SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY IN $|z| < \frac{1}{2}$

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Abstract: For analytic functions $f(z)$ with $f(0) = f'(0)-1 = 0$ in the open unit disc $\mathbb{E}$, T. H. MacGregor has considered some conditions for $f(z)$ to be starlike or convex. The object of the present paper is to discuss some interesting problems for $f(z)$ to be starlike or convex for $|z| < \frac{1}{2}$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike with respect to the origin in $\mathbb{E}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{E}).$$

Also, a function $f \in \mathcal{A}$ is called as convex in $\mathbb{E}$ if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{E}).$$

MacGregor [2] has shown the following.

Theorem A. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{f(z)}{z} - 1\right| < 1 \qquad (z \in \mathbb{E}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad \left(|z| < \frac{1}{2}\right)$$

so that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad \left(|z| < \frac{1}{2}\right).$$

Therefore, $f(z)$ is univalent and starlike for $|z| < \frac{1}{2}$.

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Also, MacGregor [3] had given the following results.

Theorem B. *If $f \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 1 \qquad (z \in \mathbb{E}),$$

then

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad \text{for } |z| < \frac{1}{2}.$$

Therefore, $f(z)$ is convex for $|z| < \frac{1}{2}$.

Theorem C. *If $f \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 1 \qquad (z \in \mathbb{E}),$$

then $f(z)$ maps $|z| < \frac{2\sqrt{5}}{5} = 0.8944\ldots$ onto a domain which is starlike with respect to the origin,

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2} \qquad \text{for } |z| < \frac{2\sqrt{5}}{5}$$

or

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \qquad \text{for } |z| < \frac{2\sqrt{5}}{5}.$$

The condition domains of Theorem A, Theorem B and Theorem C are some circular domains whose center is the point $z = 1$.

It is the purpose of the present paper to obtain some sufficient conditions for starlikeness or convexity under the hypotheses whose condition domains are annular domains centered at the origin.

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2. Starlikeness and Convexity

We start with the following result for starlikeness of functions $f(z)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ and suppose that*

$$0.10583\cdots = \exp\left(-\frac{\pi^2}{4\log 3}\right) < \left|\frac{zf'(z)}{f(z)}\right| < \exp\left(\frac{\pi^2}{4\log 3}\right) = 9.44915\ldots \qquad (z \in \mathbb{E}). \tag{2.1}$$

Then $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Proof. From the assumption (2.1), we get

$$f(z) \neq 0 \qquad (0 < |z| < 1).$$

From the harmonic function theory (cf. Duren [1]), we have

$$\log\left(\frac{zf'(z)}{f(z)}\right) = \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\frac{\zeta+z}{\zeta-z}d\varphi + i\arg\left(\frac{zf'(z)}{f(z)}\right)_{z=0}$$
$$= \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{zf'(\zeta)}{f(\zeta)}\right|\right)\frac{\zeta+z}{\zeta-z}d\varphi$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.

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It follows that

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &= \left|\frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\left(\operatorname{Im}\frac{\zeta+z}{\zeta-z}\right)d\varphi\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right|\left|\frac{2Rr\sin(\varphi-\theta)}{R^2-2Rr\cos(\varphi-\theta)+r^2}\right|d\varphi \\ &< \frac{\pi^2}{4\log 3}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{2Rr\,|\sin(\varphi-\theta)|}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi \\ &= \frac{\pi^2}{4\log 3}\,\frac{2}{\pi}\log\frac{R+r}{R-r}.\end{aligned}$$

Letting $R \to 1$, we have

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &< \frac{\pi}{2\log 3}\log\frac{1+r}{1-r} \\ &< \frac{\pi}{2\log 3}\log 3 \\ &= \frac{\pi}{2} \qquad \left(|z| = r < \frac{1}{2}\right).\end{aligned}$$

This completes the proof of the theorem. □

Next we derive the following

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Theorem 2.2. *Let $f \in \mathcal{A}$ and suppose that*

$$\begin{aligned} 0.472367\ldots = \exp\left(-\frac{3}{4}\right) &< \left|\frac{f(z)}{z}\right| \\ &< \exp\left(\frac{3}{4}\right) = 2.177\ldots \qquad (z \in \mathbb{E}). \end{aligned} \tag{2.2}$$

Then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad \left(|z| < \frac{1}{2}\right),$$

or $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Proof. From the assumption (2.2), we have

$$f(z) \neq 0 \qquad (0 < |z| < 1).$$

Applying the harmonic function theory (cf. Duren [1]), we have

$$\log\left(\frac{f(z)}{z}\right) = \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{f(\zeta)}{\zeta}\right|\right)\frac{\zeta + z}{\zeta - z}d\varphi,$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.
Then, it follows that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{f(\zeta)}{\zeta}\right|\right)\frac{2\zeta z}{(\zeta - z)^2}d\varphi.$$

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This gives us

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{1}{2\pi}\int_{|\zeta|=R}\left|\log\left|\frac{f(\zeta)}{\zeta}\right|\right|\frac{2Rr}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi$$
$$< \frac{3}{4}\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2Rr}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi$$
$$= \frac{3}{4}\frac{2Rr}{R^2-r^2}.$$

Making $R \to 1$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \frac{3}{4}\frac{2r}{1-r^2} < 1 \qquad \left(|z| = r < \frac{1}{2}\right),$$

which completes the proof of the theorem. □

For convexity of functions $f(z)$, we show the following corollary without the proof.

Corollary 2.3. *Let $f \in \mathcal{A}$ and suppose that*

$$0.472367\cdots = \exp\left(-\frac{3}{4}\right) < |f'(z)| < \exp\left(\frac{3}{4}\right) = 2.117\ldots \qquad (z \in \mathbb{E}). \tag{2.3}$$

Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Next our result for the convexity of functions $f(z)$ is contained in

Theorem 2.4. *Let $f \in \mathcal{A}$ and suppose that*
(2.4)

$$0.778801\cdots = \exp\left(-\frac{1}{4}\right) < \left|\frac{zf'(z)}{f(z)}\right| < \exp\left(\frac{1}{4}\right) = 1.28403\ldots \qquad (z \in \mathbb{E}).$$

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Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Proof. From the condition (2.4) of the theorem, we have

$$\frac{zf'(z)}{f(z)} \neq 0 \qquad \text{in } \mathbb{E}.$$

Then, it follows that

$$\log \frac{zf'(z)}{f(z)} = \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \frac{\zeta f'(\zeta)}{f(\zeta)} \right) \frac{\zeta + z}{\zeta - z} d\varphi, \tag{2.5}$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.
Differentiating (2.5) and multiplying by z, we obtain that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \frac{2\zeta z}{(\zeta - z)^2} d\varphi.$$

In view of Theorem 2.1, $f(z)$ is starlike for $|z| < \frac{1}{2}$ and therefore, we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{1-r}{1+r} \qquad \left(|z| = r < \frac{1}{2} \right).$$

Then, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &= \operatorname{Re} \frac{zf'(z)}{f(z)} + \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \left(\operatorname{Re} \frac{2\zeta z}{(\zeta - z)^2} \right) d\varphi \\ &> \frac{1-r}{1+r} - \frac{1}{2\pi} \int_{|\zeta|=R} \frac{1}{4} \frac{2Rr}{|\zeta - z|^2} d\varphi \\ &= \frac{1-r}{1+r} - \frac{1}{4} \frac{2Rr}{R^2 - r^2}. \end{aligned}$$

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Letting $R \to 1$, we see that

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &> \frac{1-r}{1+r} - \frac{1}{4} \frac{2r}{1-r^2} \\ &= \frac{1}{3} - \frac{1}{4} \cdot \frac{4}{3} \\ &= 0 \qquad \left(|z| = r < \frac{1}{2} \right), \end{aligned}$$

which completes the proof of our theorem. □

Finally, we prove

Theorem 2.5. *Let $f \in \mathcal{A}$ and suppose that*

$$\begin{aligned} 0.10583\ldots = \exp\left(-\frac{\pi^2}{4\log 3} \right) \\ < \left| \frac{zf'(z)}{f(z)} \right| < \exp\left(\frac{\pi^2}{4\log 3} \right) = 9.44915\ldots \qquad (z \in \mathbb{E}). \end{aligned}$$

Then $f(z)$ is convex in $|z| < r_0$ where r_0 is the root of the equation

$$(4\log 3)r^2 - 2(4\log 3 + \pi^2)r + 4\log 3 = 0,$$

$$r_0 = \frac{\pi^2 - 4\log 3 - \pi\sqrt{\pi^2 + 8\log 3}}{4\log 3} = 0.15787\ldots.$$

Proof. Applying the same method as the proof of Theorem 2.5, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &= \operatorname{Re} \frac{zf'(z)}{f(z)} + \frac{1}{2\pi} \int_{|\zeta|=R} \left(\log \left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \right) \left(\operatorname{Re} \frac{2\zeta z}{(\zeta - z)^2} \right) d\varphi \\ &> \frac{1-r}{1+r} - \frac{\pi^2}{4\log 3} \frac{2Rr}{R^2 - r^2} \end{aligned}$$

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where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.
Putting $R \to 1$, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} &> \frac{1-r}{1+r} - \frac{\pi^2}{4\log 3} \frac{2r}{1-r^2} \\ &= \frac{1}{(1-r^2)4\log 3} \left\{ (4\log 3)r^2 - 2(4\log 3 + \pi^2)r + 4\log 3 \right\} \\ &> 0 \qquad (|z| < r_0). \end{aligned}$$

□

Remark 1. The condition in Theorem A by MacGregor [2] implies that

$$0 < \operatorname{Re}\left(\frac{f(z)}{z}\right) < 2 \qquad (z \in \mathbb{E}).$$

However, the condition in Theorem 2.2 implies that

$$-2.117\cdots < \operatorname{Re}\left(\frac{f(z)}{z}\right) < 2.117\ldots \qquad (z \in \mathbb{E}).$$

Furthermore, the condition in Theorem B by MacGregor [3] implies that

$$0 < \operatorname{Re} f'(z) < 2 \qquad (z \in \mathbb{E}).$$

However, the condition in Corollary 2.3 implies that

$$-2.117\cdots < \operatorname{Re} f'(z) < 2.117\ldots \qquad (z \in \mathbb{E}).$$

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References

[1] P. DUREN, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics **156**, Cambridge Univ. Press, 2004.

[2] T.H. MacGREGOR, The radius of univalence of certain analytic functions. II, *Proc. Amer. Math. Soc.*, **14**(3) (1963), 521–524.

[3] T.H. MacGREGOR, A class of univalent functions, *Proc. Amer. Math. Soc.*, **15** (1964), 311–317.

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