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REARRANGEMENTS OF THE COEFFICIENTS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

We establish extremal values of a solution y of a second-order initial value problem as the coefficients vary in a nonconvex set. These results extend earlier work by M. Essen in particular by allowing a coefficient in the second derivative expression.

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Contents

1. Introduction

Let $L^1_+(0,l)$ denote the set of all nonnegative functions from $L^1(0,l)$. l is a positive number. Let $f \in L^1_+(0,l)$ and μ_f its distribution function

$$\mu_f(t) = |\{x \in (0,l) : f(x) > t\}| \quad \text{for } t \geq 0,$$

where, here and below, $|I|$ is the measure of the set I. Let f^* denote the decreasing rearrangement of f,

$$f^*(x) = \sup\{t > 0 : \mu_f(t) > x\}.$$

It is known that f^* is nonnegative, right continuous and that [2]

$$\int_0^t f\,ds \leq \int_0^t f^*\,ds, \quad t \in [0,l], \tag{1.1}$$

$$\int_0^l f\,ds = \int_0^l f^*\,ds. \tag{1.2}$$

The increasing rearrangement of f is simply f^{**} defined by $f^{**}(t) = f^*(l-t)$. A crucial property of rearrangements is that if f and g are nonnegative with $f \in L^1(0,l)$ and $g \in L^\infty(0,1)$ then

$$\int_0^l f^{**}g^*\,ds \leq \int_0^l fg\,ds \leq \int_0^l f^*g^*\,ds. \tag{1.3}$$

We will say that f and g are equimeasurable or equivalently that f is a rearrangement of g if they have the same distribution function. We will denote this equivalence relation by $f \sim g$. Let f_0 be a member of $L^1_+(0,l)$ and $C(f_0)$ its equivalence class for the relation $\sim$, i.e.,

$$C(f_0) = \{f \in L^1_+(0,l),\ f^* = f_0^*\}.$$

A function $\sigma : [0,l] \to [0,l]$ is measure-preserving if, for each measurable set $I \subset [0,l]$, $\sigma^{-1}(I)$ is measurable and $|\sigma^{-1}(I)| = |I|$. Let Σ be the class of such functions. According to Ryff [6], to each $f \in L^1_+(0,l)$ there corresponds $\sigma \in \Sigma$ such that $f = f^* \circ \sigma$. In particular, we have

$$C(f_0) = \{f \in L^1_+(0,l),\ f = f_0^* \circ \sigma,\ \sigma \in \Sigma\}.$$

Let p and q be in $L^1_+(0,l)$ and consider the second-order differential equation

$$(p^{-1}(x)y'(x))' + q(x)y(x) = 0, \quad y(0) = 1, \quad (p^{-1}y')(0) = 0. \tag{1.4}$$

[1]A solution of the equation is a function y such that y and y' are absolutely continuous and the equation is satisfied almost everywhere. In the first part of this paper we are interested in finding the supremum and the infimum of $y(l)$ when the couple (p,q) varies in the set $C = C(f_0) \times C(g_0)$, where g_0 is also a member of $L^\infty_+(0,l)$. Consider

Problem 1. *Determine* $\inf y(l)$, $(p,q) \in C$.

Problem 2. *Determine* $\sup y(l)$, $(p,q) \in C$.

To solve these problems, we shall use a kind of calculus of variations which does not work in C; this class is not convex. Following Essen [3] and [4], and

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[1]The choice of p^{-1} instead of p is essential for the study of our problems.

recalling that $C(f_0)$ and $C(g_0)$ are weakly relatively compact in $L^1(0,l)$, we introduce the set $K = K(f_0) \times K(g)$ consisting of all weak limits of sequences of C in $[L^1(0,l)]^2$. To simplify notations, we use the symbol $\prec$ introduced by Hardy, Littlewood and Polya [5]. We say that f majorates g, written $g \prec f$, if

$$\int_0^x g^*\,dt \le \int_0^x f^*\,dt, \quad x \in [0,l],$$

$$\int_0^l g^*\,dt = \int_0^l f^*\,dt.$$

We note that if $g \prec f$ (f and g are in $L_+^\infty(0,l)$) then

$$\text{ess sup } g \le \text{ ess sup } f,$$
$$\text{ess inf } f \le \text{ ess inf } g.$$

The relations $g \prec f$ and $f \prec g$ imply that $f \sim g$. In [7], it is shown that

$$K(f_0) = \{f \in L_+^1(0,l),\ f \prec f_0\},$$

and $K(f_0)$ is the convex hull of $C(f_0)$. $K(f_0)$ is closed and weakly compact in $L^1(0,l)$. More generally, $K(f_0)$ is weakly compact in $L^p(0,l)$ if $f_0 \in L_+^p(0,l)$, $1 \le p \le \infty$. According to [1], $C(f_0)$ in the set of "∞-dimensional" extreme points of $K(f_0)$. That is if $f \in K(f_0) - C(f_0)$, then for any $m \ge 1$, one can find $f_1, \dots, f_m$ linearly independent in $K(f_0)$ and $\theta_1, \dots, \theta_m \in (0,1)$ such that

$$\sum_{i=1}^m \theta_i = 1, \qquad \sum_{i=1}^m \theta_i f_i = f.$$

The following result is given in [1].

Proposition 1.1. *Let $h,\ g \in L^1_+(0,l)$. Then the following are equivalent*

(i) $g \prec f$.

(ii) For all $h \in L^\infty_+(0,l)$,

$$\int_0^x gh\,dt \le \int_0^x f^* h^*\,dt, \qquad \int_0^l g\,dt = \int_0^l f\,dt.$$

(iii) For all $h \in L^\infty_+(0,l)$,

$$\int_0^x g^* h^*\,dt \le \int_0^x f^* h^*\,dt, \qquad \int_0^l g\,dt = \int_0^l f\,dt.$$

(iv) We have

$$\int_0^l F(g)\,dt = \int_0^l F(f)\,dt,$$

for all convex, nonnegative functions F such that $F(0) = 0$, F is Lipschitz.

As previously remarked we will consider the following problems

Problem 3. *Determine* $\inf y(l)$, $(p,q) \in K$.

Problem 4. *Determine* $\sup y(l)$, $(p,q) \in K$.

Similar problems may be considered for the differential equation

$$(p^{-1}(x)y'(x))' - q(x)y(x) = 0, \quad y(0) = 1, \quad (p^{-1}y')(0) = 0. \tag{1.5}$$

Let then

Problem 5. *Determine* $\inf y(l)$, $(p,q) \in K$.

Problem 6. *Determine* $\sup y(l)$, $(p,q) \in K$.

Proposition 1.2. *Let y be the solution of* (1.4) [*resp.* (1.5)]. *Then*

$$\inf y(l) \leq \cos(Al) \leq \sup y(l),$$

resp.

$$\inf y(l) \leq \cosh(Al) \leq \sup y(l),$$

where $A = (||f_0||_{L^1} ||g_0||_{L^1})^{1/2}$.

These estimates hold since the functions

$$p \equiv l^{-1}||f_0||_{L^1} \quad \text{and} \quad q \equiv l^{-1}||g_0||_{L^1}$$

are respectively members of $K(f_0)$ and $K(g_0)$.

2. Oscillation and Nonoscillation Criteria

To simplify this section, we assume that p, p^{-1} and q are in $L^\infty_+(0,l)$.

Lemma 2.1. *If*

$$\int_0^l p(x)\,dt \int_0^l q(x)\,dt \leq 1,$$

then a solution of (1.4) *does not vanish in* $[0,l]$.

Proof. Let y_0 be a solution of (1.4) vanishing in $(0,l]$, and denote by a its smallest zero. We have

$$(2.1) \qquad (p^{-1}(x)y_0'(x))' + q(x)y_0(x) = 0, \quad (p^{-1}y_0')(0) = 0, \quad y_0(a) = 0.$$

Multiplying (2.1) by y_0, we then integrate by parts to obtain

$$\int_0^a p^{-1}(y')^2\,dx = \int_0^a qy^2\,dx \leq y_{\max}^2 \int_0^a q\,dx,$$

and then apply the inequality (y' and p are linearly independent)

$$|y_{\max}| \leq \int_0^a |y'|\,dx < \left(\int_0^a p\,dx\right)^{\frac{1}{2}} \left(\int_0^a p^{-1}(y')^2\,dx\right)^{\frac{1}{2}}.$$

By substitution of the bound for $|y_{\max}|$ into the first inequality and cancelling the term $\int_0^a p^{-1}(y')^2\,dx$, the conclusion follows (by contradiction) since $a \leq l$. □

Lemma 2.2. *If*

$$\|p\|_\infty \|q\|_\infty < \left(\frac{\pi}{2l}\right)^2, \tag{2.2}$$

then a solution of (1.4) *does not vanish in* $[0, l]$.

Proof. Let y_0 be as in the previous proof, so that $\lambda_0 = 1$ is the first eigenvalue of the problem

$$\left(p^{-1}(x)y'(x)\right)' + \lambda q(x)y(x) = 0, \quad (p^{-1}y')(0) = 0, \quad y(a) = 0.$$

According to a variational principle,

$$\begin{aligned}\lambda_0 = \inf_{y(a)=0} \frac{\int_0^a p^{-1}(x)y'(x)^2\,dx}{\int_0^a q(x)y(x)^2\,dx} &\le \|p\|_\infty^{-1}\|q\|_\infty^{-1} \inf_{y(a)=0} \frac{\int_0^a y'(x)^2\,dx}{\int_0^a y(x)^2\,dx} \\ &= \|p\|_\infty^{-1}\|q\|_\infty^{-1}\pi^2(2a)^{-2}.\end{aligned}$$

Hence,

$$a^2 \ge \left(\frac{\pi}{2}\right)^2 \|p\|_\infty^{-1}\|q\|_\infty^{-1},$$

which contradicts (2.2). □

The proof shows that if $\|p\|_\infty \|q\|_\infty = \pi^2/(2l)^2$, then a solution of (1.4) may vanish only at $x = l$. It is not difficult to show that this case holds only when p and q are constants.

The following lemma gives sufficient conditions for oscillations.

Lemma 2.3. *Assume that p is nondecreasing, $p^{-1} \in C^1[0,l]$ and $p(x) \le h^{-1}$ on $[0,l]$, where h is a positive constant. There exists a number $H > 0$ (depending on h) such that if $q \ge H$ a.e. on $(0,l)$ then every solution of (1.4) changes its sign on $(0,l)$.*

Proof. Let $z(x) = (l-x)^2(l+x)^2$. Multiplying both sides in (1.4) by $z(x)$ and integrating over $(0,l)$, we obtain

$$\int_0^l y(x)[(p^{-1}z')'(x) + q(x)z(x)]\,dx = 0. \tag{2.3}$$

As p is nondecreasing we have for all $x \in (0,l)$

$$(p^{-1}z')'(x) = (p^{-1})'(x)z'(x) + p^{-1}(x)z''(x) \ge p^{-1}(x)z''(x).$$

Let ε be a positive number such that z'' is positive on $[l-\varepsilon, l]$. Suppose that $y(x) \ge 0$ on $[0,l]$. Then (2.3) implies that

$$\int_0^{l-\varepsilon} y(x)[(p^{-1}z')'(x) + q(x)z(x)]\,dx \le 0. \tag{2.4}$$

Let

$$H > h \max_{[0,l]}(-z'')(l-\varepsilon)^{-2}(l+\varepsilon)^{-2}.$$

Then,

$$(p^{-1}z')'(x) + q(x)z(x) \ge hz''(x) + Hz(x) > 0$$

for all $x \in (0, l-\varepsilon)$, which contradicts (2.4). □

Lemma 2.4. *Any solution of* (1.5) *is positive and nondecreasing. Moreover, if* $||p||_{L^1}||q||_{L^1} < 1$ *then*

$$y(l) \leq (1 - ||p||_{L^1}||q||_{L^1})^{-1}.$$

Proof. Let y be a solution of (1.5). We have

$$y'(x) = p(x) \int_0^x q(t)y(t)\,dt,$$

which implies that $y(x) \geq 1$ and y is nondecreasing. Therefore,

$$y'(x) \leq y(l)p(x) \int_0^x q(t)\,dt.$$

Integrating both sides of the last inequality over $(0, l)$, we get

$$y(l) - 1 \leq y(l) \int_0^l p(t)\,dt \int_0^l q(t)\,dt.$$

Hence,

$$y(l) \leq (1 - ||p||_{L^1}||q||_{L^1})^{-1}.$$

□

3. Characterization of the Extremal Couples

The existence of extremal couples will be discussed at the end of this section. We suppose that $f_0,\ g_0 \in L_+^\infty(0,l)$ and $f_0 \geq h$ where h is a positive constant.

Theorem 3.1. *Assume that all solutions of (1.4) are positive when (p,q) varies in $K(f_0)\times K(g_0)$. Let (p_0,q_0) be an extremal couple for Problem 3 and y_0 the corresponding solution in (1.4). Then $q_0 = g_0^*$ and in the open set where*

$$\int_0^t p_0(s)\,ds > \int_0^t f_0^{**}(s)\,ds,$$

we have $P'(t)=0$ where

$$P(t) = \frac{y_0'^{\,2}(t)}{p_0^2(t)}\left(\int_t^l p_0(t)y_0(t)^{-2}dt\right) - \frac{y_0'(t)}{(p_0y_0)(t)}, \quad t\in[0,l].$$

If f_0 is bounded below by a positive constant then the above set is empty and $p_0 = f_0^{**}$, i.e., the infimum over the larger class K coincides with the infimum over the smallest class C.

Theorem 3.2. *Assume that all solutions of (1.4) are positive when (p,q) varies in $K(f_0)\times K(g_0)$. Let (p_0,q_0) be an extremal couple for Problem 4 and y_0 the corresponding solution in (1.4). Then $q_0 = g_0^{**}$ and in the open set where*

$$\int_0^t p_0(s)\,ds < \int_0^t f_0^{*}(s)\,ds,$$

we have $P'(t)=0$ where P is as above. If f_0 is far from zero then the above set is empty and $p_0 = f_0^*$, i.e. the supremum over the larger class K coincides with the supremum over the smallest class C.

Let a_i and b_i, $(i = 1, 2)$, be positive numbers such that $a_1 < a_2$ and $b_1 < b_2$. Define the sets E and F by

$$E = \left\{ p \in L^\infty(0,l),\ a_1 \le p \le a_2,\ \int_0^l p\,dx = A \right\}$$

and

$$F = \left\{ q \in L^\infty(0,l),\ b_1 \le p \le b_2,\ \int_0^l q\,dx = B \right\},$$

where A and B are such that $a_1 l < A < a_2 l$ and $b_1 l < B < b_2 l$. Then we have

Corollary 3.3. *If* $AB \le 1$, *then* $\inf y(l)$ *when* (p, q) *varies in* $E \times F$ *is reached by*

$$p_0(x) = \begin{cases} a_1 & \text{if } x \in (0, \alpha), \\ a_2 & \text{if } x \in (\alpha, l), \end{cases}$$

and

$$q_0(x) = \begin{cases} b_2 & \text{if } x \in (0, \beta), \\ b_1 & \text{if } x \in (\beta, l), \end{cases}$$

where α *and* β *are chosen so that* $\int_0^l p_0\,dx = A$ *and* $\int_0^l q_0\,dx = B$. *The supremum of* $y(l)$ *over* $E \times F$ *is reached by* $\bar{p} = p_0^*$ *and* $\bar{q} = q_0^{**}$.

A counterexample. We show that Theorem 3.2 does not hold if the solutions of (1.4) are allowed to vanish. Set $l = 2\pi$, and let $p_0 \equiv 1$ in $(0, l)$ and

$$q_0(x) = \begin{cases} 0 & \text{if } x \in (0, l_0), \\ 4 & \text{if } x \in (l_0, l), \end{cases}$$

where $l_0 = 3\pi/2$. Then it is easily verified that the solution in (1.4) with $(p,q) = (p_0, q_0)$ is

$$y_0(x) = \begin{cases} 1 & \text{if } x \in (0, l_0), \\ \cos 4(x - l_0) & \text{if } x \in (l_0, l). \end{cases}$$

Let $\bar{p}(x) \equiv \bar{q}(x) \equiv 1$ in $(0, 2\pi)$. The corresponding solution in (1.4) is $\bar{y}(x) = \cos x$. We see that $\bar{y}(l) > y_0(l)$ in spite of $\bar{q} \prec q_0$. The assumption in Theorem 3.1 is also necessary.

Proofs of Theorems 3.1 and 3.2. Necessary conditions on p_0. By the change of variable $u = -y'/(py)$, i.e.,

$$y(x) = e^{-\int_0^x pu\,dt} \qquad x \in [0, l], \tag{3.1}$$

equation (1.4) is changed into

$$u' - pu^2 = q, \qquad u(0) = 0. \tag{3.2}$$

The solution of (3.2) is written

$$u(t) = \int_0^t q(s) \left\{ \exp \int_s^t p(r)u(r)\,dr \right\} ds.$$

In view of (3.1), Problem 3 is equivalent to

$$\text{maximising} \quad \int_0^l pu\,dt \quad \text{subject to} \quad (p,q) \in K.$$

Let p_0 be an extremal function for the infimum problem and p an arbitrary member in $K(f_0)$. Define

$$p_\delta = (1-\delta)p_0 + \delta p, \qquad \delta \in [0,1].$$

We note that this type of variation is not possible in $C(f_0)$. Let u_δ satisfy

$$u'_\delta - p_\delta u_\delta^2 = q_0, \qquad u_\delta(0) = 0. \tag{3.3}$$

Forming the difference of (3.3) and (3.3) with $\delta = 0$, we have

$$u'_\delta - u'_0 = p_\delta(u_\delta - u_0)(u_\delta + u_0) + \delta(p - p_0)u_0^2.$$

Therefore,

$$(u_\delta - u_0)(t) = \delta \int_0^t (p - p_0)u_0^2 \left\{ \exp \int_s^t p_\delta(r)(u_\delta + u_0)(r)\, dr \right\} ds.$$

Writing $p_\delta u_\delta - p_0 u_0 = p_\delta(u_\delta - u_0) + (p_\delta - p_0)u_0$ and integrating over $(0,l)$, we obtain

$$\begin{aligned}
\int_0^l (p_\delta u - p_0 u_0) dt &= \int_0^l p_\delta \left(\delta \int_0^t (p - p_0)u_0^2 \left\{ \exp \int_s^t p_\delta(u_\delta + u_0)\, dr \right\} ds \right) dt \\
&\quad + \delta \int_0^l (p - p_0)u_0\, dt \\
&= \delta \int_0^l (p - p_0)u_0^2 \left(\int_s^l p_\delta \left\{ \exp \int_s^t p_\delta(u_\delta + u_0)\, dr \right\} dt \right) ds \\
&\quad + \delta \int_0^l (p - p_0)u_0\, dt.
\end{aligned}$$

For Problem 3 the left-hand side is nonpositive. Dividing by δ and letting $\delta \to 0^+$ brings

$$\int_0^l (p - p_0)(t)P(t)\,dt \le 0, \qquad \text{for all } p \in K(f_0), \tag{3.4}$$

where P is given in Theorem 3.1. If p_0 is an extremal coefficient for Problem 4 then we find

$$\int_0^l (p - p_0)(t)P(t)\,dt \ge 0, \qquad \text{for all } p \in K(f_0). \tag{3.5}$$

Let us first discuss (3.4). By Ryff's characterization, there exists $\sigma \in \Sigma$ such that $P = P^* \circ \sigma$. Substituting $p = p_0^* \circ \sigma$ into (3.4) we see that

$$\int_0^l P^* p_0^*\,dt = \int_0^l Pp\,dt \le \int_0^l Pp_0\,dt \le \int_0^l P^* p_0^*\,dt. \tag{3.6}$$

In the last step we used (1.3) which requires that P is nonnegative. This will be proved later. As a result, equalities hold everywhere in (3.6) and we have

$$\int_0^\infty \left\{ \int_{\{P(t)>s\}} p_0(t)\,dt \right\} ds = \int_0^\infty \left\{ \int_{\{P^*(t)>s\}} p_0^*(t)\,dt \right\} ds \tag{3.7}$$

for all s. As

$$|\{P(t) > s\}| = |\{P^*(t) > s\}|,$$

we know that

$$\int_{\{P(t)>s\}} p_0(t)\,dt \le \int_{\{P^*(t)>s\}} p_0^*(t)\,dt$$

for all s. It follows from (3.7) that

$$\int_{\{P(t)>s\}} p_0(t)\,dt = \int_{\{P^*(t)>s\}} p_0^*(t)\,dt, \tag{3.8}$$

$$\operatorname{ess}\inf_{\{P(t)>s\}} p_0(t) \geq \operatorname{ess}\inf_{\{P(t)\leq s\}} p_0(t). \tag{3.9}$$

for all s. From (3.9) one deduces that if P is increasing on the interval I, then p_0 must be nondecreasing on this interval if we neglect a set of measure zero. Similarly, if P is decreasing on some interval, p_0 will be nonincreasing. If these relations hold, we say that P and p_0 are *codependent.*

We now return to the function P. We have $P(0) = 0$ and a straightforward calculation yields

$$P'(t) = q_0 \left(1 - 2\frac{q_0}{p_0} y_0 y_0' \int_t^l p_0(s) y_0^{-2}(s)\,ds\right)$$

that is nonnegative for all $t \in (0, l)$. Choosing $p = f_0^{**}$ in the variational equation (3.4) and integrating by parts gives

$$0 \geq \int_0^l (f_0^{**} - p_0) P(t)\,dt = \int_0^l \left(\int_0^t (f_0^{**} - p_0)\,ds\right) d(-P(t)) \geq 0.$$

We used the inequality

$$\int_0^t p_0\,ds \geq \int_0^t f_0^{**}\,ds, \qquad t \in [0, l].$$

Consequently,

$$P'(t)\int_0^t (f_0^{**}-p_0)\,ds = 0, \qquad t\in[0,l],$$

and the second part of Theorem 3.1 is proved.

For the supremum problem we use the same arguments. If $P = P^*\circ\sigma$, where $\sigma\in\Sigma$, we choose $p = p_0^{**}\circ\sigma$ in (3.5) to obtain

$$\int_0^l P^* p_0^{**}\,dt = \int_0^l Pp\,dt \geq \int_0^l Pp_0\,dt \geq \int_0^l P^* p_0^{**}\,dt. \tag{3.10}$$

Thus, there is equality everywhere in (3.10) and

$$\int_0^\infty \left\{\int_{\{P(t)>s\}} p_0(t)\,dt\right\} ds = \int_0^\infty \left\{\int_{\{P^*(t)>s\}} p_0^{**}(t)\,dt\right\} ds. \tag{3.11}$$

Since

$$\int_{\{P^{**}(t)>s\}} p^{**}(t)\,dt \leq \int_{\{P(t)>s\}} p_0(t)\,dt,$$

for all s, (3.11) implies that

$$\int_{\{P(t)>s\}} p_0(t)\,dt = \int_{\{P^*(t)>s\}} p_0^{**}(t)\,dt,$$

$$\operatorname{ess}\inf_{\{P(t)>s\}} p_0(t) \geq \operatorname{ess}\inf_{\{P(t)\leq s\}} p_0(t),$$

for all s. In this case P and p_0 are *contra-dependent*, i.e. if P is increasing (resp. decreasing) on an interval I, p_0 will be nonincreasing (resp. nondecreasing) on

I. Choosing $p = f_0^*$ in the variational equation (3.5) and arguing as above, we prove the second part of Theorem 3.2.

Necessary conditions on q_0. Let q_0 be an extremal function for Problem 3. For $q \in K(g_0)$, we define

$$q_\delta = (1-\delta)q_0 + \delta q, \qquad \delta \in [0,1].$$

Let u_δ be the solution of

$$u' - p_0 u^2 = q_\delta, \qquad u(0) = 0. \tag{3.12}$$

Forming the difference of (3.12) and (3.12) with $\delta = 0$, calculations similar to those of the preceding case allow us to derive the necessary conditions of optimality

$$\int_0^l (q - q_0)(t)Q(t)\,dt \le 0 \qquad \text{for all } q \in K(g_0),$$

where

$$Q(t) = y_0^2(t)\int_t^l p_0(s)y_0^{-2}(s)\,ds.$$

We remark that $Q(l) = 0$ and

$$Q'(t) = 2y_0y_0'\int_t^l p_0(s)y_0^{-2}(s)\,ds - p_0$$

is nonpositive on $(0,l)$. For Problem 4, q_0 satisfies

$$\int_0^l (q - q_0)(t)Q(t)\,dt \ge 0 \qquad \text{for all } q \in K(g_0).$$

Reasoning as above, we deduce that q_0 and Q are codependent for the infimum problem. The argument for characterizing p_0 yields $q_0 = g_0^*$. For the supremum problem q_0 and Q are contra-dependent and we get $q_0 = g_0^{**}$ which completes the proofs. □

Existence.

Let m_0 denote the infimum of $y(l)$ when (p, q) varies in K and (p_n, q_n) a minimizing sequence in K. Let $\{u_n\}$ be an associated sequence of solutions in the differential equation (3.2) so that $\lim_{n\to\infty} \int_0^l p_n u_n \, dt = m_0$. Using weak* compactness, we find that $(p_0, q_0) \in K$ such that $p_n \to p$ and $q_n \to q$ weakly in $L^\infty(0, l)$. From the expression of u_n, we see that

$$u_n(t) \leq \int_0^l q_n(t) e^{-\int_0^l p_n u_n \, ds} \, dt \leq ||g_0||_{L^1} \, e^{-m_0}.$$

It follows from (3.2) that the sequence $\{u_n'\}$ is uniformly bounded in $L^\infty(0, l)$. By Ascoli's theorem, there exists a subsequence (we may assume that it is the original sequence) such that $u_n \to u_0$ uniformly in $[0, l]$. It is easy to check that u_0 is the solution of (3.2) for $(p, q) = (p_0, q_0)$. The proof of the supremum problem is quite the same.

4. Problem 6

Suppose that $f_0, g_0 \in L_+^\infty(0,l)$ and $f_0 \geq 1$ over $(0,l)$. The existence of extremal couples for Problems 5 and 6 may be proved as above. Let

$$P(t) = \frac{y_0'^2(t)}{p_0^2(t)} \left(\int_t^l p_0(s) y_0(s)^{-2}\, ds \right) - \frac{y_0'(t)}{(p_0 y_0)(t)},$$

$$Q(t) = y_0^2(t) \int_t^l p_0(s) y_0(s)^{-2}\, ds, \qquad t \in [0,l].$$

Theorem 4.1. *Let (p_0, q_0) be the extremal couple for Problem 6, and y_0 an associated solution in (1.5). In the open set where*

$$\int_0^t p_0\, ds > \int_0^t f_0^{**}\, ds$$

resp.

$$\int_0^t q_0\, ds < \int_0^t g_0^{*}\, ds,$$

we have $P'(t) = 0$, resp. $Q'(t) = 0$.

Proof. By the change of variable $u = y'/(py)$ equation (1.5) is changed into

$$u' + pu^2 = q, \quad u(0) = 0, \quad t \in [0,l].$$

We shall then study the equivalent problem

$$\max \int_0^l p\, u\, dt, \quad (p,q) \in K.$$

Let (p_0, q_0) be the extremal couple for Problem 6. Arguing as above, we find that p_0 and q_0 satisfy the conditions

$$\int_0^l (p - p_0)(t)P(t)\,dt \geq 0 \qquad \text{for all } p \in K(f_0), \tag{4.1}$$

$$\int_0^l (q - q_0)(t)Q(t)\,dt \leq 0 \qquad \text{for all } q \in K(g_0)n \tag{4.2}$$

where P and Q are given above. Unlike the preceding case, it is difficult here to know the sign of P and Q. We shall then proceed as above: Let y_1 be the function defined by

$$y_1(t) = y_0(t) \int_t^l p_0(s) y_0^{-2}(s)\,ds, \quad t \in [0, l].$$

y_1 is a solution of the differential equation

$$(p_0^{-1}(x)y'(x))' - q_0(x)y(x) = 0, \qquad x \in (0, l),$$

but $y_1(l) = 0$ and $y_1'(l) = -(y_0/p_0)^{-1}(l)$. Besides, it is easy to see that $y_1'(t) < 0$ for all $t \in (0, l)$. Let

$$\xi = \left(\frac{y_0'}{y_0 p_0} - \frac{y_1'}{y_1 p_0} \right) \Big/ 2, \qquad \eta = -\left(\frac{y_0'}{y_0 p_0} + \frac{y_1'}{y_1 p_0} \right) \Big/ 2.$$

Then, we have

$$(4.3)\qquad \begin{aligned} \xi' &= 2\xi\,\eta\,p_0, \\ \eta' &= p_0(\xi^2+\eta^2)-q_0, \\ \xi(0) &= \left(\int_0^l p_0(s)y_0^{-2}(s)\,ds\right)^{-1} \Big/ 2 = \eta(0). \end{aligned}$$

The key of deciding the sign of P and Q are the following relations

$$(4.4)\qquad Q(t) = \frac{1}{2}\xi(t)^{-1},$$

and

$$(4.5)\qquad P'(t) = \frac{1}{2}\frac{q_0}{p_0}\left(\frac{1}{\xi}\right)^{-1}.$$

In fact, we have

$$(4.6)\qquad \xi Q = \xi y_0 y_1 = \frac{1}{2p_0(t)}(y_0' y_1 - y_0 y_1') = \frac{1}{2},$$

and

$$\begin{aligned} P(t) &= 2\frac{q_0}{p_0}y_0y_0'\int_t^l p_0(s)y_0^{-2}(s)\,ds - q_0 \\ &= \frac{q_0}{p_0}\left(2y_0y_0'\int_t^l p_0(s)y_0^{-2}(s)\,ds - p_0\right) \\ &= \frac{q_0}{p_0}Q'(t). \end{aligned}$$

Relation (4.6) implies that ξ is positive and $\lim \xi(t) = \infty$, $t \to l-$. From (4.3) it follows that $\limsup \eta(t) \geq 0$, $t \to l-$. Assume now that η changes its sign on $(0, l)$. Since $\eta(0) > 0$, there exists an interval $[a, b] \subset [0, l)$ such that for some $c > 0$, we have

$$\eta(t) \leq \eta(a) < 0, \qquad t \in [a, a+c],$$

$$\eta(t) < 0, \quad t \in [a, b), \quad \eta(b) = 0.$$

Since η is assumed negative on (a, b), ξ will be decreasing on this interval. (4.4) and (4.5) imply that P and Q are both increasing on $[a, b]$. From (4.1) and (4.2) we see that p_0 is nonincreasing and q_0 is nondecreasing on this interval. As a result, we have

$$\begin{aligned} 0 &\geq \eta(t) - \eta(a) \\ &= \int_a^t (p_0\xi^2 - q_0) + \int_a^t p_0\eta^2 \\ &\geq (t-a)\left(p_0(t)\xi^2(t) - q_0(t) + \eta(a)^2\right), \end{aligned}$$

$$t \in (a, a+c),$$

since $\operatorname{ess\,inf}_{(0,l)} p_0(t) \geq 1$. Arguing as in [4], we arrive at the following contradiction: $\eta(b) \leq \eta(a) < 0$. Hence, η is nonnegative and ξ is nondecreasing. Taking $p = f_0^{**}$ in the variational equation (4.1), we obtain

$$0 \leq \int_0^l (f_0^{**} - p_0)P(t)\,dt = \int_0^l \left(\int_0^t (f_0^{**} - p_0)\,ds\right) d(-P(t)) \leq 0,$$

and therefore

$$P'(t)\int_0^t (f_0^{**} - p_0)\,ds = 0, \qquad t \in [0, l]$$

which proves the first part of Theorem 4.1. To complete the proof, we choose $q = g_0^*$ in (4.2). ☐

Remark 1. *For Problem 5, the arguments for deciding the sign of η on $(0, l)$ break down and the problem requires the development of other arguments.*

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