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NOTES ON CERTAIN SUBCLASS OF p-VALENTLY BAZILEVIČ FUNCTIONS

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ABSTRACT. In the present paper, we discuss a subclass $\mathcal{M}_p(\lambda, \mu, A, B)$ of p-valently Bazilevič functions, which was introduced and investigated recently by Patel [5]. Such results as inclusion relationship, coefficient inequality and radius of convexity for this class are proved. The results presented here generalize and improve some earlier results. Several other new results are also obtained.

Key words and phrases: Analytic functions, Multivalent functions, Bazilevič functions, subordination between analytic functions, Briot-Bouquet differential subordination.

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1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For simplicity, we write

$$\mathcal{A}_1 =: \mathcal{A}.$$

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For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{M}_p(\lambda, \mu, A, B)$ denote the class of functions in $\mathcal{A}_p$ satisfying the following subordination condition:

$$\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right] \prec p\frac{1+Az}{1+Bz} \tag{1.1}$$
$$(-1 \leqq B < A \leqq 1;\ z \in \mathbb{U})$$

for some real μ $(\mu \geqq 0),\ \lambda$ $(\lambda \geqq 0)$ and $g \in \mathcal{S}_p^*$, where $\mathcal{S}_p^*$ denotes the usual class of p-valently starlike functions in $\mathbb{U}$.

For simplicity, we write

$$\mathcal{M}_p\left(\lambda, \mu, 1 - \frac{2\alpha}{p}, -1\right) = \mathcal{M}_p(\lambda, \mu, \alpha)$$
$$:= \left\{ f(z) \in \mathcal{A}_p :\ \Re\left(\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right]\right) > \alpha \right\},$$

for some α $(0 \leqq \alpha < p)$ and $z \in \mathbb{U}$.

The class $\mathcal{M}_p(\lambda, \mu, A, B)$ was introduced and investigated recently by Patel [5]. The author obtained some interesting properties for this class in the case $\lambda > 0$, he also proved the following result:

Theorem 1.1. *Let*

$$\mu \geqq 0,\ \lambda > 0 \quad \textit{and} \quad -1 \leqq B < A \leqq 1.$$

If $f \in \mathcal{M}_p(\lambda, \mu, A, B)$, then

$$\frac{zf'(z)}{pf^{1-\mu}(z)g^{\mu}(z)} \prec \frac{\lambda}{pQ(z)} = q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \tag{1.2}$$

where

$$Q(z) = \begin{cases} \int_0^1 s^{\frac{p}{\lambda}-1}\left(\frac{1+Bsz}{1+Bz}\right)^{\frac{p(A-B)}{\lambda B}} ds & (B \neq 0), \\ \int_0^1 s^{\frac{p}{\lambda}-1} \exp\left(\frac{p}{\lambda}(s-1)Az\right) ds & (B = 0), \end{cases}$$

and $q(z)$ is the best dominant of (1.2).

In the present paper, we shall derive such results as inclusion relationship, coefficient inequality and radius of convexity for the class $\mathcal{M}_p(\lambda, \mu, A, B)$ by making use of the techniques of Briot-Bouquet differential subordination. The results presented here generalize and improve some known results. Several other new results are also obtained.

2. PRELIMINARY RESULTS

In order to prove our main results, we shall require the following lemmas.

Lemma 2.1. *Let*

$$\mu \geqq 0,\ \lambda \geqq 0 \quad \textit{and} \quad -1 \leqq B < A \leqq 1.$$

Then

$$\mathcal{M}_p(\lambda, \mu, A, B) \subset \mathcal{M}_p(0, \mu, A, B).$$

Proof. Suppose that $f \in \mathcal{M}_p(\lambda, \mu, A, B)$. By virtue of (1.2), we know that

$$\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} \prec p\frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

which implies that $f \in \mathcal{M}_p(0, \mu, A, B)$. Therefore, the assertion of Lemma 2.1 holds true. $\square$

Lemma 2.2 (see [3])**.** *Let*

$$-1 \leqq B_1 \leqq B_2 < A_2 \leqq A_1 \leqq 1.$$

Then

$$\frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}.$$

Lemma 2.3 (see [4])**.** *Let F be analytic and convex in $\mathbb{U}$. If*

$$f,\ g \in \mathcal{A} \quad \textit{and} \quad f,\ g \prec F,$$

then

$$\lambda f + (1-\lambda)g \prec F \qquad (0 \leqq \lambda \leqq 1).$$

Lemma 2.4 (see [6])**.** *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be analytic in $\mathbb{U}$ and

$$g(z) = \sum_{k=0}^{\infty} b_k z^k$$

be analytic and convex in $\mathbb{U}$. If $f \prec g$, then

$$|a_k| \leqq |b_1| \qquad (k \in \mathbb{N}).$$

3. MAIN RESULTS

We begin by stating our first inclusion relationship given by Theorem 3.1 below.

Theorem 3.1. *Let*

$$\mu \geqq 0,\ \lambda_2 \geqq \lambda_1 \geqq 0 \quad \textit{and} \quad -1 \leqq B_1 \leqq B_2 < A_2 \leqq A_1 \leqq 1.$$

Then

$$\mathcal{M}_p(\lambda_2, \mu, A_2, B_2) \subset \mathcal{M}_p(\lambda_1, \mu, A_1, B_1).$$

Proof. Suppose that $f \in \mathcal{M}_p(\lambda_2, \mu, A_2, B_2)$. We know that

$$\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} + \lambda_2\left[1 + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}\right] \prec p\frac{1+A_2z}{1+B_2z}.$$

Since

$$-1 \leqq B_1 \leqq B_2 < A_2 \leqq A_1 \leqq 1,$$

it follows from Lemma 2.2 that

$$\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}+\lambda_2\left[1+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}\right]\prec p\frac{1+A_1z}{1+B_1z}. \tag{3.1}$$

that is, that $f\in\mathcal{M}_p(\lambda_2,\mu,A_1,B_1)$. Thus, the assertion of Theorem 3.1 holds true for

$$\lambda_2=\lambda_1\geqq 0.$$

If

$$\lambda_2>\lambda_1\geqq 0,$$

by virtue of Lemma 2.1 and (3.1), we know that $f\in\mathcal{M}_p(0,\mu,A_1,B_1)$, that is

$$\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}\prec p\frac{1+A_1z}{1+B_1z}. \tag{3.2}$$

At the same time, we have

$$\begin{aligned}&\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}+\lambda_1\left[1+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}\right]\\&\quad=\left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}\\&\qquad+\frac{\lambda_1}{\lambda_2}\left\{\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}+\lambda_2\left[1+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}\right]\right\}.\end{aligned} \tag{3.3}$$

It is obvious that

$$h_1(z)=\frac{1+A_1z}{1+B_1z}$$

is analytic and convex in $\mathbb{U}$. Thus, we find from Lemma 2.3, (3.1), (3.2) and (3.3) that

$$\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}+\lambda_1\left[1+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}\right]\prec p\frac{1+A_1z}{1+B_1z},$$

that is, that $f\in\mathcal{M}_p(\lambda_1,\mu,A_1,B_1)$. This implies that

$$\mathcal{M}_p(\lambda_2,\mu,A_2,B_2)\subset\mathcal{M}_p(\lambda_1,\mu,A_1,B_1).$$

□

Remark 1. Setting $A_1=A_2=A$ and $B_1=B_2=B$ in Theorem 3.1, we get the corresponding result obtained by Guo and Liu [2].

Corollary 3.2. *Let*

$$\mu\geqq 0,\ \lambda_2\geqq\lambda_1\geqq 0\quad\textit{and}\quad p>\alpha_2\geqq\alpha_1\geqq 0.$$

Then

$$\mathcal{M}_p(\lambda_2,\mu,\alpha_2)\subset\mathcal{M}_p(\lambda_1,\mu,\alpha_1).$$

Theorem 3.3. *If $f\in\mathcal{A}_p$ satisfies the following conditions:*

$$\Re\left(\frac{f(z)}{z^p}\right)>0\quad\textit{and}\quad\left|\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)}-p\right|<\nu p\qquad\left(0\leqq\mu<\frac{1}{2};\ 0<\nu\leqq 1;\ z\in\mathbb{U}\right)$$

for $g\in\mathcal{S}_p^*$, then f is p-valently convex (univalent) in $|z|<R(p,\mu,\nu)$, where

$$R(p,\mu,\nu)=\frac{2p\mu+2\mu-\nu-2+\sqrt{(2p\mu+2\mu-\nu-2)^2+4(\nu+p)(p-2p\mu)}}{2(\nu+p)}.$$

Proof. Suppose that

$$h(z) := \frac{zf'(z)}{pf^{1-\mu}(z)g^{\mu}(z)} - 1 \qquad (z \in \mathbb{U}).$$

Then h is analytic in $\mathbb{U}$ with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1 \quad (z \in \mathbb{U}).$$

Thus, by applying Schwarz's Lemma, we get

$$h(z) = \nu z\psi(z) \qquad (0 < \nu \leqq 1),$$

where ψ is analytic in $\mathbb{U}$ with

$$|\psi(z)| \leqq 1 \qquad (z \in \mathbb{U}).$$

Therefore, we have

$$zf'(z) = pf^{1-\mu}(z)g^{\mu}(z)(1 + \nu z\psi(z)). \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z logarithmically, we obtain

$$1 + \frac{zf''(z)}{f'(z)} = (1-\mu)\frac{zf'(z)}{f(z)} + \mu\frac{zg'(z)}{g(z)} + \frac{\nu z(\psi(z) + z\psi'(z))}{1 + \nu z\psi(z)}. \tag{3.5}$$

We now suppose that

$$\phi(z) := \frac{f(z)}{z^p} = 1 + c_1 z + c_2 z^2 + \cdots , \tag{3.6}$$

by hypothesis, we know that

$$\Re(\phi(z)) > 0 \qquad (z \in \mathbb{U}). \tag{3.7}$$

It follows from (3.6) that

$$\frac{zf'(z)}{f(z)} = p + \frac{z\phi'(z)}{\phi(z)}. \tag{3.8}$$

Upon substituting (3.8) into (3.5), we get

$$1 + \frac{zf''(z)}{f'(z)} = (1-\mu)p + (1-\mu)\frac{z\phi'(z)}{\phi(z)} + \mu\frac{zg'(z)}{g(z)} + \frac{\nu z(\psi(z) + z\psi'(z))}{1 + \nu z\psi(z)}. \tag{3.9}$$

Now, by using the following well known estimates (see [1]):

$$\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) \geqq -\frac{2r}{1-r^2}, \qquad \Re\left(\frac{zg'(z)}{g(z)}\right) \geqq -p\frac{1-r}{1+r},$$

and

$$\Re\left(\frac{z(\psi(z) + z\psi'(z))}{1 + z\psi(z)}\right) \geqq -\frac{r}{1-r}$$

for $|z| = r < 1$ in (3.9), we obtain

$$\begin{aligned}
\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) &\geqq (1-\mu)p - \frac{2(1-\mu)r}{1-r^2} - \frac{p\mu(1-r)}{1+r} - \frac{\nu r}{1-\nu r} \\
&\geqq (1-\mu)p - \frac{2(1-\mu)r}{1-r^2} - \frac{p\mu(1-r)}{1+r} - \frac{\nu r}{1-r} \\
&\geqq -\frac{(\nu+p)r^2 - (2p\mu + 2\mu - \nu - 2)r - (p - 2p\mu)}{1-r^2},
\end{aligned}$$

which is certainly positive if $r < R(p, \mu, \nu)$. □

Putting $\nu = 1$ in Theorem 3.3, we get the following result.

Corollary 3.4. *If $f \in \mathcal{A}_p$ satisfies the following conditions:*

$$\Re\left(\frac{f(z)}{z^p}\right) > 0 \quad \textit{and} \quad \left|\frac{zf'(z)}{f^{1-\mu}(z)g^{\mu}(z)} - p\right| < p \quad \left(0 \leqq \mu < \frac{1}{2};\ z \in \mathbb{U}\right)$$

for $g \in \mathcal{S}_p^*$, then f is p-valently convex (univalent) in $|z| < R(p,\mu)$, where

$$R(p,\mu) = \frac{2p\mu + 2\mu - 3 + \sqrt{(2p\mu + 2\mu - 3)^2 + 4(1+p)(p-2p\mu)}}{2(1+p)}.$$

Remark 2. Corollary 3.4 corrects the mistakes of Theorem 3.8 which was obtained by Patel [5].

Theorem 3.5. *Let*

$$\mu \geqq 0,\ \lambda \geqq 0 \quad \textit{and} \quad -1 \leqq B < A \leqq 1.$$

If

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (z \in \mathbb{U})$$

satisfies the following subordination condition:

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \lambda\left[\frac{zf''(z)}{f'(z)} + (1-\mu)\left(1 - \frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1+Az}{1+Bz}, \tag{3.10}$$

then

$$|a_{n+1}| \leqq \frac{A-B}{(1+n\lambda)(n+\mu)} \qquad (n \in \mathbb{N}). \tag{3.11}$$

Proof. Suppose that

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (z \in \mathbb{U})$$

satisfies (3.10). It follows that

$$\begin{aligned} f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} &+ \lambda\left[\frac{zf''(z)}{f'(z)} + (1-\mu)\left(1 - \frac{zf'(z)}{f(z)}\right)\right] \\ &= 1 + (1+n\lambda)(n+\mu)a_{n+1}z^n + \cdots \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}). \end{aligned} \tag{3.12}$$

Therefore, we find from Lemma 2.4, (3.12) and $-1 \leqq B < A \leqq 1$ that

$$|(1+n\lambda)(n+\mu)a_{n+1}| \leqq A - B. \tag{3.13}$$

The assertion (3.11) of Theorem 3.5 can now easily be derived from (3.13). □

Taking $A = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$ and $B = -1$ in Theorem 3.5, we get the following result.

Corollary 3.6. *Let*

$$\mu \geqq 0,\ \lambda \geqq 0 \quad \textit{and} \quad 0 \leqq \alpha < 1.$$

If

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

satisfies the following inequality:

$$\Re\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \lambda\left[\frac{zf''(z)}{f'(z)} + (1-\mu)\left(1 - \frac{zf'(z)}{f(z)}\right)\right]\right) > \alpha,$$

then

$$|a_{n+1}| \leqq \frac{2(1-\alpha)}{(1+n\lambda)(n+\mu)} \qquad (n \in \mathbb{N}).$$

Remark 3. Corollary 3.6 provides an extension of the corresponding result obtained by Guo and Liu [2].

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