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NORMIC INEQUALITY OF TWO-DIMENSIONAL VISCOSITY OPERATOR

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ABSTRACT. A relation between the coefficients of Legendre expansions of two-dimensional function and those for the derived function is given. With this relation the normic inequality of two-dimensional viscosity operator is obtained.

Key words and phrases: Legendre expansion, Two-dimension, Viscosity operator.

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1. INTRODUCTION

Spectral methods employ various orthogonal systems of infinitely differentiable functions to represent an approximate projection of the exact solution sought. The resulting high accuracy of spectral algorithms was a major motivation behind their rapid development in the past three decades, e.g., see Gottlieb and Orszag [1] and Guo [2] .

For nonperiodic problems, it is natural to use Legendre spectral methods or Legendre pseudospectral methods. More attention has been paid to these two methods recently due to the appearance of the Fast Legendre Transformation. In studying the spectral methods for nonlinear conservation laws, we have to face those equations whose solutions may develop spontaneous jump discontinuities, i.e., shock waves. To overcome these difficulties, the spectral viscosity (SV) method was introduced by Tadmor [3] . Maday, Ould Kaber and Tadmor [4] firstly considered the nonperiodic Legendre pseudospectral viscosity method for an initial-boundary value problem, and Ma [5] , Guo, Ma and Tadmor [6] recently developed the nonperiodic Chebyshev-Legendre approximation. So far, however, few works have been done in multiple dimensions. This paper will study the relation between the coefficients of Legendre expansions of the two-dimensional function and those for the derived function, and gives the normic inequality of the two-dimensional SV operator, which plays important roles in the SV method.

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2. Notations and Lemmas

Let $x = (x_1, x_2), \Lambda = (-1, 1)^2$. We define the space $L^p(\Lambda)$ and its norm $\|\cdot\|_{L^p}$ in the usual way. If $p = 2$, we denote the norm of space $L^2(\Lambda)$ by $\|\cdot\|$, that is

$$\|v\| = \left(\int_\Lambda |v(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Let $\mathcal{N}$ be the set of all non-negative integers and $\mathcal{P}_N$ be the set of all algebraic polynomials of degree at most N in all variables. Let $l = (l_1, l_2) \in \mathcal{N}^2$, $|l| = \max\{l_1, l_2\}$, the Legendre polynomial of degree l is $L_l(x) = L_{l_1}(x_1)L_{l_2}(x_2)$. The Legndre transformation of a function $v \in L^2(\Lambda)$ is

$$Sv(x) = \sum_{|l|=0}^{\infty} \hat{v}_l L_l(x),$$

with the Legendre coefficients

$$\hat{v}_l = \left(l_1 + \frac{1}{2}\right)\left(l_2 + \frac{1}{2}\right)\int_\Lambda v(x)L_l(x)\mathrm{d}x, \qquad |l| = 0, 1, \ldots.$$

Lemma 2.1 ([2]). *For any $\phi \in \mathcal{P}_N, 2 \le p \le \infty$ and $|k| = m$, we have*

$$\|\partial_x^k \phi\|_{L^p} \le cN^{2m}\|\phi\|_{L^p}.$$

Here, c is a generic positive constant independent of any function and N.

A viscosity operator Q is defined by

$$Qv(x) := \sum_{|l|=0}^{N} \hat{q}_l \hat{v}_l L_l(x), \quad v = \sum_{|l|=0}^{\infty} \hat{v}_l L_l(x). \tag{2.1}$$

Here, $\hat{q}_l$ are the so-called viscosity coefficients,

$$\begin{cases} \hat{q}_l = 0 & \text{for} \quad |l| \le m \\ \hat{q}_l \ge 1 - \frac{m^2}{l_1^2 + l_2^2} & \text{for} \quad m < |l| \le N. \end{cases}$$

Observe that the Q operator is activated by only the high mode numbers, $\ge m$. In particular, if we let $m \longrightarrow \infty$, the Q operator is spectrally small (in the sense that $\|Qv\|_{H^{-s}} \le cm^{-s}\|v\|$).

Let R denote the corresponding low modes filter

$$Rv(x) := \sum_{|l|=0}^{N} \hat{r}_l \hat{v}_l L_l(x), \qquad \text{here } \hat{r}_l = 1 - \hat{q}_l. \tag{2.2}$$

Clearly,

$$\begin{cases} \hat{r}_l = 1 & \text{for} \quad |l| \le m \\ \hat{r}_l \le \frac{m^2}{l_1^2 + l_2^2} & \text{for} \quad m < |l| \le N. \end{cases}$$

3. Main Results

Firstly , we consider the relation between the coefficients of Legendre expansions of $v(x)$ and those for $\partial_x v(x)$.

Theorem 3.1. *For any $v(x) \in H^1(\Lambda)$, let $\hat{v}_k, \hat{v}_k^{(1)}$ be the coefficients of Legendre expansions of $v(x), \partial_x v(x),\ |k| = 0, 1, 2, \dots,$*

$$\hat{v}_k^{(1)} = \hat{v}_{(k_1,k_2)}^{(1)} = (2k_1+1) \sum_{\substack{p=k_1+1 \\ k_1+p \ odd}}^{\infty} \hat{v}_{(p,k_2)} + (2k_2+1) \sum_{\substack{q=k_2+1 \\ k_2+q \ odd}}^{\infty} \hat{v}_{(k_1,q)}.$$

Proof. By the property of the one-dimensional Legendre polynomial:

$$(2k+1)L_k(x) = L'_{k+1}(x) - L'_{k-1}(x), \qquad k \geq 1,$$

we have

$$L'_k(x) = \sum_{\substack{l=0 \\ k+l \text{ odd}}}^{k-1} (2l+1)L_l(x), \quad k = 0, 1, 2, \dots.$$

Then, for $x = (x_1, x_2)$,

$$\begin{aligned}
\partial_x L_k(x) &= L'_{k_1}(x_1)L_{k_2}(x_2) + L_{k_1}(x_1)L'_{k_2}(x_2) \\
&= \sum_{\substack{l_1=0 \\ k_1+l_1 \text{ odd}}}^{k_1-1} (2l_1+1)L_{l_1}(x_1)L_{k_2}(x_2) + \sum_{\substack{l_2=0 \\ k_2+l_2 \text{ odd}}}^{k_2-1} (2l_2+1)L_{l_2}(x_2)L_{k_1}(x_1);
\end{aligned}$$

$$\begin{aligned}
\partial_x v &= \sum_{|k|=0}^{\infty} \hat{v}_k \partial_x L_k(x) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \hat{v}_{(k_1,k_2)} \left[\sum_{\substack{l_1=0 \\ k_1+l_1 \text{ odd}}}^{k_1-1} (2l_1+1)L_{l_1}(x_1)L_{k_2}(x_2) \right. \\
&\qquad \left. + \sum_{\substack{l_2=0 \\ k_2+l_2 \text{ odd}}}^{k_2-1} (2l_2+1)L_{l_2}(x_2)L_{k_1}(x_1) \right].
\end{aligned}$$

On the other hand,

$$\partial_x v = \sum_{|k|=0}^{\infty} \hat{v}_k^{(1)} L_k(x) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \hat{v}_{(k_1,k_2)}^{(1)} L_{k_1}(x_1)L_{k_2}(x_2).$$

We can obtain

$$\hat{v}_{(k_1,k_2)}^{(1)} = (2k_1+1) \sum_{\substack{p=k_1+1 \\ k_1+p \text{ odd}}}^{\infty} \hat{v}_{(p,k_2)} + (2k_2+1) \sum_{\substack{q=k_2+1 \\ k_2+q \text{ odd}}}^{\infty} \hat{v}_{(k_1,q)}.$$

□

Specially, for any $v \in \mathcal{P}_N$, we have

$$\hat{v}_{(k_1,k_2)}^{(1)} = (2k_1+1) \sum_{\substack{p=k_1+1 \\ k_1+p \text{ odd}}}^{N} \hat{v}_{(p,k_2)} + (2k_2+1) \sum_{\substack{q=k_2+1 \\ k_2+q \text{ odd}}}^{N} \hat{v}_{(k_1,q)}. \tag{3.1}$$

Let $J_{k,N} = \{j \big| k+1 \leq j \leq N, k+j \text{ odd}\}$, then

$$\hat{v}_{(k_1,k_2)}^{(1)} = (2k_1+1) \sum_{p \in J_{k_1,N}} \hat{v}_{(p,k_2)} + (2k_2+1) \sum_{q \in J_{k_2,N}} \hat{v}_{(k_1,q)} \qquad \forall\, v \in \mathcal{P}_N$$

Theorem 3.2. *Consider the SV operator $Q = Q_m$, (2.1) with the parametrization and operator R, (2.2). For any $\phi \in \mathcal{P}_N$, the following inequalities hold:*

$$\begin{aligned}\|\partial_x(R\phi)\|^2 &\le cm^4 \ln N \|\phi\|^2 \\ \|\partial_x\phi\|^2 &\le 2\|\partial_x(Q\phi)\|^2 + cm^4 \ln N\|\phi\|^2 \\ \|\partial_x(Q\phi)\|^2 &\le 2\|\partial_x\phi\|^2 + cm^4 \ln N\|\phi\|^2\end{aligned}$$

Proof. We decompose $\partial_x(R\phi(x)) = A_1(x) + A_2(x)$, where

$$A_1(x) := \partial_x\left(\sum_{|k|=0}^{m} \hat{r}_k\hat{\phi}_k L_k(x)\right), \quad A_2(x) := \partial_x\left(\sum_{|k|=m+1}^{N} \hat{r}_k\hat{\phi}_k L_k(x)\right).$$

By Lemma , $\|\partial_x\phi\| \le cN^2\|\phi\|, \ \forall\, \phi(x) \in \mathcal{P}_N$, and hence $\|A_1(x)\|^2 \le cm^4\|\phi\|^2$.

Further let $J^{(0)}_{k,N} = \{j \big| j \in J_{k,N}, j > m\}$, $J^{(d)}_{k,N} = \{j\big| j \in J_{k,N}, \max\{j, k_d\} > m\}$, here, $m \in \mathcal{N}, d = 1, 2$. Then

$$\begin{aligned}\|A_2(x)\|^2 &\le \sum_{|k|=0}^{N}\left[(2k_1+1)\sum_{p\in J^{(2)}_{k_1,N}} \hat{r}_{(p,k_2)}\hat{\phi}_{(p,k_2)} + (2k_2+1)\sum_{q\in J^{(1)}_{k_2,N}} \hat{r}_{(k_1,q)}\hat{\phi}_{(k_1,q)}\right]^2 \|L_k\|^2 \\ &\le 8\,(A_{2,1} + A_{2,2})\,,\end{aligned}$$

in which

$$\begin{aligned}A_{2,1} &= \sum_{k_1=0}^{N}\sum_{k_2=0}^{N}(2k_1+1)^2\left(\sum_{p\in J^{(2)}_{k_1,N}} \hat{r}_{(p,k_2)}\hat{\phi}_{(p,k_2)}\right)^2 \frac{1}{(2k_1+1)(2k_2+1)} \\ &= \sum_{k_1=0}^{N}\sum_{k_2=0}^{N}\frac{2k_1+1}{2k_2+1}\left(\sum_{p\in J^{(2)}_{k_1,N}} \hat{r}_{(p,k_2)}\hat{\phi}_{(p,k_2)}\right)^2 \\ &\le \sum_{k_1=0}^{N}\sum_{k_2=0}^{N}\frac{2k_1+1}{2k_2+1}\left(\sum_{p\in J^{(2)}_{k_1,N}} |\hat{r}_{(p,k_2)}|^2\|L_{(p,k_2)}\|^{-2}\right)\left(\sum_{p\in J^{(2)}_{k_1,N}} |\hat{\phi}_{(p,k_2)}|^2\|L_{(p,k_2)}\|^2\right) \\ &\le \sum_{k_1=0}^{N}\sum_{k_2=0}^{N}\frac{2k_1+1}{2k_2+1}\left(\sum_{p\in J^{(2)}_{k_1,N}} \frac{m^4(2p+1)(2k_2+1)}{4(p^2+k_2^2)^2}\sum_{p\in J^{(2)}_{k_1,N}} |\hat{\phi}_{(p,k_2)}|^2\|L_{(p,k_2)}\|^2\right) \\ &\le c_1m^4\sum_{k_2=0}^{N}\left(\sum_{k_1=0}^{N}(2k_1+1)\left(\sum_{p\in J^{(2)}_{k_1,N}} \frac{2p+1}{(p^2+k_2^2)^2}\right)\left(\sum_{p=0}^{N} |\hat{\phi}_{(p,k_2)}|^2\|L_{(p,k_2)}\|^2\right)\right) \\ &\le c_2m^4\sum_{k_2=0}^{N}\left(\sum_{p=0}^{N} |\hat{\phi}_{(p,k_2)}|^2\|L_{(p,k_2)}\|^2\right)\sum_{k_1=0}^{N}(2k_1+1)\sum_{p\in J^{(0)}_{k_1,N}} p^{-3}\end{aligned}$$

$$\begin{aligned}
&\leq 2c_2 m^4 \sum_{k_2=0}^{N}\sum_{p=0}^{N} |\hat{\phi}_{(p,k_2)}|^2 \|L_{(p,k_2)}\|^2 \left(m^{-2} \sum_{k_1=0}^{m} (2k_1+1) + \sum_{k_1=m+1}^{N} (2k_1+1) k_1^{-2} \right) \\
&\leq c_3 m^4 \ln N \sum_{k_2=0}^{N}\sum_{p=0}^{N} |\hat{\phi}_{(p,k_2)}|^2 \|L_{(p,k_2)}\|^2 \\
&\leq c_3 m^4 \ln N \|\phi\|^2;
\end{aligned}$$

Similarly,

$$A_{2,2} = \sum_{k_1=0}^{N}\sum_{k_2=0}^{N} \frac{2k_2+1}{2k_1+1} \left(\sum_{q \in J^{(1)}_{k_2,N}} \hat{r}_{(k_1,q)} \hat{\phi}_{(k_1,q)} \right)^2 \leq c_4 m^4 \ln N \|\phi\|^2.$$

Thus

$$\|A_2(x)\|^2 \leq c_5 m^4 \ln N \|\phi\|^2.$$

$$\|\partial_x (R\phi)\|^2 \leq 2\|A_1(x)\|^2 + 2\|A_2(x)\|^2 \leq c m^4 \ln N \|\phi\|^2.$$

Since $\partial_x \phi \equiv \partial_x (Q\phi) + \partial_x (R\phi)$, the desired estimates follow. $\square$

Remark 3.3. The theorem shows the equivalence of the H^1 norm before and after application of the SV operator , $Q = Q_m$ for moderate size of $m_{_N} \ll N^{1/4}$. This holds despite the fact that for $m = m_{_N} \sim cN^{\beta} \longrightarrow \infty,\ 0 < 4\beta < 1$, the corresponding SV operator Q_m is spectrally small.

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