimates that are known are based on summing over one index at a time. For instance, according to the same source, the formula

$$\binom{n}{k} \sim \frac{2^n e^{-\frac{(n-2k)^2}{2n}}}{\sqrt{\frac{n\pi}{2}}}$$

is valid only when $|2n-k| \in o(n^{\frac{3}{4}})$.

We raise the question of getting good bounds for the binomial coefficient, which should be valid for any n, k.

In the August-September 2000 issue of American Mathematical Monthly, O. Krafft proposed the following problem (P10819):

For $m \geq 2$, $n \geq 1$, we have

$$\binom{mn}{n} \geq \frac{m^{m(n-1)+1}}{(m-1)^{(m-1)(n-1)}} n^{-\frac{1}{2}}.$$

In this note, we are able to improve this inequality (by replacing 1 in the right-hand side by a better absolute constant) and also generalize the inequality to $\binom{mn}{pn}$.

We also employ a method of Sasvári [5] (see also [2]), to derive better lower and upper bounds, with the absolute constants replaced by appropriate functions of m, n, p.

2. The Results

The following double inequality for the factorial was shown by H. Robbins in [3] (1955), a step in a proof of Stirling's formula $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$.

Lemma 2.1 (Robbins). *For* $n \geq 1$,

$$n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+r(n)}, \tag{2.1}$$

where $r(n)$ *satisfies* $\frac{1}{12n+1} < r(n) < \frac{1}{12n}$.

One approach to get approximations for the binomial coefficient $\binom{mn}{pn}$, $m \geq p$, would be to use Stirling's approximation for the factorial of Lemma 2.1, namely

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}. \tag{2.2}$$

Thus

(2.3)

$$\begin{aligned}
&\binom{mn}{pn} \\
&= \frac{(mn)!}{(pn)!((m-p)n)!} \\
&> \frac{\sqrt{2\pi}\,(mn)^{mn+\frac{1}{2}}\, e^{-mn+\frac{1}{12mn+1}}}{\sqrt{2\pi}\,(pn)^{pn+\frac{1}{2}}\, e^{-pn+\frac{1}{12pn}}\,\sqrt{2\pi}\,((m-p)n)^{(m-p)n+\frac{1}{2}}\, e^{-(m-p)n+\frac{1}{12n(m-p)}}} \\
&= \frac{1}{\sqrt{2\pi}}\, n^{-\frac{1}{2}}\, \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}}\, p^{pn+\frac{1}{2}}}\, e^{\frac{1}{12nm+1}-\frac{1}{12pn}-\frac{1}{12n(m-p)}}
\end{aligned}$$

and

(2.4)
$$\binom{mn}{pn} < \frac{\sqrt{2\pi}\,(mn)^{mn+\frac{1}{2}}\,e^{-mn+\frac{1}{12mn}}}{\sqrt{2\pi}\,(pn)^{pn+\frac{1}{2}}\,e^{-pn+\frac{1}{12pn+1}}\,\sqrt{2\pi}\,((m-p)n)^{(m-p)n+\frac{1}{2}}\,e^{-(m-p)n+\frac{1}{12n(m-p)+1}}}$$
$$= \frac{1}{\sqrt{2\pi}}\,n^{-\frac{1}{2}}\,\frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}}p^{pn+\frac{1}{2}}}\,e^{\frac{1}{12nm}-\frac{1}{12pn+1}-\frac{1}{12n(m-p)+1}}.$$

However, we can improve the lower bound, by employing a method of Sasvári [5] (see also [2]). Let

$$D_N(n,m,p) = \sum_{j=1}^{N} \frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{(mn)^{2j-1}} - \frac{1}{(np)^{2j-1}} - \frac{1}{((m-p)n)^{2j-1}}\right),$$

with B_{2j}, the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}t^{2j}$$

and

$$\Delta(n,m,p) = r(mn) - r(pn) - r((m-p)n).$$

We show that $\Delta(n,m,p) - D_N(n,m,p)$ is an increasing (decreasing) function of n if N is even (respectively, odd). We proceed to the proof of the above fact.

By the Binet formula (see [2]), we get

$$r(x) = \int_0^\infty \frac{1}{t^2}\left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right) e^{-tx} d\,x, \qquad x \in (0,\infty),$$

and using $j! = \int_0^\infty t^j e^{-t} d\,t$, we get

$$\Delta(n,m,p) - D_N(n,m,p) = \int_0^\infty \frac{1}{t^2} P_N(t) Q_n(t) d\,t,$$

where

$$P_N(t) = \frac{t}{e^t-1} - 1 + \frac{t}{2} - \sum_{j=1}^{N} \frac{B_{2j}}{(2j)!} t^{2j}$$

and

$$Q_n(t) = e^{-mnt} - e^{(m-p)nt} - e^{-pnt}.$$

Sasvári proved that $P_N(t)$ is positive (negative) if N is even (respectively, odd). So we need to show that $Q_n(t)$ is increasing with respect to n, if $t > 0$ and $m > p \geq 1$. Since $Q_n(t) = f(e^{-nt})$, for $f(u) = u^m - u^{m-p} - u^p$, it suffices to show that f is decreasing on $(0,1)$, that is $f'(u) < 0$ on $(0,1)$. Now, $f'(u) < 0$ is equivalent to $mu^{m-1} - (m-p)u^{m-p-1} - pu^{p-1} < 0$, which is equivalent to $g(u) = u^{m-2p}(mu^p - m + p) < p$. If $m \geq 2p$, then $g(u) \leq mu^p - m + p < p$. If $1 < m < 2p$, then

$$\begin{aligned} g'(u) &= (m-2p)u^{m-2p-1}(mu^p - m + p) + mpu^{m-p-1} \\ &= u^{m-2p-1}(m-2p)(mu^p - m + 2p) > 0. \end{aligned}$$

Therefore, for $0 < u < 1$, we have $g(u) < g(1) = p$ and the claim is proved. Thus, we have

Theorem 2.2.

$$\frac{1}{\sqrt{2\pi}}\, e^{D_{2N+1}(n,m,p)}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} < \binom{m\,n}{p\,n} < \frac{1}{\sqrt{2\pi}}\, e^{D_{2N}(n,m,p)}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}}. \tag{2.5}$$

Taking $N = 0$ and observing that $B_2 = \frac{1}{6}$, we get

Corollary 2.3.

$$\frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{12n}\left(\frac{1}{m}-\frac{1}{p}-\frac{1}{m-p}\right)}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} < \binom{m\,n}{p\,n} < \frac{1}{\sqrt{2\pi}}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}}. \tag{2.6}$$

By using (2.4), the upper bound can be improved and we get

Corollary 2.4.

$$\frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{12n}\left(\frac{1}{m}-\frac{1}{p}-\frac{1}{m-p}\right)}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} < \binom{m\,n}{p\,n} < \frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{12nm}-\frac{1}{12pn+1}-\frac{1}{12n(m-p)+1}}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} \tag{2.7}$$

To show that the upper bound of Corollary 2.4 improves upon the one of Corollary 2.3 we use (2.4) and prove that

$$\frac{1}{12nm} - \frac{1}{12pn+1} - \frac{1}{12n(m-p)+1} < 0 \tag{2.8}$$

by rewriting as

$$\begin{aligned}
&\frac{1}{12nm} - \frac{1}{12pn+1} - \frac{1}{12n(m-p)+1} \\
&\quad = \frac{144mnp(m-p) + 12n(m-p) + 12pm + 1 - 144mn^2(m-p) - 12mn - 144m^2np - 12mn}{12mn(12pn+1)(12n(m-p)+1)} \\
&\quad = \frac{-144mnp^2 - 12np + 12pm + 1 - 144mn^2(m-p) - 12mn}{12mn(12pn+1)(12n(m-p)+1)} < 0.
\end{aligned}$$

Remark 2.1. *The left side of Corollary 2.3 differs slightly from (2.3), in that* $12mn+1$ *is replaced by* $12mn$*. Therefore, the left side of (2.6) is an improvement of (2.3).*

Next, we prove another result, where the expressions given by exponential powers are replaced by functions of n only. We prove

Theorem 2.5. *Let* m, n, p *be positive integers, with* $m > p \geq 1$ *and* $n \geq 1$.

Then

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{8n}}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} < \binom{mn}{pn} < \frac{1}{\sqrt{2\pi}}\, n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} \tag{2.9}$$

Proof. Using Corollary 2.3, we need to show that

$$\frac{1}{12nm} - \frac{1}{12np} - \frac{1}{12n(m-p)} \geq -\frac{1}{8n}. \tag{2.10}$$

The inequality (2.10) is equivalent to

$$\frac{1}{m} + \frac{m}{p(m-p)} \leq \frac{3}{2}. \tag{2.11}$$

Let $x = m - p$. Thus, $x \geq 1$. We show first that the left side of (2.11), $g(x,p) = \frac{x^2+px+p^2}{px(p+x)}$ is decreasing with respect to x, that is

$$\frac{d\,g(x,p)}{d\,x} = -\frac{1}{x^2} + \frac{1}{(p+x)^2} < 0,$$

which is certainly true. Therefore,

$$g(x,p) \leq g(1,p) = \frac{p^2+p+1}{p(p+1)} (= h(p)).$$

Since $h'(p) = -\frac{2p+1}{p^2(p+1)^2} < 0$, we get that h is decreasing with respect to p, so

$$g(x,p) \le h(p) \le h(1) = \frac{3}{2}.$$

□

Now we provide a further simplification of Theorem 2.5. The following lemma proves to be very useful.

Lemma 2.6. *Let $p \ge 1$ be a fixed natural number and $m \ge p+1$. Then the function $\left(\frac{m}{m-p}\right)^{m-\frac{1}{2}}$ is decreasing (with respect to m) and*

$$\lim_{m\to\infty}\left(\frac{m}{m-p}\right)^{m-\frac{1}{2}} = e^p.$$

Proof. It suffices to prove that the function $h(x) = \log\left(\frac{x}{x-p}\right)^{x-\frac{1}{2}}$, $x \ge p+1$, is decreasing and its limit is e^p. By differentiation

$$h'(x) = \log\frac{x}{x-p} - \frac{2xp-p}{2x(x-p)}.$$

Since

$$\log\frac{x}{x-p} = -\log\left(1-\frac{p}{x}\right) < \frac{p}{x} + \frac{p^2}{2x^2}$$

(by Taylor expansion), we get

$$h'(x) < \frac{p}{x} + \frac{p^2}{2x^2} - \frac{p}{x} - \frac{2p^2-p}{2x(x-p)} = \frac{x-px-p^2}{x(x-p)} < 0,$$

since $p \geq 1$, so h is decreasing. The lower bound of this function is its limit, which is e^p, since $\left(1-\frac{p}{x}\right)^x \to e^{-p}$, and $\left(\frac{x-p}{x}\right)^{-\frac{1}{2}} \to 1$ as $x \to \infty$. □

Using Theorem 2.5 and Lemma 2.6, we get

Theorem 2.7. *We have, for $m > p \geq 1$ and $n \geq 2$,*

$$\binom{m\,n}{p\,n} > \frac{1}{\sqrt{2\pi}} e^{p-\frac{1}{8n}} n^{-\frac{1}{2}} \frac{m^{m(n-1)+1}}{(m-p)^{(m-p)(n-1)-p+1} p^{pn+\frac{1}{2}}}. \tag{2.12}$$

Taking $p = 1$, we obtain a stronger version of the inequality P10819, namely

Corollary 2.8. *We have, for $m > 1$ and $n \geq 2$,*

$$\binom{mn}{n} > 1.08444\, e^{-\frac{1}{8n}} n^{-\frac{1}{2}} \frac{m^{m(n-1)+1}}{(m-1)^{(m-1)(n-1)}}. \tag{2.13}$$

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