Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 4, Article 62, 2002

ON SOME INEQUALITIES OF LOCAL TIMES OF ITERATED STOCHASTIC INTEGRALS

LITAN YAN

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TOYAMA UNIVERSITY
3190 GOFUKU, TOYAMA 930-8555
JAPAN.
yan@math.toyama-u.ac.jp
litanyan@dhu.edu.cn
litanyan@hotmail.com

Received 06 July, 2001; accepted 25 June, 2002
Communicated by N.S. Barnett

ABSTRACT. Let $X = (X_t, \mathcal{F}_t)_{t\geq 0}$ be a continuous local martingale with quadratic variation process $\langle X\rangle$ and $X_0 = 0$. Define iterated stochastic integrals $I_n(X) = (I_n(t, X), \mathcal{F}_t)$ $(n \geq 0)$, inductively by

$$I_n(t,X) = \int_0^t I_{n-1}(s,X)dX_s$$

with $I_0(t, X) = 1$ and $I_1(t, X) = X_t$. In this paper, we obtain some martingale inequalities for $I_n(X)$, $n = 1, 2, \ldots$ and their local times at any random time.

Key words and phrases: Continuous local martingale, Continuous semimartingale, Iterated stochastic integrals, Local time, Random time, Burkholder-Davis-Gundy inequalities, Barlow-Yor inequalities.

2000 *Mathematics Subject Classification.* 60H05, 60G44, 60J55.

1. INTRODUCTION

Let $X = (X_t)_{t\geq 0}$ be a continuous local martingale with quadratic variation process $\langle X\rangle$ and $X_0 = 0$, defined on some filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Consider the corresponding sequence of iterated stochastic integrals,

$$I_n(X) = (I_n(t,X), \mathcal{F}_t) \quad (n \geq 0),$$

ISSN (electronic): 1443-5756

The author would like to thank Professor N. Kazamaki for his guidance and encouragement in the study of martingales and related fields. The author wishes also to thank Professor N. Barnett and an anonymous earnest referee for a careful reading of the manuscript and many helpful comments.

The author's present address : Department of Mathematics, College of Science, Donghua University, 1882 West Yan'an Rd., Shanghai 200051, China.

055-01

defined inductively by

$$I_n(t,X)=\int_0^t I_{n-1}(s,X)dX_s, \tag{1.1}$$

where $I_0(t,X)=1$ and $I_1(t,X)=X_t$.

It is known that there exist positive constants $B_{n,p}$ and $A_{n,p}$ depending only on n and p, such that the inequalities (see [2, 8])

$$A_{n,p}\left\|\langle X\rangle_T^{\frac{n}{2}}\right\|_p \le \left\|\sup_{0\le t\le T}|I_n(t,X)|\right\|_p \le B_{n,p}\left\|\langle X\rangle_T^{\frac{n}{2}}\right\|_p \quad (0<p<\infty), \tag{1.2}$$

hold for all continuous local martingales X with $X_0=0$ and all $(\mathcal{F}_t)$–stopping time T.

On the other hand, M.T. Barlow and M. Yor have established in [1] (see also Theorem 2.4 in [7, p.457]) the following martingale inequalities for local times:

$$c_p\left\|\langle X\rangle_\infty^{\frac{1}{2}}\right\|_p \le \|\mathcal{L}_\infty^*(X)\|_p \le C_p\left\|\langle X\rangle_\infty^{\frac{1}{2}}\right\|_p \quad (0<p<\infty),$$

where $(\mathcal{L}_t^x(X);t\ge 0)$ is the local time of X at x and $\mathcal{L}_t^*(X)=\sup_{x\in\mathbb{R}}\mathcal{L}_t^x(X)$. It follows that for all $0<p<\infty$

$$c_{n,p}\left\|\langle X\rangle_T^{\frac{n}{2}}\right\|_p \le \|\mathcal{L}_T^*(n,X)\|_p \le C_{n,p}\left\|\langle X\rangle_T^{\frac{n}{2}}\right\|_p \tag{1.3}$$

for all $(\mathcal{F}_t)$–stopping times T, where $(\mathcal{L}_t^x(n,X);t\ge 0)$ stands for the local time of $I_n(X)$ at x.

However, it is clear that the inequalities (1.2) and (1.3) are not true when T is replaced by an arbitrary $\mathbb{R}_+$–valued random time (see, for example, [12] when $n=1$). In this paper we extend (1.2) and (1.3) to any random time.

2. PRELIMINARIES

Throughout this paper, we fix a filtered complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ with the usual conditions. For any process $X=(X_t)_{t\ge 0}$, denote $X_\tau^*=\sup_{0\le t\le\tau}|X_t|$ and $X^*=\sup_{0\le t<\infty}|X_t|$. Let c stand for some positive constant depending only on the subscripts whose value may be different in different appearances, and this assumption is also made for $\hat{c}$.

From now on an $\mathcal{F}$–measurable non–negative random variable $L:\Omega\to\mathbb{R}_+$ is called a random time and we denote by $\mathbb{L}$ the collection of all random times, i.e.,

$$\mathbb{L}=\{L\ :\ L \text{ is an } \mathcal{F}\text{–measurable, non–negative, random variable}\}.$$

For any $L\in\mathbb{L}$, let (G_t^L) be the smallest filtration satisfying the usual conditions which both contains $(\mathcal{F}_t)$ and makes L a (G_t^L)–stopping time. Define

$$Z_t^L=E\left[1_{\{L>t\}}|\mathcal{F}_t\right] \qquad \text{and} \qquad J_L=\inf_{s<L}Z_s^L.$$

Then $Z^L=(Z_t^L)$ is a potential of class (D). Assume that the Doob–Meyer decomposition for Z^L is

$$Z^L=M-A. \tag{2.1}$$

For simplicity, in the present paper we assume throughout that $L\in\mathbb{L}$ avoids $(\mathcal{F}_t)$–stopping times, i.e.,

for every $(\mathcal{F}_t)$–stopping time T, $P(L=T)=0$.

Thus, under the condition, Z^L is continuous and so M is also continuous. Furthermore, for any continuous $(\mathcal{F}_t)$–local martingale X there exists a continuous (G_t^L)–local martingale $\widetilde{X}$ with $\langle X\rangle_{L\wedge t} = \langle \widetilde{X}\rangle_t$ such that

$$X_{L\wedge t} = \widetilde{X}_t + \int_0^{L\wedge t} \frac{d\langle X, M\rangle_s}{Z_s^L},$$

where $L\wedge t = \min\{L, t\}$. For more information on $X^L = (X_{L\wedge t})_{t\geq 0}$ and (G_t^L), see [10, 11, 12].

Lemma 2.1 ([10])**.** *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

$$E\left[(X_L^*)^p\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{p}{2}}\right], \tag{2.2}$$

$$E\left[\langle X\rangle_L^{\frac{p}{2}}\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)(X^*)_L^p\right] \tag{2.3}$$

hold for all continuous $(\mathcal{F}_t)$–local martingales X vanishing at zero.

It is known that the inequalities in Lemma 2.1 are the extensions to the Burkholder-Davis-Gundy inequalities. For the proof, see Proposition 4 in [10, p.122] (or Theorem 13.4 in [12, p.57]).

Let X now be a continuous semimartingale. Then for every $x \in \mathbb{R}$ the following Meyer–Tanaka formula may be considered as a definition of the local time $\{\mathcal{L}_t^x(X); t \geq 0\}$ of X at x

$$|X_t - x| - |X_0 - x| = \int_0^t \operatorname{sgn}(X_s - x)dX_s + \mathcal{L}_t^x(X).$$

One may take a version $\mathcal{L} : (x, t, \omega) \to \mathcal{L}_t^x(\omega)$ which is right continuous and has a left limit at x, and is continuous in t. In particular, if X is a continuous local martingale, then $\mathcal{L}_t^x(X)$ has a continuous version in both variables. In this paper, we use such a version of local time.

The fundamental formula of occupation density for a continuous semimartingale is:

$$\int_0^t \Phi(X_s)d\langle X\rangle_s = \int_{-\infty}^{\infty} \Phi(x)\mathcal{L}_t^x(X)dx \tag{2.4}$$

for all bounded, Borel functions $\Phi : \mathbb{R} \to \mathbb{R}$, which gives

$$\langle X\rangle_\infty \leq 2X_\infty^* \mathcal{L}_\infty^*(X) \tag{2.5}$$

since $\mathcal{L}_\infty^x = 0$ for all $x \notin [-X^*, X^*]$. It follows that (see [3]) for all continuous $(\mathcal{F}_t)$–local martingales X, and all $t \geq 0, x \in \mathbb{R}$ and $L \in \mathbb{L}$

$$\mathcal{L}_{L\wedge t}^x(X) = \mathcal{L}_t^x(X^L) \tag{2.6}$$

if M is continuous, where $X^L = (X_{L\wedge t})$. So, we have

$$\langle X\rangle_L = \langle X^L\rangle_\infty \leq 2\mathcal{L}_\infty^*(X^L)X_L^* = 2\mathcal{L}_L^*(X)X_L^* \tag{2.7}$$

by (2.5). Furthermore, the following lemma which can be found in [3] extends the Barlow–Yor inequalities.

Lemma 2.2. *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

$$E\left[\left(\mathcal{L}_L^*(X)\right)^p\right] \leq c_p \min\left\{E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{p}{2}}\right],\ E\left[\left(1 + \log^p \frac{1}{J_L}\right)(X_L^*)^p\right]\right\} \tag{2.8}$$

and

$$\max\left\{E\left[(X_L^*)^p\right],\ E\left[\langle X\rangle_L^{\frac{p}{2}}\right]\right\} \leq c_p E\left[\left(1 + \log^p \frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^p\right] \tag{2.9}$$

hold.

Remark 2.3. In [3], C. S. Chou proved that (2.8) and (2.9) hold for $1 \leq p < \infty$. In fact, when $0 < p < 1$ (2.8) and (2.9) are also true from the proof in [3].

3. INEQUALITIES AND PROOFS

In this section, we shall extend (1.2) and (1.3) to any random time $L \in \mathbb{L}$.

Theorem 3.1. *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

$$E\Big[\big(I_n^*(L,X)\big)^p\Big] \leq c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}} \frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right], \tag{3.1}$$

$$E\Big[\big(I_n^*(L,X)\big)^p\Big] \leq c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}} \frac{1}{J_L}\Big)(X_L^*)^{np}\right], \tag{3.2}$$

$$E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \leq c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}} \frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right], \tag{3.3}$$

$$E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \leq c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}} \frac{1}{J_L}\Big)(X_L^*)^{np}\right] \tag{3.4}$$

hold for all continuous local martingales X with $X_0 = 0$ and $n = 1, 2, \ldots$.

Proof. Let $n \geq 1$, $L \in \mathbb{L}$ and let X be a continuous local martingale.

(3.1) can be verified by induction. In fact, when $n = 1$ (3.1) is true from (2.2). Now suppose that (3.1) is true for $2, \ldots, n-1$. Then we have

$$E\left[\big(I_{n-1}^*(L,X)\big)^{\frac{np}{n-1}}\right] \leq c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}} \frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right].$$

On the other hand, from (1.1) we see that

$$\langle I_n(X)\rangle_t = \int_0^t \big(I_{n-1}(s,X)\big)^2 d\langle X\rangle_s \leq \sup_{0\leq s\leq t} \big(I_{n-1}(s,X)\big)^2 \langle X\rangle_t$$

for all $t \geq 0$, which gives

$$\langle I_n(X)\rangle_L \leq \big(I_{n-1}^*(L,X)\big)^2 \langle X\rangle_L. \tag{3.5}$$

Thus, by applying (2.2), (3.5) and then applying the Hölder inequality with exponents $s = n$ and $r = \frac{n}{n-1}$, and noting

$$(a+b)^n \leq c_n(a^n + b^n) \qquad (a, b \geq 0),$$

we find

$$\begin{aligned}
E\Big[\big(I_n^*(L,X)\big)^p\Big] &\leq c_p E\left[\Big(1+\log^{\frac{p}{2}} \frac{1}{J_L}\Big)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \\
&\leq c_p E\left[\Big(1+\log^{\frac{p}{2}} \frac{1}{J_L}\Big)\big(I_{n-1}^*(L,X)\big)^p\langle X\rangle_L^{\frac{p}{2}}\right] \\
&\leq c_p E\left[\Big(1+\log^{\frac{p}{2}} \frac{1}{J_L}\Big)^n\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}} E\left[\big(I_{n-1}^*(L,X)\big)^{\frac{np}{n-1}}\right]^{\frac{n-1}{n}} \\
&\leq c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}} \frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right].
\end{aligned}$$

This establishes (3.1).

Now, we verify (3.2). From the well-known correspondence of iterated stochastic integral $I_n(X)$ and the Hermite polynomial of degree n (see [4, 7])

$$I_n(t,X) = \frac{1}{n!} H_n(X_t, \langle X \rangle_t),$$

where $H_n(x,y) = y^{\frac{n}{2}} h_n\left(\frac{x}{\sqrt{y}}\right)$ $(y>0)$ and $h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ is the Hermite polynomial of degree n, more generally, $H_n(x,y)$ can be defined as

$$H_n(x,y) = (-y)^n e^{\frac{x^2}{2y}} \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2y}},$$

we see that iterated stochastic integrals $I_n(X)$, $n = 1, 2, \dots$ have the representation

$$I_n(t,X) = \sum_{j=0}^{[\frac{n}{2}]} C_n^{(j)} X_t^{n-2j} \langle X \rangle_t^j, \tag{3.6}$$

where $C_n^{(j)} = \left(-\frac{1}{2}\right)^j \frac{1}{(n-2j)!j!}$ and $[x]$ stands for the integer part of x.

On the other hand, for $0 < j < \frac{n}{2}$, by using the Hölder inequality with exponents $s = \frac{n}{n-2j}$ and $r = \frac{n}{2j}$, we get

$$\begin{aligned}
E\Big[(X_L^*)^{(n-2j)p} \langle X \rangle_L^{jp}\Big] &\le E\big[(X_L^*)^{np}\big]^{\frac{n-2j}{n}} E\left[\langle X \rangle_L^{\frac{np}{2}}\right]^{\frac{2j}{n}} \\
&\le c_{n,p} E\big[(X_L^*)^{np}\big]^{\frac{n-2j}{n}} E\left[\Big(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\Big)(X_L^*)^{np}\right]^{\frac{2j}{n}} \\
&\le c_{n,p} E\left[\Big(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\Big)(X_L^*)^{np}\right].
\end{aligned}$$

Clearly, the inequality above is also true for $j = \frac{n}{2}$ and $j = 0$.

Combining this with (3.6), we get for $0 < p \le 1$

$$\begin{aligned}
E\Big[\big(I_n^*(L,X)\big)^p\Big] &\le c_p \sum_{j=0}^{[\frac{n}{2}]} |C_n^{(j)}|^p E\Big[(X_L^*)^{(n-2j)p} (\langle X \rangle_L)^{jp}\Big] \\
&\le c_{n,p} E\left[\Big(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\Big)(X_L^*)^{np}\right]
\end{aligned}$$

and for $1 < p < \infty$

$$\begin{aligned}
E\Big[\big(I_n^*(L,X)\big)^p\Big]^{\frac{1}{p}} &\le \sum_{j=0}^{[\frac{n}{2}]} |C_n^{(j)}| E\Big[(X_L^*)^{(n-2j)p} \langle X \rangle_L^{jp}\Big]^{\frac{1}{p}} \\
&\le c_{n,p} E\left[\Big(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\Big)(X_L^*)^{np}\right]^{\frac{1}{p}}.
\end{aligned}$$

This gives (3.2).

Next, from (3.5) and (3.1) we see that

$$\begin{aligned} E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] &\le E\left[\left(I^*_{n-1}(L,X)\right)^p\langle X\rangle_L^{\frac{p}{2}}\right] \\ &\le E\left[\left(I^*_{n-1}(L,X)\right)^{\frac{np}{n-1}}\right]^{\frac{n-1}{n}} E\left[\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}} \\ &\le c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{n-1}{n}} E\left[\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}} \\ &\le c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]. \end{aligned}$$

Finally, from (3.5), (3.2) and (2.3), we have

$$\begin{aligned} E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] &\le E\left[\left(I^*_{n-1}(L,X)\right)^p\langle X\rangle_L^{\frac{p}{2}}\right] \\ &\le c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)(X_L^*)^{np}\right]^{\frac{n-1}{n}} E\left[\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}} \\ &\le c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)(X_L^*)^{np}\right]. \end{aligned}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

(3.7) $$E\left[\left(\mathcal{L}^*_L(n,X)\right)^p\right] \le c_{n.p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right],$$

(3.8) $$E\left[\left(\mathcal{L}^*_L(n,X)\right)^p\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)(X_L^*)^{np}\right],$$

(3.9) $$E\left[\left(\mathcal{L}^*_L(n,X)\right)^p\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}^*_L(X)\right)^{np}\right],$$

(3.10) $$E\left[\left(I^*_n(L,X)\right)^p\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}^*_L(X)\right)^{np}\right],$$

(3.11) $$E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}^*_L(X)\right)^{np}\right]$$

hold for all continuous local martingales X with $X_0 = 0$ and $n = 1, 2, \ldots$.

Proof. Let $n \ge 2$, $0 < p < \infty$ and let X be a continuous local martingale.

First we prove (3.7). From (2.8), (3.5), (3.1) and the Hölder inequality with exponents $s = n$ and $r = \frac{n}{n-1}$, we have

$$\begin{aligned} E\left[\left(\mathcal{L}^*_L(n,X)\right)^p\right] &\le c_pE\left[\left(1+\log^{\frac{p}{2}}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \\ &\le c_pE\left[\left(1+\log^{\frac{p}{2}}\frac{1}{J_L}\right)\left(I^*_{n-1}(L,X)\right)^p\langle X\rangle_L^{\frac{p}{2}}\right] \\ &\le c_pE\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}} E\left[\left(I^*_{n-1}(L,X)\right)^{\frac{np}{n-1}}\right]^{\frac{n-1}{n}} \\ &\le c_{n.p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]. \end{aligned}$$

Now, by using (3.7), (2.7) and Lemma 2.2, we have

$$\begin{aligned} E\Big[\big(\mathcal{L}_L^*(n,X)\big)^p\Big] &\le c_p E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right] \\ &\le c_p E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)(X_L^*)^{\frac{np}{2}}\big(\mathcal{L}_L^*(X)\big)^{\frac{np}{2}}\right] \\ &\le c_p E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)^2(X_L^*)^{np}\right]^{\frac{1}{2}} E\Big[\big(\mathcal{L}_L^*(X)\big)^{np}\Big]^{\frac{1}{2}} \\ &\le c_{n,p} E\left[\Big(1+\log^{np}\frac{1}{J_L}\Big)(X_L^*)^{np}\right] \end{aligned}$$

and

$$\begin{aligned} E\Big[\big(\mathcal{L}_L^*(n,X)\big)^p\Big] &\le c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right] \\ &\le c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)\big(\mathcal{L}_L^*(X)\big)^{\frac{np}{2}}(X_L^*)^{\frac{np}{2}}\right] \\ &\le c_{n,p} E\left[\Big(1+\log^{np}\frac{1}{J_L}\Big)\big(\mathcal{L}_L^*(X)\big)^{np}\right], \end{aligned}$$

which give (3.8) and (3.9).

Next, from (3.1), (2.7) and (2.9), we have

$$\begin{aligned} E\Big[\big(I_n^*(L,X)\big)^p\Big] &\le c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right] \\ &\le c_{n,p} E\left[\Big(1+\log^{np}\frac{1}{J_L}\Big)\big(\mathcal{L}_L^*(X)\big)^{np}\right]. \end{aligned}$$

Finally, from (3.3), (2.7) and (2.9) we have

$$\begin{aligned} E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] &\le c_{n,p} E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)\langle X\rangle_L^{\frac{np}{2}}\right] \\ &\le c_{n,p} E\left[\Big(1+\log^{np}\frac{1}{J_L}\Big)\big(\mathcal{L}_L^*(X)\big)^{np}\right]. \end{aligned}$$

This completes the proof of Theorem 3.2. □

Now, we consider the reverse of the inequalities in Theorem 3.1 and Theorem 3.2. Let $L \in \mathbb{L}$ and $0 < p < \infty$. Then the inequalities

$$E\left[\Big(1+\log^{\frac{np}{2}}\frac{1}{J_L}\Big)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le c_{n,p} E\left[\Big(1+\log^{np}\frac{1}{J_L}\Big)\big(I_n^*(L,X)\big)^p\right] \qquad (n \ge 1) \tag{3.12}$$

follow from (2.5) and Lemma 2.2 for all continuous local martingales X with $X_0 = 0$. Furthermore, in [11, p.161] M. Yor showed that for any non-increasing continuous function $g : (0,1] \to \mathbb{R}_+$ the inequality

$$E\Big[g(J_L)X_L^*\Big] \le c_g E\Big[(gg_{\frac{1}{2}})(J_L)\langle X\rangle_L^{\frac{1}{2}}\Big] \tag{3.13}$$

holds for all continuous local martingales X with $X_0 = 0$, where $g_\gamma(x) = 1 + \log^\gamma \frac{1}{x}$ ($\gamma \ge 0,\ x \in (0,1]$). As a consequence of the inequality, we have

Lemma 3.3. *Let* $0 < p < \infty$ *and* $L \in \mathbb{L}$*. Then the inequality*

$$E\left[\left(1+\log^{\gamma p}\frac{1}{J_L}\right)(X_L^*)^p\right] \le c_{\gamma,p} E\left[\left(1+\log^{(\gamma+\frac{1}{2})p}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{p}{2}}\right] \qquad (\gamma \ge 0) \tag{3.14}$$

holds for all continuous local martingales X *with* $X_0 = 0$*.*

Proof. Let $\gamma \ge 0$ and let X be a continuous local martingale. Then we have from (3.13)

$$E\left[\left(1+\log^{\gamma}\frac{1}{J_L}\right)X_L^*\right] \le c_\gamma E\left[\left(1+\log^{\gamma+\frac{1}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{1}{2}}\right], \tag{3.15}$$

since g_γ is non-increasing and $(g_\gamma g_{\frac{1}{2}})(x) \le c_\gamma\left(1+\log^{\gamma+\frac{1}{2}}\frac{1}{x}\right)$.

Now, denote for $t \ge 0$

$$A_t = \left(1+\log^{\gamma}\frac{1}{J_L}\right)X_{L\wedge t}^* \quad \text{and} \quad B_t = \left(1+\log^{\gamma+\frac{1}{2}}\frac{1}{J_L}\right)\langle X\rangle_{L\wedge t}^{\frac{1}{2}}.$$

Then for any couple (S,T) of stopping times S,T with $T \ge S \ge 0$

$$\begin{aligned} E\big[A_T - A_S\big] &= E\left[\left(1+\log^{\gamma}\frac{1}{J_L}\right)(X_{L\wedge T}^* - X_{L\wedge S}^*)\right] \\ &\le E\left[\left(1+\log^{\gamma}\frac{1}{J_L}\right)\sup_{S\le t\le T}|X_{L\wedge t} - X_{L\wedge S}|1_{\{S<T\}}\right] \\ &= E\left[\left(1+\log^{\gamma}\frac{1}{J_L}\right)\sup_{t\ge 0}|X_{T\wedge(S+t)}^L - X_S^L|1_{\{S<T\}}\right] \\ &\equiv E\left[\left(1+\log^{\gamma}\frac{1}{J_L}\right)\sup_{t\ge 0}|(X_{T\wedge(S+t)} - X_S)^L|1_{\{S<T\}}\right], \end{aligned}$$

where $X_t^L \equiv X_{t\wedge L}$.

Observe that $(X_{(S+t)\wedge T} - X_S)1_{\{S<T\}}, t \ge 0$ is a continuous $(\mathcal{F}_{S+t})$–local martingale, we find by (3.15)

$$\begin{aligned} E\big[A_T - A_S\big] &\le c_\gamma E\left[\left(1+\log^{\gamma+\frac{1}{2}}\frac{1}{J_L}\right)\langle X\rangle_{L\wedge T}^{\frac{1}{2}}1_{\{S<T\}}\right] \\ &= E\Big[c_\gamma B_T 1_{\{S<T\}}\Big] \le \big\|c_\gamma B_T\big\|_\infty P(S<T). \end{aligned}$$

It follows from Lemma 7 and Lemma 8 in [5] with $C = c_\gamma B$, $\alpha = \beta = 1$ that for all $0 < p < \infty$

$$E\left[\left(1+\log^{\gamma}\frac{1}{J_L}\right)^p(X_L^*)^p\right] \le c_{\gamma,p}E\left[\left(1+\log^{\gamma+\frac{1}{2}}\frac{1}{J_L}\right)^p\langle X\rangle_L^{\frac{p}{2}}\right].$$

Thus, (3.14) follows from the inequalities

$$\hat{c}_p(a^p + b^p) \le (a+b)^p \le c_p(a^p+b^p) \qquad (p,a,b \ge 0).$$

This completes the proof. □

On the other hand, in [2], E. Carlen and P. Krée obtained the identity

$$I_n(t,X)I_{n-2}(t,X) = I_{n-1}^2(t,X) - \sum_{j=1}^{n}\frac{(n-j)!}{n!}I_{n-j}^2(t,X)\langle X\rangle_t^{j-1} \quad (n \ge 2)$$

for all $t \ge 0$ and all continuous local martingales X with $X_0 = 0$. It follows that

$$\frac{1}{n!}\langle X\rangle_t^{n-1} \le \frac{n-1}{n}I_{n-1}^2(t,X) - I_n(t,X)I_{n-2}(t,X) \quad (n \ge 2).$$

Integrating both sides of the inequality above on $[0,t]$ with respect to the measure $d\langle X\rangle_t$, we get

$$\frac{1}{n!}\langle X\rangle_t^n \leq (n-1)\langle I_n(X)\rangle_t^2 - n\int_0^t I_n(s,X)I_{n-2}(s,X)d\langle X\rangle_s \quad (n\geq 2)$$

since $\langle I_n(X)\rangle_t = \int_0^t I_{n-1}^2(s,X)d\langle X\rangle_s$, which gives

$$\frac{\langle X\rangle_t^{\frac{n}{2}}}{\sqrt{n!}} \leq \sqrt{n-1}\big\langle I_n(X)\big\rangle_t^{\frac{1}{2}} + \sqrt{n}\left(I_n^*(t,X)I_{n-2}^*(t,X)\langle X\rangle_t\right)^{\frac{1}{2}} \quad (n\geq 2). \tag{3.16}$$

Theorem 3.4. *Let $0<p<\infty$ and $L\in\mathbb{L}$. If V is one of the three random variables X_L^*, $\langle X\rangle_L^{\frac{1}{2}}$ and $\mathcal{L}_L^*(X)$, then the inequalities*

$$E\big[V^{np}\big] \leq c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right], \tag{3.17}$$

$$E\big[V^{np}\big] \leq c_{n,p}E\left[\left(1+\log^{(n+\frac{1}{2})p}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right], \tag{3.18}$$

$$E\big[V^{np}\big] \leq c_{n,p}E\left[\left(1+\log^{(2n+1)p}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(n,X)\right)^p\right] \tag{3.19}$$

hold for all continuous local martingales X with $X_0=0$ and $n=1,2,\ldots$.

Proof. Let $n\geq 2$, $0<p<\infty$ and let X be a continuous local martingale.

For $n\geq 3$, by applying the Hölder inequality with exponents $s=n$ and $r=\frac{n}{n-2}$ and Theorem 3.1 we have

$$\begin{aligned}E\Big[\left(I_{n-2}^*(L,X)\langle X\rangle_L\right)^p\Big] &\leq E\Big[\left(I_{n-2}^*(L,X)\right)^{\frac{np}{n-2}}\Big]^{\frac{n-2}{n}}E\Big[\langle X\rangle_L^{\frac{np}{2}}\Big]^{\frac{2}{n}}\\ &\leq c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right].\end{aligned}$$

Clearly, the inequality above is also true for $n=2$.

It follows from (3.16) that for $n\geq 2$

$$\begin{aligned}&\left(\frac{1}{\sqrt{n!}}\right)^p E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]\\ &\leq E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\left(\sqrt{n-1}\langle I_n(X)\rangle_L^{\frac{1}{2}}+\sqrt{n}\left(I_n^*(L,X)I_{n-2}^*(L,X)\langle X\rangle_L\right)^{\frac{1}{2}}\right)^p\right]\\ &\leq \hat{c}_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]\\ &\qquad + c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]^{\frac{1}{2}}E\Big[\left(I_{n-2}^*(L,X)\langle X\rangle_L\right)^p\Big]^{\frac{1}{2}}\\ &\leq \hat{c}_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]\\ &\qquad + c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]^{\frac{1}{2}}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{2}}.\end{aligned}$$

Combining this with (3.12), we get the quadratic inequality as follows

$$E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$
$$\le \hat{c}_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]+c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]^{\frac{1}{2}}$$
$$\times E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{2}}.$$

Solving the above quadratic inequality leads to the inequality

$$E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]\le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]. \tag{3.20}$$

Consequently, by Lemma 3.3

$$E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]\le c_{n,p}E\left[\left(1+\log^{(n+\frac{1}{2})p}\frac{1}{J_L}\right)\big\langle I_n(X)\big\rangle_L^{\frac{p}{2}}\right], \tag{3.21}$$

and so by (2.5) and (2.9)

$$E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]\le c_{n,p}E\left[\left(1+\log^{(2n+1)p}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(n,X)\right)^p\right]. \tag{3.22}$$

Now, the inequalities (3.17) – (3.19) are consequences of (3.20) – (3.22) by Lemma 2.1 and Lemma 2.2. This completes the proof. □

Remark 3.5. Let $0<p<\infty$ and $L\in\mathbb{L}$. As some special cases of the inequalities in Theorem 3.4, we can show that the inequalities

$$E\Big[\langle X\rangle_L^p\Big]\le c_pE\left[\left(1+\log^{\frac{p}{2}}\frac{1}{J_L}\right)\left(I_2^*(L,X)\right)^p\right], \tag{3.23}$$

$$E\Big[\langle X\rangle_L^p\Big]\le c_pE\left[\left(1+\log^{\frac{p}{2}}\frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right], \tag{3.24}$$

$$E\Big[(X_L^*)^{2p}\Big]\le c_pE\left[\left(1+\log^{\frac{p}{2}}\frac{1}{J_L}\right)\left(I_2^*(L,X)\right)^p\right], \tag{3.25}$$

$$E\Big[(X_L^*)^{2p}\Big]\le c_pE\left[\left(1+\log^{\frac{p}{2}}\frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right], \tag{3.26}$$

$$E\Big[(X_L^*)^{2p}\Big]\le c_pE\left[\left(1+\log^{p}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(2,X)\right)^p\right], \tag{3.27}$$

$$E\Big[\langle X\rangle_L^p\Big]\le c_pE\left[\left(1+\log^{p}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(2,X)\right)^p\right], \tag{3.28}$$

$$E\Big[\left(\mathcal{L}_L^*(X)\right)^{2p}\Big]\le c_pE\left[\left(1+\log^{2p}\frac{1}{J_L}\right)\left(I_2^*(L,X)\right)^p\right], \tag{3.29}$$

$$E\Big[\left(\mathcal{L}_L^*(X)\right)^{2p}\Big]\le c_pE\left[\left(1+\log^{\frac{3p}{2}}\frac{1}{J_L}\right)\big\langle I_2(X)\big\rangle_L^{\frac{p}{2}}\right], \tag{3.30}$$

$$E\Big[\left(\mathcal{L}_L^*(X)\right)^{2p}\Big]\le c_pE\left[\left(1+\log^{3p}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(2,X)\right)^p\right] \tag{3.31}$$

hold for all continuous local martingales X with $X_0 = 0$. In fact, from (3.16) we have

$$E\Big[\langle X\rangle_L^p\Big] \le \hat{c}_p E\left[\big\langle I_2(X)\big\rangle_L^{\frac{p}{2}}\right] + c_p E\left[\big(I_2^*(L,X)\langle X\rangle_L\big)^{\frac{p}{2}}\right]$$

for $0 < p < \infty$ and so

$$E\Big[\langle X\rangle_L^p\Big] \le \hat{c}_p E\Big[\big\langle I_2(X)\big\rangle_L^{\frac{p}{2}}\Big] + c_p E\Big[\big(I_2^*(L,X)\big)^p\Big]^{\frac{1}{2}} E\Big[\langle X\rangle_L^p\Big]^{\frac{1}{2}}.$$

Combining this with Lemma 2.1, we find

$$E\Big[\langle X\rangle_L^p\Big] \le \hat{c}_p E\left[\Big(1+\log^{\frac{p}{2}}\frac{1}{I_L}\Big)\big(I_2^*(L,X)\big)^p\right] + c_p E\left[\Big(1+\log^{\frac{p}{2}}\frac{1}{I_L}\Big)\big(I_2^*(L,X)\big)^p\right]^{\frac{1}{2}} E\Big[\langle X\rangle_L^p\Big]^{\frac{1}{2}}$$

and

$$E\Big[\langle X\rangle_L^p\Big] \le \hat{c}_p E\left[\Big(1+\log^{\frac{p}{2}}\frac{1}{I_L}\Big)\big\langle I_2(X)\big\rangle_L^{\frac{p}{2}}\right] + c_p E\left[\Big(1+\log^{\frac{p}{2}}\frac{1}{I_L}\Big)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]^{\frac{1}{2}} E\Big[\langle X\rangle_L^p\Big]^{\frac{1}{2}}.$$

The above quadratic inequalities lead to (3.23) and (3.24).

Next, observe that from (3.6)

$$(X_L^*)^2 \le 2I_2^*(L,X) + \langle X\rangle_L,$$

we obtain the inequalities (3.25) – (3.28).

Finally, combining (3.16) with Lemma 3.3, we get

$$\begin{aligned}
E&\left[\Big(1+\log^p\frac{1}{J_L}\Big)\langle X\rangle_L^p\right]\\
&\le E\left[\Big(1+\log^p\frac{1}{J_L}\Big)\Big(\sqrt{2}\langle I_2(X)\rangle_L^{\frac{1}{2}} + 2\big(I_2^*(L,X)\langle X\rangle_L\big)^{\frac{1}{2}}\Big)^p\right]\\
&\le \hat{c}_p E\left[\Big(1+\log^p\frac{1}{J_L}\Big)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]\\
&\quad + c_p E\left[\Big(1+\log^p\frac{1}{J_L}\Big)\big(I_2^*(L,X)\big)^p\right]^{\frac{1}{2}} E\left[\Big(1+\log^p\frac{1}{J_L}\Big)\langle X\rangle_L^p\right]^{\frac{1}{2}}.\\
&\le \hat{c}_p E\left[\Big(1+\log^{\frac{3p}{2}}\frac{1}{J_L}\Big)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]\\
&\quad + c_p E\left[\Big(1+\log^{\frac{3p}{2}}\frac{1}{J_L}\Big)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]^{\frac{1}{2}} E\left[\Big(1+\log^p\frac{1}{J_L}\Big)\langle X\rangle_L^p\right]^{\frac{1}{2}},
\end{aligned}$$

which gives a quadratic inequality

$$x^2 - \hat{c}_p y^2 - c_p xy \le 0 \qquad (\hat{c}_p, c_p \ge 0)$$

with

$$x = E\left[\Big(1+\log^p\frac{1}{J_L}\Big)\langle X\rangle_L^p\right]^{\frac{1}{2}} \quad \text{and} \quad y = E\left[\Big(1+\log^{\frac{3p}{2}}\frac{1}{J_L}\Big)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]^{\frac{1}{2}}.$$

Solving the quadratic inequality leads to

$$E\left[\left(1+\log^{p}\frac{1}{J_L}\right)\langle X\rangle_L^p\right]\le c_pE\left[\left(1+\log^{\frac{3p}{2}}\frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right],$$

which gives (3.30) and (3.31).

Thus, we obtain the inequalities (3.23) – (3.31).

REFERENCES

[1] M.T. BARLOW AND M. YOR, (Semi–)Martingale inequalities and local times, *Z. W.*, **55** (1981), 237–254.

[2] E. CARLEN AND P. KRÉE, L^p–estimates on iterated stochastic integrals, *Ann. Probab.*, **19** (1991), 354–368.

[3] C.S. CHOU, On some inequalities of local time, *J. Theoret. Probab.*, **8**(1) (1995), 17–22.

[4] K. L. CHUNG AND R. J. WILLIAMS, *Introduction to stochastic integration*, Second Edition, Boston, Basel and Stuttgart, Birkhäuser 1990.

[5] S. D. JACKA AND M. YOR, Inequalities for non-moderate functions of a pair of stochastic processes, *Proc. London Math. Soc.*, **67** (1993), 649–672.

[6] E. LENGLART, D. LÉPINGLE AND M. PRATELLI, *Présentation unifiée de certaines inégalités de la théorie des martingales*, Sém. Proba. XIV, Lect. Notes in Math., **784**, Berlin, Heidelberg and New York, Springer 1980.

[7] D. REVUZ AND M. YOR, *Continuous Martingales and Brownian Motion*, Third edition, Berlin, Heidelberg and New York, Springer–Varlag 1999.

[8] L. YAN, Some inequalities for continuous martingales associated with the Hermite polynomials, *Kobe J. Math.*, **17** (2000), 191–200.

[9] L. YAN, Two inequalities for iterated stochastic integrals, *Archiv der Mathematik*, to appear.

[10] M. YOR, Inégalités de martingales continues arrétées á un temps quelconque (I): théorémes géraux. *Lect. Notes in Math.*, **1118**, 110–146, Springer, Berlin 1985.

[11] M. YOR, Inégalités de martingales continues arrétées á un temps quelconque (II): le rôle de certains espaces BMO, *Lect. Notes in Math.*, **1118**, 147–171, Springer, Berlin 1985.

[12] M. YOR, *Some Aspects of Brownian motion, Part II: Some recent martingale problems*, Lect. in Math. ETH Zürich, Birkhäuser 1997.