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SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY IN $|z| < \frac{1}{2}$

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Abstract: For analytic functions $f(z)$ with $f(0) = f'(0) - 1 = 0$ in the open unit disc \mathbb{E} , T. H. MacGregor has considered some conditions for $f(z)$ to be starlike or convex. The object of the present paper is to discuss some interesting problems for $f(z)$ to be starlike or convex for $|z| < \frac{1}{2}$.

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Conditions for Starlikeness and Convexity

Mamoru Nunokawa, Shigeyoshi Owa,
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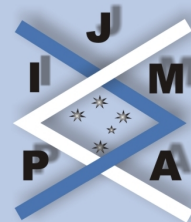
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1. Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike with respect to the origin in \mathbb{E} if it satisfies

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{E}).$$

Also, a function $f \in \mathcal{A}$ is called as convex in \mathbb{E} if it satisfies

$$\operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{E}).$$

MacGregor [2] has shown the following.

Theorem A. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{f(z)}{z} - 1 \right| < 1 \quad (z \in \mathbb{E}),$$

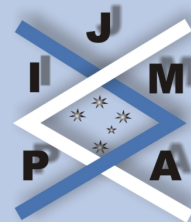
then

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1 \quad \left(|z| < \frac{1}{2} \right)$$

so that

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > 0 \quad \left(|z| < \frac{1}{2} \right).$$

Therefore, $f(z)$ is univalent and starlike for $|z| < \frac{1}{2}$.



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Also, MacGregor [3] had given the following results.

Theorem B. *If $f \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 1 \quad (z \in \mathbb{E}),$$

then

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad \text{for } |z| < \frac{1}{2}.$$

Therefore, $f(z)$ is convex for $|z| < \frac{1}{2}$.

Theorem C. *If $f \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 1 \quad (z \in \mathbb{E}),$$

then $f(z)$ maps $|z| < \frac{2\sqrt{5}}{5} = 0.8944\dots$ onto a domain which is starlike with respect to the origin,

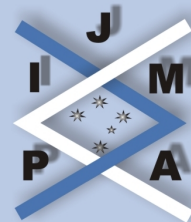
$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \quad \text{for } |z| < \frac{2\sqrt{5}}{5}$$

or

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{for } |z| < \frac{2\sqrt{5}}{5}.$$

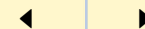
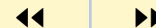
The condition domains of Theorem A, Theorem B and Theorem C are some circular domains whose center is the point $z = 1$.

It is the purpose of the present paper to obtain some sufficient conditions for starlikeness or convexity under the hypotheses whose condition domains are annular domains centered at the origin.



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2. Starlikeness and Convexity

We start with the following result for starlikeness of functions $f(z)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ and suppose that*

$$(2.1) \quad \begin{aligned} 0.10583 \dots &= \exp\left(-\frac{\pi^2}{4 \log 3}\right) \\ &< \left| \frac{zf'(z)}{f(z)} \right| \\ &< \exp\left(\frac{\pi^2}{4 \log 3}\right) = 9.44915 \dots \quad (z \in \mathbb{E}). \end{aligned}$$

Then $f(z)$ is starlike for $|z| < \frac{1}{2}$.

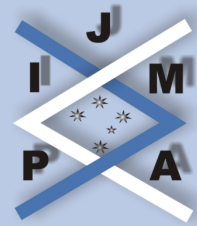
Proof. From the assumption (2.1), we get

$$f(z) \neq 0 \quad (0 < |z| < 1).$$

From the harmonic function theory (cf. Duren [1]), we have

$$\begin{aligned} \log \left(\frac{zf'(z)}{f(z)} \right) &= \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \frac{\zeta+z}{\zeta-z} d\varphi + i \arg \left(\frac{zf'(z)}{f(z)} \right)_{z=0} \\ &= \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{zf'(\zeta)}{f(\zeta)} \right| \right) \frac{\zeta+z}{\zeta-z} d\varphi \end{aligned}$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.



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It follows that

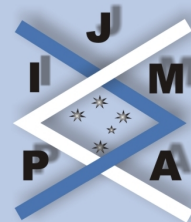
$$\begin{aligned} \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| &= \left| \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \left(\operatorname{Im} \frac{\zeta + z}{\zeta - z} \right) d\varphi \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right| \left| \frac{2Rr \sin(\varphi - \theta)}{R^2 - 2Rr \cos(\varphi - \theta) + r^2} \right| d\varphi \\ &< \frac{\pi^2}{4 \log 3} \frac{1}{2\pi} \int_0^{2\pi} \frac{2Rr |\sin(\varphi - \theta)|}{R^2 - 2Rr \cos(\varphi - \theta) + r^2} d\varphi \\ &= \frac{\pi^2}{4 \log 3} \frac{2}{\pi} \log \frac{R+r}{R-r}. \end{aligned}$$

Letting $R \rightarrow 1$, we have

$$\begin{aligned} \left| \arg \frac{zf'(z)}{f(z)} \right| &< \frac{\pi}{2 \log 3} \log \frac{1+r}{1-r} \\ &< \frac{\pi}{2 \log 3} \log 3 \\ &= \frac{\pi}{2} \quad \left(|z| = r < \frac{1}{2} \right). \end{aligned}$$

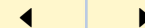
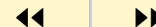
This completes the proof of the theorem. □

Next we derive the following



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Theorem 2.2. Let $f \in \mathcal{A}$ and suppose that

$$(2.2) \quad \begin{aligned} 0.472367 \dots &= \exp\left(-\frac{3}{4}\right) \\ &< \left| \frac{f(z)}{z} \right| \\ &< \exp\left(\frac{3}{4}\right) = 2.177 \dots \quad (z \in \mathbb{E}). \end{aligned}$$

Then we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad \left(|z| < \frac{1}{2} \right),$$

or $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Proof. From the assumption (2.2), we have

$$f(z) \neq 0 \quad (0 < |z| < 1).$$

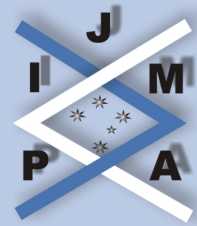
Applying the harmonic function theory (cf. Duren [1]), we have

$$\log \left(\frac{f(z)}{z} \right) = \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{\zeta + z}{\zeta - z} d\varphi,$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.

Then, it follows that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{2\zeta z}{(\zeta - z)^2} d\varphi.$$



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This gives us

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &\leq \frac{1}{2\pi} \int_{|\zeta|=R} \left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \frac{2Rr}{R^2 - 2Rr \cos(\varphi - \theta) + r^2} d\varphi \\ &< \frac{3}{4} \frac{1}{2\pi} \int_{|\zeta|=R} \frac{2Rr}{R^2 - 2Rr \cos(\varphi - \theta) + r^2} d\varphi \\ &= \frac{3}{4} \frac{2Rr}{R^2 - r^2}. \end{aligned}$$

Making $R \rightarrow 1$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{3}{4} \frac{2r}{1 - r^2} < 1 \quad \left(|z| = r < \frac{1}{2} \right),$$

which completes the proof of the theorem. \square

For convexity of functions $f(z)$, we show the following corollary without the proof.

Corollary 2.3. *Let $f \in \mathcal{A}$ and suppose that*

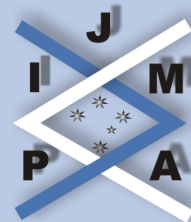
$$(2.3) \quad 0.472367 \dots = \exp\left(-\frac{3}{4}\right) < |f'(z)| < \exp\left(\frac{3}{4}\right) = 2.117 \dots \quad (z \in \mathbb{E}).$$

Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Next our result for the convexity of functions $f(z)$ is contained in

Theorem 2.4. *Let $f \in \mathcal{A}$ and suppose that*

$$(2.4) \quad 0.778801 \dots = \exp\left(-\frac{1}{4}\right) < \left| \frac{zf'(z)}{f(z)} \right| < \exp\left(\frac{1}{4}\right) = 1.28403 \dots \quad (z \in \mathbb{E}).$$



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Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Proof. From the condition (2.4) of the theorem, we have

$$\frac{zf'(z)}{f(z)} \neq 0 \quad \text{in } \mathbb{E}.$$

Then, it follows that

$$(2.5) \quad \log \frac{zf'(z)}{f(z)} = \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \frac{\zeta f'(\zeta)}{f(\zeta)} \right) \frac{\zeta + z}{\zeta - z} d\varphi,$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.

Differentiating (2.5) and multiplying by z , we obtain that

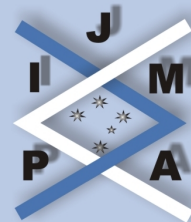
$$1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \frac{2\zeta z}{(\zeta - z)^2} d\varphi.$$

In view of Theorem 2.1, $f(z)$ is starlike for $|z| < \frac{1}{2}$ and therefore, we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{1-r}{1+r} \quad \left(|z| = r < \frac{1}{2} \right).$$

Then, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &= \operatorname{Re} \frac{zf'(z)}{f(z)} + \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \left(\operatorname{Re} \frac{2\zeta z}{(\zeta - z)^2} \right) d\varphi \\ &> \frac{1-r}{1+r} - \frac{1}{2\pi} \int_{|\zeta|=R} \frac{1}{4} \frac{2Rr}{|\zeta - z|^2} d\varphi \\ &= \frac{1-r}{1+r} - \frac{1}{4} \frac{2Rr}{R^2 - r^2}. \end{aligned}$$



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Letting $R \rightarrow 1$, we see that

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &> \frac{1-r}{1+r} - \frac{1}{4} \frac{2r}{1-r^2} \\ &= \frac{1}{3} - \frac{1}{4} \cdot \frac{4}{3} \\ &= 0 \quad \left(|z| = r < \frac{1}{2} \right), \end{aligned}$$

which completes the proof of our theorem. □

Finally, we prove

Theorem 2.5. *Let $f \in \mathcal{A}$ and suppose that*

$$\begin{aligned} 0.10583\dots &= \exp\left(-\frac{\pi^2}{4\log 3}\right) \\ &< \left| \frac{zf'(z)}{f(z)} \right| < \exp\left(\frac{\pi^2}{4\log 3}\right) = 9.44915\dots \quad (z \in \mathbb{E}). \end{aligned}$$

Then $f(z)$ is convex in $|z| < r_0$ where r_0 is the root of the equation

$$\begin{aligned} (4\log 3)r^2 - 2(4\log 3 + \pi^2)r + 4\log 3 &= 0, \\ r_0 &= \frac{\pi^2 - 4\log 3 - \pi\sqrt{\pi^2 + 8\log 3}}{4\log 3} = 0.15787\dots \end{aligned}$$

Proof. Applying the same method as the proof of Theorem 2.5, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &= \operatorname{Re} \frac{zf'(z)}{f(z)} + \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \left(\operatorname{Re} \frac{2\zeta z}{(\zeta - z)^2} \right) d\varphi \\ &> \frac{1-r}{1+r} - \frac{\pi^2}{4\log 3} \frac{2Rr}{R^2 - r^2} \end{aligned}$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.

Putting $R \rightarrow 1$, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &> \frac{1-r}{1+r} - \frac{\pi^2}{4 \log 3} \frac{2r}{1-r^2} \\ &= \frac{1}{(1-r^2)4 \log 3} \left\{ (4 \log 3)r^2 - 2(4 \log 3 + \pi^2)r + 4 \log 3 \right\} \\ &> 0 \quad (|z| < r_0). \end{aligned}$$

□

Remark 1. The condition in Theorem A by MacGregor [2] implies that

$$0 < \operatorname{Re} \left(\frac{f(z)}{z} \right) < 2 \quad (z \in \mathbb{E}).$$

However, the condition in Theorem 2.2 implies that

$$-2.117 \dots < \operatorname{Re} \left(\frac{f(z)}{z} \right) < 2.117 \dots \quad (z \in \mathbb{E}).$$

Furthermore, the condition in Theorem B by MacGregor [3] implies that

$$0 < \operatorname{Re} f'(z) < 2 \quad (z \in \mathbb{E}).$$

However, the condition in Corollary 2.3 implies that

$$-2.117 \dots < \operatorname{Re} f'(z) < 2.117 \dots \quad (z \in \mathbb{E}).$$



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