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AN OSTROWSKI TYPE INEQUALITY FOR $p-$NORMS

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ABSTRACT. In this paper, we establish general form of an inequality of Ostrowski type for twice differentiable mappings in terms of L_p-norm, with first derivative absolutely continuous. The integral inequality of similar type already pointed out in literature is a special case of ours. The already established inequality contains a mistake and as a result incorrect consequences and applications. The corrected version of the inequality is pointed out and the inequality is also applied to special means and numerical integration.

Key words and phrases: Ostrowski inequality, Numerical integration, Special means.

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1. INTRODUCTION

We establish here the general form of an inequality of Ostrowski type, different to that of Cerone, Dragomir and Roumeliotis [1], for twice differentiable mappings in terms of L_p-norm. The integral inequality of similar type already pointed out by N.S. Barnett, P. Cerone, S.S. Dragomir, J. Roumeliotis and A. Sofo [2], contains a mistake which has already been reported by N.A. Mir and A. Rafiq in their research work [3]. The same mistake has been carried out in their other research article, namely Theorem 20 of [2] and as a result incorrect consequences and applications of this theorem. The corrected form of the theorem is as follows:

Theorem 1.1. *Let $g : [a, b] \longrightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$. If we assume that the second derivative $g'' \in L_p(a, b), 1 < p < \infty$, then we have the*

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inequality

$$\text{(1.1)}\quad \left|\int_a^b g(t)dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2}\right](b-a) + \frac{1}{2}(b-a)\left(x - \frac{a+b}{2}\right)g'(x)\right|$$
$$\leq \frac{1}{2}\left(\frac{b-a}{2}\right)^{2+\frac{1}{q}} \|g''\|_p$$
$$\times \begin{cases} [B(q+1,q+1) + B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1)]^{\frac{1}{q}} & \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ [B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)]^{\frac{1}{q}} & \text{for } x \in (\frac{a+b}{2}, b], \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $q > 1$, *and* $B(\cdot,\cdot)$ *is the Beta function of Euler given by*

$$B(l,s) = \int_0^1 t^{l-1}(1-t)^{s-1}dt, \quad l,s > 0.$$

Further

$$B_r(l,s) = \int_0^r t^{l-1}(1-t)^{s-1}dt$$

is the incomplete Beta function,

$$\Psi_r(l,s) = \int_0^r t^{l-1}(1+t)^{s-1}dt$$

is the real positive valued integral,

$$x_1 = \frac{2(x-a)}{b-a}, \quad x_2 = 1 - x_1, \quad x_3 = x_1 - 1, \quad x_4 = 2 - x_1$$

and

$$\|g''\|_p := \left(\int_a^b |g''(t)|^p\, dt\right)^{\frac{1}{p}}.$$

If we assume that $g'' \in L_1(a,b)$, *then we have*

$$\text{(1.2)}\quad \left|\int_a^b g(t)dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2}\right](b-a) + \frac{1}{2}(b-a)\left(x - \frac{a+b}{2}\right)g'(x)\right| \leq \frac{\|g''\|_1}{8}(b-a)^2,$$

where

$$\|g''\|_1 := \int_a^b |g''(t)|\, dt.$$

2. MAIN RESULTS

The following theorem is now proved and subsequently applied to numerical integration and special means.

Theorem 2.1. *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$. *If we assume that the second derivative* $g'' \in L_p(a,b), 1 < p < \infty$, *then we have the* *inequality*

$$\begin{aligned}
(2.1)\quad & \left|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)\right. \\
&\quad -\frac{1}{2}g(x)-\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right. \\
&\qquad \left.\left.+\ \frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\right| \\
\leq \left(\frac{b-a}{2}\right)^{2+\frac{1}{q}}\|g''\|_p & \begin{cases}
\left[\left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B(q+1,q+1)+\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_1}(q+1,q+1)\right. \\
\qquad \left.+\left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}} \quad \text{for } x\in\left[a,\frac{a+b}{2}\right], \\
\left[\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B(q+1,q+1)+\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_3}(q+1,q+1)\right. \\
\qquad \left.+\left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}} \quad \text{for } x\in\left(\frac{a+b}{2},b\right],
\end{cases}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $q>1$, *and* $B(\cdot,\cdot)$ *is the Beta function of Euler given by*

$$B(l,s)=\int_0^1 t^{l-1}(1-t)^{s-1}dt, l,s>0.$$

Further,

$$B_r(l,s)=\int_0^r t^{l-1}(1-t)^{s-1}dt$$

is the incomplete Beta function,

$$\Psi_r(l,s)=\int_0^r t^{l-1}(1+t)^{s-1}dt$$

is a real positive valued integral,

$$x_1=\frac{2(x-a)}{b-a},\quad x_2=1-x_1,\quad x_3=x_1-1,\quad x_4=2-x_1$$

and

$$\|g''\|_p:=\left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

If we assume that $g''\in L_1(a,b)$, *then we have*

$$\begin{aligned}
(2.2)\quad & \left|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)-\frac{1}{2}g(x)\right. \\
&\quad -\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right. \\
&\quad \left.\left.+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\right| \\
&\qquad \leq \frac{1}{2}\|g''\|_1\|K(x,t)\|_\infty,
\end{aligned}$$

where

$$\|g''\|_1=\int_a^b |g''(t)|\,dt,$$

and

$$\|K(x,t)\|_\infty = \frac{1}{\alpha+\beta}\max\left(\frac{\alpha}{x-a},\frac{\beta}{b-x}\right)\frac{(b-a)^2}{4} \quad \textit{for } x\in[a,b].$$

Proof. We begin by recalling the following integral equality proved by N.A. Mir and A. Rafiq [3] which is generalization of an integral equality proved by Dragomir and Wang [4].

$$\begin{aligned}(2.3)\quad &\left|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)-\frac{1}{2}g(x)\right.\\ &-\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right.\\ &\left.\left.+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\right|\\ &=\frac{1}{2}\left|\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt\right|\end{aligned}$$

whose left hand side is equivalent to that of (2.1). From the right hand side of (2.3) we have, by Hölder's inequality, that

$$\begin{aligned}&\left|\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt\right|\\ &\leq\left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}\\ &=\|g''\|_p\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}},\end{aligned}$$

and from (2.3) we obtain the inequality

$$\begin{aligned}(2.4)\quad &\left|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)\right.\\ &-\frac{1}{2}g(x)-\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right.\\ &\left.\left.+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\right|\\ &\leq\frac{1}{2}\|g''\|_p\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

From the right hand side of (2.4) we may define

$$\begin{aligned}I &:= \int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\\ &=\left(\frac{\alpha}{\alpha+\beta}\cdot\frac{1}{x-a}\right)^q\int_a^x (t-a)^q\left|t-\frac{a+b}{2}\right|^q dt\\ &\quad+\left(\frac{\beta}{\alpha+\beta}\cdot\frac{1}{b-x}\right)^q\int_x^b |t-b|^q\left|t-\frac{a+b}{2}\right|^q dt \qquad (2.5)\end{aligned}$$

such that we can identify two distinct cases.

(a) For $x \in \left[a, \frac{a+b}{2}\right]$

$$I_A = \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q \int_a^x (t-a)^q \left(\frac{a+b}{2}-t\right)^q dt + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \int_x^{\frac{a+b}{2}} (b-t)^q \left(\frac{a+b}{2}-t\right)^q dt + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \int_{\frac{a+b}{2}}^b (b-t)^q \left(t-\frac{a+b}{2}\right)^q dt.$$

Investigating the three separate integrals, we may evaluate as follows:

$$I_1 = \int_a^x (t-a)^q \left(\frac{a+b}{2}-t\right)^q dt,$$

making the change of variable $t = a + \left(\frac{b-a}{2}\right) w$, we arrive at

$$I_1 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_1} w^q (1-w)^q dw, = \left(\frac{b-a}{2}\right)^{2q+1} Bx_1(q+1,q+1),$$

where $B_{x_1}(\cdot,\cdot)$ is the incomplete Beta function and $x_1 = \frac{2(x-a)}{b-a}$.

$$I_2 = \int_x^{\frac{a+b}{2}} (b-t)^q \left(\frac{a+b}{2}-t\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} - \left(\frac{b-a}{2}\right) w$, we obtain

$$I_2 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_2} w^q (1+w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} \Psi_{x_2}(q+1,q+1),$$

where

$$\Psi_{x_2} := \int_0^{x_2} w^q (1+w)^q dw$$

and $x_2 = \frac{a+b-2x}{b-a} = 1 - x_{1.}$

$$I_3 = \int_{\frac{a+b}{2}}^b (b-t)^q \left(t-\frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right) w$, we get

$$I_3 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^1 w^q (1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B(q+1,q+1),$$

where $B(\cdot,\cdot)$ is the Beta function.

We may now write

$$I_A = I_1 + I_2 + I_3 = \left(\frac{b-a}{2}\right)^{2q+1} \left[\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q Bx_1(q+1,q+1) + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \Psi_{x_2}(q+1,q+1) + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B(q+1,q+1)\right]$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

(b) For $x \in \left(a, \frac{a+b}{2}\right]$

$$I_B = \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q \int_a^{\frac{a+b}{2}} (t-a)^q \left(\frac{a+b}{2}-t\right)^q dt + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q \int_{\frac{a+b}{2}}^{x} (t-a)^q \left(t-\frac{a+b}{2}\right)^q dt + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \int_x^b (b-t)^q \left(t-\frac{a+b}{2}\right)^q dt.$$

In a similar fashion to the previous case, we have

$$I_4 = \int_a^{\frac{a+b}{2}} (t-a)^q \left(\frac{a+b}{2}-t\right)^q dt.$$

Letting $t = a + \left(\frac{b-a}{2}\right) w$, we obtain

$$I_4 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^1 w^q(1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B(q+1,q+1),$$

where $B\left(\cdot,\cdot\right)$ is the Beta function.

$$I_5 = \int_{\frac{a+b}{2}}^{x} (t-a)^q \left(t-\frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right) w$, we arrive at

$$I_5 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_3} w^q(1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B_{x_3}(q+1,q+1),$$

where $B_{x_3}\left(\cdot,\cdot\right)$ is the incomplete Beta function and $x_3 = x_1 - 1$.

$$I_6 = \int_x^b (b-t)^q \left(t-\frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = b - \left(\frac{b-a}{2}\right) w$, we get

$$I_6 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_4} w^q(1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B_{x_4}(q+1,q+1),$$

where $B_{x_4}\left(\cdot,\cdot\right)$ is the incomplete Beta function and $x_4 = 2 - x_1$.

$$\begin{aligned} I_B &= I_4 + I_5 + I_6 \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \left[\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B(q+1,q+1) + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_3}(q+1,q+1) \right. \\ &\qquad \left. + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B_{x_4}(q+1,q+1)\right] \end{aligned}$$

for $x \in (\frac{a+b}{2}, b]$.

Also from (2.5)

$$I = I_A + I_B$$

$$= \left(\frac{b-a}{2}\right)^{2q+1} \begin{cases} \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_1}(q+1,q+1) + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \Psi_{x_2}(q+1,q+1) \\ \qquad + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B(q+1,q+1) \text{ for } x \in \left[a, \frac{a+b}{2}\right], \\ \\ \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B(q+1,q+1) + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_3}(q+1,q+1) \\ \qquad + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B_{x_4}(q+1,q+1) \text{ for } x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Using (2.4), we obtain the result (2.1). Using the inequality (2.3), we can also state that

$$\Bigg| \frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt + \frac{\beta}{b-x}\int_x^b g(t)dt\right) - \frac{1}{2}g(x) - \frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right) + \frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right) - (\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\Bigg| \le \frac{1}{2}\left\|g''\right\|_1 \left\|K(x,t)\right\|_\infty,$$

where

$$\left\|K(x,t)\right\|_\infty = p(x,t)\left(t-\frac{a+b}{2}\right).$$

As it is easy to see that

$$\left\|K(x,t)\right\|_\infty = \frac{1}{\alpha+\beta}\cdot\max\left(\frac{\alpha}{x-a},\frac{\beta}{b-x}\right)\cdot\frac{(b-a)^2}{4} \quad \text{for } x \in [a,b],$$

we deduce (2.2). □

Remark 2.2. Putting $\alpha = x-a$ and $\beta = b-x$ in (2.1) and (2.2), we get the inequalities (1.1) and (1.2).

Remark 2.3. Simple manipulation of (2.1) will allow for the corrected result of (1.1) and (1.2), owing to a missing factor of $\frac{1}{2}$ in the third term of the original result (1.1) of the Barnett, Cerone, Dragomir, Roumeliotis and Sofo, this will not be done here.

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