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## GOOD LOWER AND UPPER BOUNDS ON BINOMIAL COEFFICIENTS

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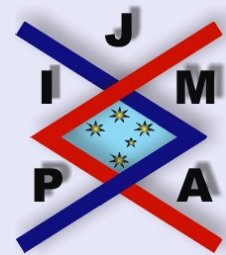
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Abstract

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## Abstract

We provide good bounds on binomial coefficients, generalizing known ones, using some results of H. Robbins and of Sasvári.

*2000 Mathematics Subject Classification:* 05A20, 11B65, 26D15.

*Key words:* Binomial Coefficients, Stirling's Formula, Inequalities.

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# 1. Motivation

Analytic techniques can be often used to obtain asymptotics for simply-indexed sequences. Asymptotic estimates for doubly(multiply)-indexed sequences are considerably more difficult to obtain (cf. [4], p. 204). Very little is known about how to obtain asymptotic estimates of these sequences. The estimates that are known are based on summing over one index at a time. For instance, according to the same source, the formula

$$\binom{n}{k} \sim \frac{2^n e^{-\frac{(n-2k)^2}{2n}}}{\sqrt{\frac{n\pi}{2}}}$$

is valid only when  $|2n - k| \in o(n^{\frac{3}{4}})$ .

We raise the question of getting good bounds for the binomial coefficient, which should be valid for any  $n, k$ .

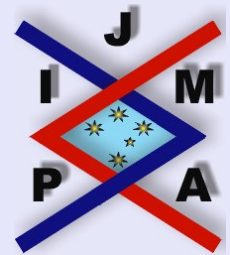
In the August-September 2000 issue of American Mathematical Monthly, O. Krafft proposed the following problem (P10819):

*For  $m \geq 2, n \geq 1$ , we have*

$$\binom{mn}{n} \geq \frac{m^{m(n-1)+1}}{(m-1)^{(m-1)(n-1)}} n^{-\frac{1}{2}}.$$

In this note, we are able to improve this inequality (by replacing 1 in the right-hand side by a better absolute constant) and also generalize the inequality to  $\binom{mn}{pn}$ .

We also employ a method of Sasvári [5] (see also [2]), to derive better lower and upper bounds, with the absolute constants replaced by appropriate functions of  $m, n, p$ .



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## 2. The Results

The following double inequality for the factorial was shown by H. Robbins in [3] (1955), a step in a proof of Stirling's formula  $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ .

**Lemma 2.1 (Robbins).** For  $n \geq 1$ ,

$$(2.1) \quad n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+r(n)},$$

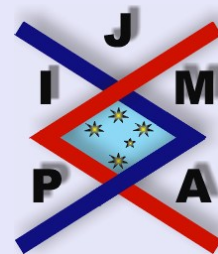
where  $r(n)$  satisfies  $\frac{1}{12n+1} < r(n) < \frac{1}{12n}$ .

One approach to get approximations for the binomial coefficient  $\binom{mn}{pn}$ ,  $m \geq p$ , would be to use Stirling's approximation for the factorial of Lemma 2.1, namely

$$(2.2) \quad \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}.$$

Thus

$$(2.3) \quad \binom{mn}{pn} = \frac{(mn)!}{(pn)!((m-p)n)!} > \frac{\sqrt{2\pi} (mn)^{mn+\frac{1}{2}} e^{-mn+\frac{1}{12mn+1}}}{\sqrt{2\pi} (pn)^{pn+\frac{1}{2}} e^{-pn+\frac{1}{12pn}} \sqrt{2\pi} ((m-p)n)^{(m-p)n+\frac{1}{2}} e^{-(m-p)n+\frac{1}{12n(m-p)}}} = \frac{1}{\sqrt{2\pi}} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} e^{\frac{1}{12nm+1} - \frac{1}{12pn} - \frac{1}{12n(m-p)}}$$



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and

$$\begin{aligned}
 (2.4) \quad & \binom{mn}{pn} \\
 & < \frac{\sqrt{2\pi} (mn)^{mn+\frac{1}{2}} e^{-mn+\frac{1}{12mn}}}{\sqrt{2\pi} (pn)^{pn+\frac{1}{2}} e^{-pn+\frac{1}{12pn+1}} \sqrt{2\pi} ((m-p)n)^{(m-p)n+\frac{1}{2}} e^{-(m-p)n+\frac{1}{12n(m-p)+1}}} \\
 & = \frac{1}{\sqrt{2\pi}} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} e^{\frac{1}{12nm} - \frac{1}{12pn+1} - \frac{1}{12n(m-p)+1}}.
 \end{aligned}$$

However, we can improve the lower bound, by employing a method of Sasvári [5] (see also [2]). Let

$$D_N(n, m, p) = \sum_{j=1}^N \frac{B_{2j}}{2j(2j-1)} \left( \frac{1}{(mn)^{2j-1}} - \frac{1}{(np)^{2j-1}} - \frac{1}{((m-p)n)^{2j-1}} \right),$$

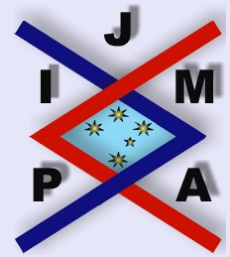
with  $B_{2j}$ , the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} t^{2j}$$

and

$$\Delta(n, m, p) = r(mn) - r(pn) - r((m-p)n).$$

We show that  $\Delta(n, m, p) - D_N(n, m, p)$  is an increasing (decreasing) function of  $n$  if  $N$  is even (respectively, odd). We proceed to the proof of the above fact.



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By the Binet formula (see [2]), we get

$$r(x) = \int_0^\infty \frac{1}{t^2} \left( \frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) e^{-tx} dx, \quad x \in (0, \infty),$$

and using  $j! = \int_0^\infty t^j e^{-t} dt$ , we get

$$\Delta(n, m, p) - D_N(n, m, p) = \int_0^\infty \frac{1}{t^2} P_N(t) Q_n(t) dt,$$

where

$$P_N(t) = \frac{t}{e^t - 1} - 1 + \frac{t}{2} - \sum_{j=1}^N \frac{B_{2j}}{(2j)!} t^{2j}$$

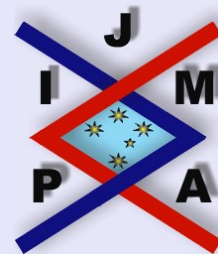
and

$$Q_n(t) = e^{-mnt} - e^{(m-p)nt} - e^{-pnt}.$$

Sasvári proved that  $P_N(t)$  is positive (negative) if  $N$  is even (respectively, odd). So we need to show that  $Q_n(t)$  is increasing with respect to  $n$ , if  $t > 0$  and  $m > p \geq 1$ . Since  $Q_n(t) = f(e^{-nt})$ , for  $f(u) = u^m - u^{m-p} - u^p$ , it suffices to show that  $f$  is decreasing on  $(0, 1)$ , that is  $f'(u) < 0$  on  $(0, 1)$ . Now,  $f'(u) < 0$  is equivalent to  $mu^{m-1} - (m-p)u^{m-p-1} - pu^{p-1} < 0$ , which is equivalent to  $g(u) = u^{m-2p}(mu^p - m + p) < p$ . If  $m \geq 2p$ , then  $g(u) \leq mu^p - m + p < p$ . If  $1 < m < 2p$ , then

$$\begin{aligned} g'(u) &= (m-2p)u^{m-2p-1}(mu^p - m + p) + mpu^{m-p-1} \\ &= u^{m-2p-1}(m-2p)(mu^p - m + 2p) > 0. \end{aligned}$$

Therefore, for  $0 < u < 1$ , we have  $g(u) < g(1) = p$  and the claim is proved. Thus, we have



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## Theorem 2.2.

$$(2.5) \quad \frac{1}{\sqrt{2\pi}} e^{D_{2N+1}(n,m,p)} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} \\ < \binom{m n}{p n} < \frac{1}{\sqrt{2\pi}} e^{D_{2N}(n,m,p)} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}}.$$

Taking  $N = 0$  and observing that  $B_2 = \frac{1}{6}$ , we get

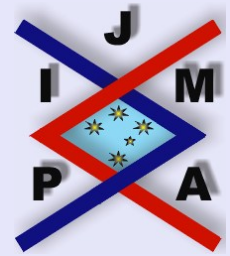
## Corollary 2.3.

$$(2.6) \quad \frac{1}{\sqrt{2\pi}} e^{\frac{1}{12n}(\frac{1}{m}-\frac{1}{p}-\frac{1}{m-p})} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} \\ < \binom{m n}{p n} < \frac{1}{\sqrt{2\pi}} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}}.$$

By using (2.4), the upper bound can be improved and we get

## Corollary 2.4.

$$(2.7) \quad \frac{1}{\sqrt{2\pi}} e^{\frac{1}{12n}(\frac{1}{m}-\frac{1}{p}-\frac{1}{m-p})} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} \\ < \binom{m n}{p n} < \frac{1}{\sqrt{2\pi}} e^{\frac{1}{12nm} - \frac{1}{12pn+1} - \frac{1}{12n(m-p)+1}} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}}$$



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To show that the upper bound of Corollary 2.4 improves upon the one of Corollary 2.3 we use (2.4) and prove that

$$(2.8) \quad \frac{1}{12nm} - \frac{1}{12pn + 1} - \frac{1}{12n(m - p) + 1} < 0$$

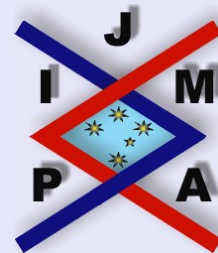
by rewriting as

$$\begin{aligned} & \frac{1}{12nm} - \frac{1}{12pn + 1} - \frac{1}{12n(m - p) + 1} \\ &= \frac{144mnp(m - p) + 12n(m - p) + 12pm + 1}{12mn(12pn + 1)(12n(m - p) + 1)} \\ & \quad - \frac{144mn^2(m - p) - 12mn - 144m^2np - 12mn}{12mn(12pn + 1)(12n(m - p) + 1)} \\ &= \frac{-144mnp^2 - 12np + 12pm + 1 - 144mn^2(m - p) - 12mn}{12mn(12pn + 1)(12n(m - p) + 1)} < 0. \end{aligned}$$

**Remark 2.1.** The left side of Corollary 2.3 differs slightly from (2.3), in that  $12mn + 1$  is replaced by  $12mn$ . Therefore, the left side of (2.6) is an improvement of (2.3).

Next, we prove another result, where the expressions given by exponential powers are replaced by functions of  $n$  only. We prove

**Theorem 2.5.** Let  $m, n, p$  be positive integers, with  $m > p \geq 1$  and  $n \geq 1$ .



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Then

$$(2.9) \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{8n}} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}} < \binom{m}{pn} < \frac{1}{\sqrt{2\pi}} n^{-\frac{1}{2}} \frac{m^{mn+\frac{1}{2}}}{(m-p)^{(m-p)n+\frac{1}{2}} p^{pn+\frac{1}{2}}}$$

*Proof.* Using Corollary 2.3, we need to show that

$$(2.10) \quad \frac{1}{12nm} - \frac{1}{12np} - \frac{1}{12n(m-p)} \geq -\frac{1}{8n}.$$

The inequality (2.10) is equivalent to

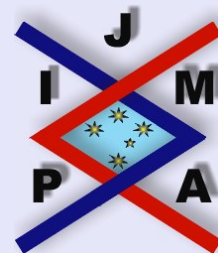
$$(2.11) \quad \frac{1}{m} + \frac{m}{p(m-p)} \leq \frac{3}{2}.$$

Let  $x = m - p$ . Thus,  $x \geq 1$ . We show first that the left side of (2.11),  $g(x, p) = \frac{x^2+px+p^2}{px(p+x)}$  is decreasing with respect to  $x$ , that is

$$\frac{d g(x, p)}{d x} = -\frac{1}{x^2} + \frac{1}{(p+x)^2} < 0,$$

which is certainly true. Therefore,

$$g(x, p) \leq g(1, p) = \frac{p^2 + p + 1}{p(p+1)} (= h(p)).$$



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Since  $h'(p) = -\frac{2p+1}{p^2(p+1)^2} < 0$ , we get that  $h$  is decreasing with respect to  $p$ , so

$$g(x, p) \leq h(p) \leq h(1) = \frac{3}{2}.$$

□

Now we provide a further simplification of Theorem 2.5. The following lemma proves to be very useful.

**Lemma 2.6.** *Let  $p \geq 1$  be a fixed natural number and  $m \geq p + 1$ . Then the function  $\left(\frac{m}{m-p}\right)^{m-\frac{1}{2}}$  is decreasing (with respect to  $m$ ) and*

$$\lim_{m \rightarrow \infty} \left(\frac{m}{m-p}\right)^{m-\frac{1}{2}} = e^p.$$

*Proof.* It suffices to prove that the function  $h(x) = \log\left(\frac{x}{x-p}\right)^{x-\frac{1}{2}}$ ,  $x \geq p + 1$ , is decreasing and its limit is  $e^p$ . By differentiation

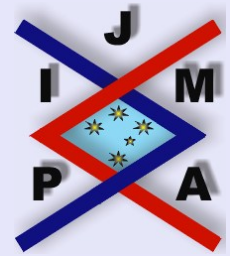
$$h'(x) = \log \frac{x}{x-p} - \frac{2xp-p}{2x(x-p)}.$$

Since

$$\log \frac{x}{x-p} = -\log\left(1 - \frac{p}{x}\right) < \frac{p}{x} + \frac{p^2}{2x^2}$$

(by Taylor expansion), we get

$$h'(x) < \frac{p}{x} + \frac{p^2}{2x^2} - \frac{p}{x} - \frac{2p^2-p}{2x(x-p)} = \frac{x-px-p^2}{x(x-p)} < 0,$$



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since  $p \geq 1$ , so  $h$  is decreasing. The lower bound of this function is its limit, which is  $e^p$ , since  $(1 - \frac{p}{x})^x \rightarrow e^{-p}$ , and  $(\frac{x-p}{x})^{-\frac{1}{2}} \rightarrow 1$  as  $x \rightarrow \infty$ .  $\square$

Using Theorem 2.5 and Lemma 2.6, we get

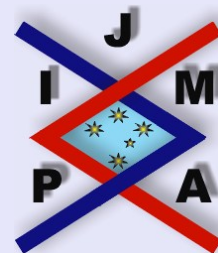
**Theorem 2.7.** *We have, for  $m > p \geq 1$  and  $n \geq 2$ ,*

$$(2.12) \quad \binom{m}{p} \binom{n}{n-p} > \frac{1}{\sqrt{2\pi}} e^{p - \frac{1}{8n}} n^{-\frac{1}{2}} \frac{m^{m(n-1)+1}}{(m-p)^{(m-p)(n-1)-p+1} p^{pn+\frac{1}{2}}}.$$

Taking  $p = 1$ , we obtain a stronger version of the inequality P10819, namely

**Corollary 2.8.** *We have, for  $m > 1$  and  $n \geq 2$ ,*

$$(2.13) \quad \binom{mn}{n} > 1.08444 e^{-\frac{1}{8n}} n^{-\frac{1}{2}} \frac{m^{m(n-1)+1}}{(m-1)^{(m-1)(n-1)}}.$$



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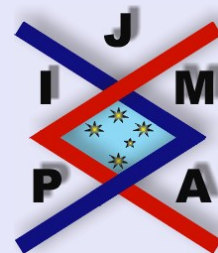
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