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ON SOME INEQUALITIES OF LOCAL TIMES OF ITERATED STOCHASTIC INTEGRALS

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Abstract

Contents

◀◀ ▶▶

◀ ▶

Home Page

Go Back

Close

Quit

Abstract

Let $X = (X_t, \mathcal{F}_t)_{t \geq 0}$ be a continuous local martingale with quadratic variation process $\langle X \rangle$ and $X_0 = 0$. Define iterated stochastic integrals $I_n(X) = (I_n(t, X), \mathcal{F}_t)$ $(n \geq 0)$, inductively by
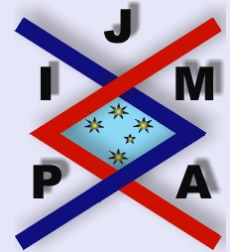
$$I_n(t, X) = \int_0^t I_{n-1}(s, X) dX_s$$

with $I_0(t, X) = 1$ and $I_1(t, X) = X_t$. In this paper, we obtain some martingale inequalities for $I_n(X)$, $n = 1, 2, \ldots$ and their local times at any random time.

Contents

On Some Inequalities of Local Times of Iterated Stochastic Integrals

Litan Yan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

1. Introduction

Let $X = (X_t)_{t \geq 0}$ be a continuous local martingale with quadratic variation process $\langle X \rangle$ and $X_0 = 0$, defined on some filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Consider the corresponding sequence of iterated stochastic integrals,

$$I_n(X) = (I_n(t, X), \mathcal{F}_t) \quad (n \geq 0),$$

defined inductively by

$$(1.1) \qquad I_n(t, X) = \int_0^t I_{n-1}(s, X) dX_s,$$
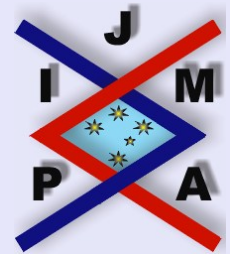
where $I_0(t, X) = 1$ and $I_1(t, X) = X_t$.

It is known that there exist positive constants $B_{n,p}$ and $A_{n,p}$ depending only on n and p, such that the inequalities (see [2, 8])

$$(1.2) \qquad A_{n,p} \left\| \langle X \rangle_T^{\frac{n}{2}} \right\|_p \leq \left\| \sup_{0 \leq t \leq T} |I_n(t, X)| \right\|_p$$
$$\leq B_{n,p} \left\| \langle X \rangle_T^{\frac{n}{2}} \right\|_p \quad (0 < p < \infty),$$

hold for all continuous local martingales X with $X_0 = 0$ and all $(\mathcal{F}_t)$–stopping time T.

On the other hand, M.T. Barlow and M. Yor have established in [1] (see also Theorem 2.4 in [7, p.457]) the following martingale inequalities for local times:

$$c_p \left\| \langle X \rangle_\infty^{\frac{1}{2}} \right\|_p \leq \| \mathcal{L}_\infty^*(X) \|_p \leq C_p \left\| \langle X \rangle_\infty^{\frac{1}{2}} \right\|_p \quad (0 < p < \infty),$$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

where $(\mathcal{L}_t^x(X); t \geq 0)$ is the local time of X at x and $\mathcal{L}_t^*(X) = \sup_{x \in \mathbb{R}} \mathcal{L}_t^x(X)$. It follows that for all $0 < p < \infty$

(1.3) $$c_{n,p} \left\| \langle X \rangle_T^{\frac{n}{2}} \right\|_p \leq \left\| \mathcal{L}_T^*(n, X) \right\|_p \leq C_{n,p} \left\| \langle X \rangle_T^{\frac{n}{2}} \right\|_p$$

for all $(\mathcal{F}_t)$–stopping times T, where $(\mathcal{L}_t^x(n, X); t \geq 0)$ stands for the local time of $I_n(X)$ at x.

However, it is clear that the inequalities (1.2) and (1.3) are not true when T is replaced by an arbitrary $\mathbb{R}_+$–valued random time (see, for example, [12] when $n = 1$). In this paper we extend (1.2) and (1.3) to any random time.
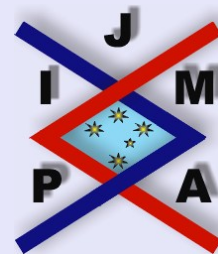
Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

2. Preliminaries

Throughout this paper, we fix a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with the usual conditions. For any process $X = (X_t)_{t \geq 0}$, denote $X_\tau^* = \sup_{0 \leq t \leq \tau} |X_t|$ and $X^* = \sup_{0 \leq t < \infty} |X_t|$. Let c stand for some positive constant depending only on the subscripts whose value may be different in different appearances, and this assumption is also made for $\hat{c}$.

From now on an $\mathcal{F}$–measurable non–negative random variable $L : \Omega \to \mathbb{R}_+$ is called a random time and we denote by $\mathbb{L}$ the collection of all random times, i.e.,

$$\mathbb{L} = \{L \; : \; L \text{ is an } \mathcal{F}\text{–measurable, non–negative, random variable}\} .$$

For any $L \in \mathbb{L}$, let (G_t^L) be the smallest filtration satisfying the usual conditions which both contains $(\mathcal{F}_t)$ and makes L a (G_t^L)–stopping time. Define
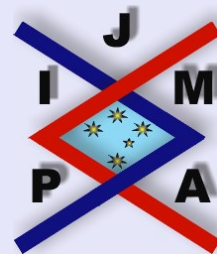
$$Z_t^L = E\left[1_{\{L > t\}} | \mathcal{F}_t\right] \qquad \text{and} \qquad J_L = \inf_{s < L} Z_s^L.$$

Then $Z^L = (Z_t^L)$ is a potential of class (D). Assume that the Doob–Meyer decomposition for Z^L is

$$(2.1) \qquad\qquad Z^L = M - A.$$

For simplicity, in the present paper we assume throughout that $L \in \mathbb{L}$ avoids $(\mathcal{F}_t)$–stopping times, i.e.,

for every $(\mathcal{F}_t)$–stopping time T, $P(L = T) = 0$.

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

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Thus, under the condition, Z^L is continuous and so M is also continuous. Furthermore, for any continuous $(\mathcal{F}_t)$–local martingale X there exists a continuous (G_t^L)–local martingale $\widetilde{X}$ with $\langle X \rangle_{L \wedge t} = \langle \widetilde{X} \rangle_t$ such that

$$X_{L \wedge t} = \widetilde{X}_t + \int_0^{L \wedge t} \frac{d\langle X, M \rangle_s}{Z_s^L},$$

where $L \wedge t = \min\{L, t\}$. For more information on $X^L = (X_{L \wedge t})_{t \geq 0}$ and (G_t^L), see [10, 11, 12].

Lemma 2.1 ([10]). *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

$$(2.2) \qquad E\big[(X_L^*)^p\big] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right) \langle X \rangle_L^{\frac{p}{2}}\right],$$

$$(2.3) \qquad E\left[\langle X \rangle_L^{\frac{p}{2}}\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)(X^*)_L^p\right]$$

hold for all continuous $(\mathcal{F}_t)$–local martingales X vanishing at zero.

It is known that the inequalities in Lemma 2.1 are the extensions to the Burkholder-Davis-Gundy inequalities. For the proof, see Proposition 4 in [10, p.122] (or Theorem 13.4 in [12, p.57]).

Let X now be a continuous semimartingale. Then for every $x \in \mathbb{R}$ the following Meyer–Tanaka formula may be considered as a definition of the local time $\{\mathcal{L}_t^x(X); t \geq 0\}$ of X at x

$$|X_t - x| - |X_0 - x| = \int_0^t \operatorname{sgn}(X_s - x) dX_s + \mathcal{L}_t^x(X).$$
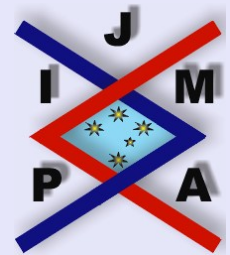
On Some Inequalities of Local Times of Iterated Stochastic Integrals

Litan Yan

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

Page 6 of 27

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

http://jipam.vu.edu.au

One may take a version $\mathcal{L} : (x, t, \omega) \to \mathcal{L}_t^x(\omega)$ which is right continuous and has a left limit at x, and is continuous in t. In particular, if X is a continuous local martingale, then $\mathcal{L}_t^x(X)$ has a continuous version in both variables. In this paper, we use such a version of local time.

The fundamental formula of occupation density for a continuous semimartingale is:

$$(2.4) \qquad \int_0^t \Phi(X_s) d\langle X \rangle_s = \int_{-\infty}^{\infty} \Phi(x) \mathcal{L}_t^x(X) dx$$

for all bounded, Borel functions $\Phi : \mathbb{R} \to \mathbb{R}$, which gives

$$(2.5) \qquad \langle X \rangle_\infty \leq 2 X_\infty^* \mathcal{L}_\infty^*(X)$$

since $\mathcal{L}_\infty^x = 0$ for all $x \notin [-X^*, X^*]$. It follows that (see [3]) for all continuous $(\mathcal{F}_t)$–local martingales X, and all $t \geq 0, x \in \mathbb{R}$ and $L \in \mathbb{L}$

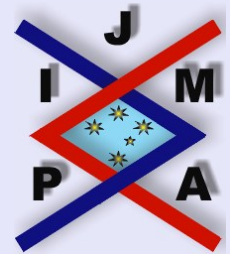$$(2.6) \qquad \mathcal{L}_{L \wedge t}^x(X) = \mathcal{L}_t^x(X^L)$$

if M is continuous, where $X^L = (X_{L \wedge t})$. So, we have

$$(2.7) \qquad \langle X \rangle_L = \langle X^L \rangle_\infty \leq 2 \mathcal{L}_\infty^*(X^L) X_L^* = 2 \mathcal{L}_L^*(X) X_L^*$$

by (2.5). Furthermore, the following lemma which can be found in [3] extends the Barlow–Yor inequalities.

Lemma 2.2. *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

$$(2.8) \quad E\left[\left(\mathcal{L}_L^*(X) \right)^p \right]$$
$$\leq c_p \min \left\{ E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L} \right) \langle X \rangle_L^{\frac{p}{2}} \right], \ E\left[\left(1 + \log^p \frac{1}{J_L} \right) (X_L^*)^p \right] \right\}$$

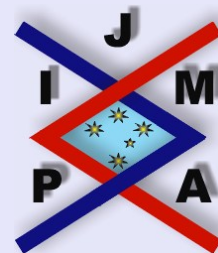*J. Ineq. Pure and Appl. Math.* **3**(4) Art. 62, 2002

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and

$$(2.9) \quad \max\left\{ E\left[(X_L^*)^p\right],\ E\left[\langle X\rangle_L^{\frac{p}{2}}\right] \right\} \le c_p E\left[\left(1 + \log^p \frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^p\right]$$

hold.

Remark 2.1. In [3], C. S. Chou proved that (2.8) and (2.9) hold for $1 \le p < \infty$. In fact, when $0 < p < 1$ (2.8) and (2.9) are also true from the proof in [3].

Title Page

Contents

◀◀　　▶▶

◀　　　▶

Go Back

Close

Quit

3. Inequalities and proofs

In this section, we shall extend (1.2) and (1.3) to any random time $L \in \mathbb{L}$.

Theorem 3.1. *Let* $0 < p < \infty$ *and* $L \in \mathbb{L}$. *Then the inequalities*

$$(3.1) \qquad E\left[\left(I_n^*(L, X)\right)^p\right] \le c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right],$$

$$(3.2) \qquad E\left[\left(I_n^*(L, X)\right)^p\right] \le c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)(X_L^*)^{np}\right],$$

$$(3.3) \qquad E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right],$$

$$(3.4) \qquad E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)(X_L^*)^{np}\right]$$

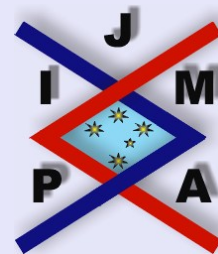*hold for all continuous local martingales* X *with* $X_0 = 0$ *and* $n = 1, 2, \ldots$.

Proof. Let $n \ge 1$, $L \in \mathbb{L}$ and let X be a continuous local martingale.

(3.1) can be verified by induction. In fact, when $n = 1$ (3.1) is true from (2.2). Now suppose that (3.1) is true for $2, \ldots, n-1$. Then we have

$$E\left[\left(I_{n-1}^*(L, X)\right)^{\frac{np}{n-1}}\right] \le c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right].$$

On the other hand, from (1.1) we see that

$$\langle I_n(X)\rangle_t = \int_0^t \left(I_{n-1}(s, X)\right)^2 d\langle X\rangle_s \le \sup_{0 \le s \le t}\left(I_{n-1}(s, X)\right)^2 \langle X\rangle_t$$

for all $t \geq 0$, which gives

$$(3.5) \qquad \langle I_n(X) \rangle_L \leq \left(I_{n-1}^*(L, X) \right)^2 \langle X \rangle_L.$$

Thus, by applying (2.2), (3.5) and then applying the Hölder inequality with exponents $s = n$ and $r = \frac{n}{n-1}$, and noting

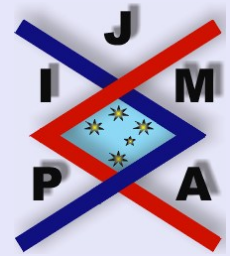$$(a + b)^n \leq c_n(a^n + b^n) \qquad (a, b \geq 0),$$

we find

$$
\begin{aligned}
E\left[\left(I_n^*(L, X) \right)^p \right] &\leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L} \right) \langle I_n(X) \rangle_L^{\frac{p}{2}} \right] \\
&\leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L} \right) \left(I_{n-1}^*(L, X) \right)^p \langle X \rangle_L^{\frac{p}{2}} \right] \\
&\leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L} \right)^n \langle X \rangle_L^{\frac{np}{2}} \right]^{\frac{1}{n}} E\left[\left(I_{n-1}^*(L, X) \right)^{\frac{np}{n-1}} \right]^{\frac{n-1}{n}} \\
&\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L} \right) \langle X \rangle_L^{\frac{np}{2}} \right].
\end{aligned}
$$

This establishes (3.1).

Now, we verify (3.2). From the well-known correspondence of iterated stochastic integral $I_n(X)$ and the Hermite polynomial of degree n (see [4, 7])

$$I_n(t, X) = \frac{1}{n!} H_n(X_t, \langle X \rangle_t),$$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 10 of 27

where $H_n(x, y) = y^{\frac{n}{2}} h_n\left(\frac{x}{\sqrt{y}}\right)$ $(y > 0)$ and $h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ is the Hermite polynomial of degree n, more generally, $H_n(x, y)$ can be defined as

$$H_n(x, y) = (-y)^n e^{\frac{x^2}{2y}} \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2y}},$$

we see that iterated stochastic integrals $I_n(X)$, $n = 1, 2, \ldots$ have the representation
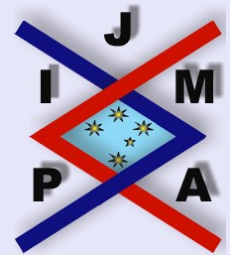
$$(3.6) \qquad I_n(t, X) = \sum_{j=0}^{\left[\frac{n}{2}\right]} C_n^{(j)} X_t^{n-2j} \langle X \rangle_t^j,$$

where $C_n^{(j)} = \left(-\frac{1}{2}\right)^j \frac{1}{(n-2j)! j!}$ and $[x]$ stands for the integer part of x.

On the other hand, for $0 < j < \frac{n}{2}$, by using the Hölder inequality with exponents $s = \frac{n}{n-2j}$ and $r = \frac{n}{2j}$, we get

$$E\left[(X_L^*)^{(n-2j)p} \langle X \rangle_L^{jp}\right] \le E\left[(X_L^*)^{np}\right]^{\frac{n-2j}{n}} E\left[\langle X \rangle_L^{\frac{np}{2}}\right]^{\frac{2j}{n}}$$

$$\le c_{n,p} E\left[(X_L^*)^{np}\right]^{\frac{n-2j}{n}} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)(X_L^*)^{np}\right]^{\frac{2j}{n}}$$

$$\le c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)(X_L^*)^{np}\right].$$

Clearly, the inequality above is also true for $j = \frac{n}{2}$ and $j = 0$.

Combining this with (3.6), we get for $0 < p \leq 1$

$$E\left[\left(I_n^*(L, X)\right)^p\right] \leq c_p \sum_{j=0}^{\left[\frac{n}{2}\right]} |C_n^{(j)}|^p E\left[(X_L^*)^{(n-2j)p}(\langle X \rangle_L)^{jp}\right]$$

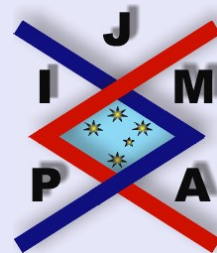$$\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)(X_L^*)^{np}\right]$$

and for $1 < p < \infty$

$$E\left[\left(I_n^*(L, X)\right)^p\right]^{\frac{1}{p}} \leq \sum_{j=0}^{\left[\frac{n}{2}\right]} |C_n^{(j)}| E\left[(X_L^*)^{(n-2j)p}\langle X \rangle_L^{jp}\right]^{\frac{1}{p}}$$

$$\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)(X_L^*)^{np}\right]^{\frac{1}{p}}.$$

This gives (3.2).

Next, from (3.5) and (3.1) we see that

$$E\left[\langle I_n(X) \rangle_L^{\frac{p}{2}}\right] \leq E\left[\left(I_{n-1}^*(L, X)\right)^p \langle X \rangle_L^{\frac{p}{2}}\right]$$

$$\leq E\left[\left(I_{n-1}^*(L, X)\right)^{\frac{np}{n-1}}\right]^{\frac{n-1}{n}} E\left[\langle X \rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}}$$

$$\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X \rangle_L^{\frac{np}{2}}\right]^{\frac{n-1}{n}} E\left[\langle X \rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}}$$

$$\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X \rangle_L^{\frac{np}{2}}\right].$$

Finally, from (3.5), (3.2) and (2.3), we have

$$E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le E\left[\left(I_{n-1}^*(L,X)\right)^p \langle X\rangle_L^{\frac{p}{2}}\right]$$

$$\le c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)(X_L^*)^{np}\right]^{\frac{n-1}{n}} E\left[\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}}$$

$$\le c_{n,p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)(X_L^*)^{np}\right].$$

This completes the proof of Theorem 3.1. $\qquad\square$

Theorem 3.2. *Let $0 < p < \infty$ and $L \in \mathbb{L}$. Then the inequalities*

$$(3.7) \qquad E\left[\left(\mathcal{L}_L^*(n,X)\right)^p\right] \le c_{n.p}E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right],$$
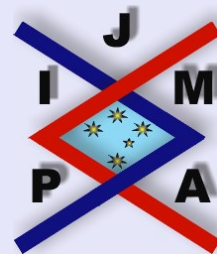
$$(3.8) \qquad E\left[\left(\mathcal{L}_L^*(n,X)\right)^p\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)(X_L^*)^{np}\right],$$

$$(3.9) \qquad E\left[\left(\mathcal{L}_L^*(n,X)\right)^p\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{np}\right],$$

$$(3.10) \qquad E\left[\left(I_n^*(L,X)\right)^p\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{np}\right],$$

$$(3.11) \qquad E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le c_{n,p}E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{np}\right]$$

hold for all continuous local martingales X with $X_0 = 0$ and $n = 1, 2, \ldots$.
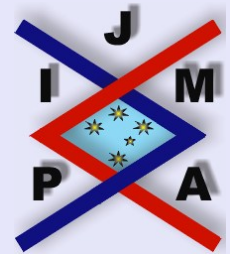
Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

Proof. Let $n \geq 2$, $0 < p < \infty$ and let X be a continuous local martingale.

First we prove (3.7). From (2.8), (3.5), (3.1) and the Hölder inequality with exponents $s = n$ and $r = \frac{n}{n-1}$, we have

$$
\begin{aligned}
E\left[\left(\mathcal{L}_L^*(n, X)\right)^p\right] &\leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right) \langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \\
&\leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right) \left(I_{n-1}^*(L, X)\right)^p \langle X\rangle_L^{\frac{p}{2}}\right] \\
&\leq c_p E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right) \langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{n}} E\left[\left(I_{n-1}^*(L, X)\right)^{\frac{np}{n-1}}\right]^{\frac{n-1}{n}} \\
&\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right) \langle X\rangle_L^{\frac{np}{2}}\right].
\end{aligned}
$$

Now, by using (3.7), (2.7) and Lemma 2.2, we have

$$
\begin{aligned}
E\left[\left(\mathcal{L}_L^*(n, X)\right)^p\right] &\leq c_p E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right) \langle X\rangle_L^{\frac{np}{2}}\right] \\
&\leq c_p E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right) (X_L^*)^{\frac{np}{2}} \left(\mathcal{L}_L^*(X)\right)^{\frac{np}{2}}\right] \\
&\leq c_p E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)^2 (X_L^*)^{np}\right]^{\frac{1}{2}} E\left[\left(\mathcal{L}_L^*(X)\right)^{np}\right]^{\frac{1}{2}} \\
&\leq c_{n,p} E\left[\left(1 + \log^{np} \frac{1}{J_L}\right) (X_L^*)^{np}\right]
\end{aligned}
$$

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

http://jipam.vu.edu.au

and

$$E\left[\left(\mathcal{L}_L^*(n,X)\right)^p\right] \le c_{n,p} E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$

$$\le c_{n,p} E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{\frac{np}{2}}\left(X_L^*\right)^{\frac{np}{2}}\right]$$

$$\le c_{n,p} E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{np}\right],$$
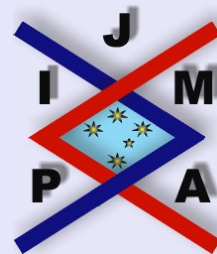
which give (3.8) and (3.9).

Next, from (3.1), (2.7) and (2.9), we have

$$E\left[\left(I_n^*(L,X)\right)^p\right] \le c_{n,p} E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$

$$\le c_{n,p} E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{np}\right].$$

Finally, from (3.3), (2.7) and (2.9) we have

$$E\left[\langle I_n(X)\rangle_L^{\frac{p}{2}}\right] \le c_{n,p} E\left[\left(1+\log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$

$$\le c_{n,p} E\left[\left(1+\log^{np}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(X)\right)^{np}\right].$$

This completes the proof of Theorem 3.2. $\qquad\square$

Now, we consider the reverse of the inequalities in Theorem 3.1 and Theorem 3.2. Let $L \in \mathbb{L}$ and $0 < p < \infty$. Then the inequalities

$$(3.12) \quad E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]$$
$$\leq c_{n,p} E\left[\left(1 + \log^{np} \frac{1}{J_L}\right)\left(I_n^*(L, X)\right)^p\right] \qquad (n \geq 1)$$

follow from (2.5) and Lemma 2.2 for all continuous local martingales X with $X_0 = 0$. Furthermore, in [11, p.161] M. Yor showed that for any non-increasing continuous function $g : (0, 1] \to \mathbb{R}_+$ the inequality

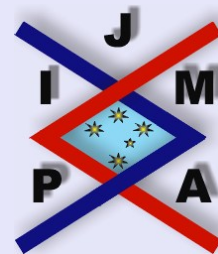$$(3.13) \qquad E\left[g(J_L)X_L^*\right] \leq c_g E\left[(gg_{\frac{1}{2}})(J_L)\langle X\rangle_L^{\frac{1}{2}}\right]$$

holds for all continuous local martingales X with $X_0 = 0$, where $g_\gamma(x) = 1 + \log^\gamma \frac{1}{x}$ ($\gamma \geq 0$, $x \in (0, 1]$). As a consequence of the inequality, we have

Lemma 3.3. *Let* $0 < p < \infty$ *and* $L \in \mathbb{L}$. *Then the inequality*

$$(3.14) \quad E\left[\left(1 + \log^{\gamma p} \frac{1}{J_L}\right)(X_L^*)^p\right]$$
$$\leq c_{\gamma,p} E\left[\left(1 + \log^{(\gamma+\frac{1}{2})p} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{p}{2}}\right] \qquad (\gamma \geq 0)$$

holds for all continuous local martingales X *with* $X_0 = 0$.

On Some Inequalities of Local Times of Iterated Stochastic Integrals

Litan Yan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 16 of 27

Proof. Let $\gamma \geq 0$ and let X be a continuous local martingale. Then we have from (3.13)

$$(3.15) \qquad E\left[\left(1 + \log^\gamma \frac{1}{J_L}\right) X_L^*\right] \leq c_\gamma E\left[\left(1 + \log^{\gamma + \frac{1}{2}} \frac{1}{J_L}\right) \langle X\rangle_L^{\frac{1}{2}}\right],$$

since g_γ is non-increasing and $(g_\gamma g_{\frac{1}{2}})(x) \leq c_\gamma\left(1 + \log^{\gamma + \frac{1}{2}} \frac{1}{x}\right)$.
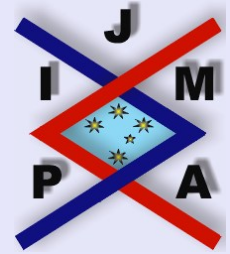
Now, denote for $t \geq 0$

$$A_t = \left(1 + \log^\gamma \frac{1}{J_L}\right) X_{L \wedge t}^* \quad \text{and} \quad B_t = \left(1 + \log^{\gamma + \frac{1}{2}} \frac{1}{J_L}\right) \langle X\rangle_{L \wedge t}^{\frac{1}{2}}.$$

Then for any couple (S, T) of stopping times S, T with $T \geq S \geq 0$

$$
\begin{aligned}
E\big[A_T - A_S\big] &= E\left[\left(1 + \log^\gamma \frac{1}{J_L}\right)(X_{L \wedge T}^* - X_{L \wedge S}^*)\right] \\
&\leq E\left[\left(1 + \log^\gamma \frac{1}{J_L}\right) \sup_{S \leq t \leq T} |X_{L \wedge t} - X_{L \wedge S}| 1_{\{S < T\}}\right] \\
&= E\left[\left(1 + \log^\gamma \frac{1}{J_L}\right) \sup_{t \geq 0} |X_{T \wedge (S+t)}^L - X_S^L| 1_{\{S < T\}}\right] \\
&\equiv E\left[\left(1 + \log^\gamma \frac{1}{J_L}\right) \sup_{t \geq 0} |(X_{T \wedge (S+t)} - X_S)^L| 1_{\{S < T\}}\right],
\end{aligned}
$$

where $X_t^L \equiv X_{t \wedge L}$.

Observe that $(X_{(S+t) \wedge T} - X_S) 1_{\{S < T\}}, t \geq 0$ is a continuous $(\mathcal{F}_{S+t})$–local

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

http://jipam.vu.edu.au

martingale, we find by (3.15)

$$E\big[A_T - A_S\big] \le c_\gamma E\left[\left(1 + \log^{\gamma + \frac{1}{2}} \frac{1}{J_L}\right)\langle X\rangle_{L \wedge T}^{\frac{1}{2}} 1_{\{S < T\}}\right]$$
$$= E\left[c_\gamma B_T 1_{\{S < T\}}\right] \le \big\|c_\gamma B_T\big\|_\infty P(S < T).$$

It follows from Lemma 7 and Lemma 8 in [5] with $C = c_\gamma B$, $\alpha = \beta = 1$ that for all $0 < p < \infty$

$$E\left[\left(1 + \log^\gamma \frac{1}{J_L}\right)^p (X_L^*)^p\right] \le c_{\gamma,p} E\left[\left(1 + \log^{\gamma + \frac{1}{2}} \frac{1}{J_L}\right)^p \langle X\rangle_L^{\frac{p}{2}}\right].$$

Thus, (3.14) follows from the inequalities

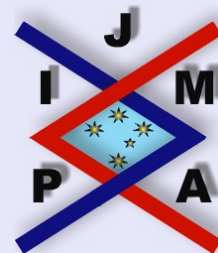$$\hat{c}_p(a^p + b^p) \le (a + b)^p \le c_p(a^p + b^p) \qquad (p, a, b \ge 0).$$

This completes the proof. □

On the other hand, in [2], E. Carlen and P. Krée obtained the identity

$$I_n(t, X)I_{n-2}(t, X) = I_{n-1}^2(t, X) - \sum_{j=1}^n \frac{(n-j)!}{n!} I_{n-j}^2(t, X)\langle X\rangle_t^{j-1} \quad (n \ge 2)$$

for all $t \ge 0$ and all continuous local martingales X with $X_0 = 0$. It follows that

$$\frac{1}{n!}\langle X\rangle_t^{n-1} \le \frac{n-1}{n} I_{n-1}^2(t, X) - I_n(t, X)I_{n-2}(t, X) \quad (n \ge 2).$$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Integrating both sides of the inequality above on $[0, t]$ with respect to the measure $d\langle X\rangle_t$, we get

$$\frac{1}{n!}\langle X\rangle_t^n \leq (n-1)\langle I_n(X)\rangle_t^2 - n\int_0^t I_n(s, X)I_{n-2}(s, X)d\langle X\rangle_s \quad (n \geq 2)$$

since $\langle I_n(X)\rangle_t = \int_0^t I_{n-1}^2(s, X)d\langle X\rangle_s$, which gives

$$(3.16) \quad \frac{\langle X\rangle_t^{\frac{n}{2}}}{\sqrt{n!}} \leq \sqrt{n-1}\langle I_n(X)\rangle_t^{\frac{1}{2}}$$
$$+ \sqrt{n}\left(I_n^*(t, X)I_{n-2}^*(t, X)\langle X\rangle_t\right)^{\frac{1}{2}} \quad (n \geq 2).$$

Theorem 3.4. *Let $0 < p < \infty$ and $L \in \mathbb{L}$. If V is one of the three random variables X_L^*, $\langle X\rangle_L^{\frac{1}{2}}$ and $\mathcal{L}_L^*(X)$, then the inequalities*
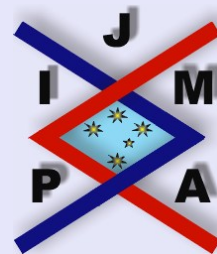
$$(3.17) \qquad E\left[V^{np}\right] \leq c_{n,p}E\left[\left(1 + \log^{np}\frac{1}{J_L}\right)\left(I_n^*(L, X)\right)^p\right],$$

$$(3.18) \qquad E\left[V^{np}\right] \leq c_{n,p}E\left[\left(1 + \log^{(n+\frac{1}{2})p}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right],$$

$$(3.19) \qquad E\left[V^{np}\right] \leq c_{n,p}E\left[\left(1 + \log^{(2n+1)p}\frac{1}{J_L}\right)\left(\mathcal{L}_L^*(n, X)\right)^p\right]$$

hold for all continuous local martingales X with $X_0 = 0$ and $n = 1, 2, \ldots$.

*Proof.* Let $n \geq 2$, $0 < p < \infty$ and let X be a continuous local martingale.

For $n \geq 3$, by applying the Hölder inequality with exponents $s = n$ and $r = \frac{n}{n-2}$ and Theorem 3.1 we have

$$E\left[\left(I_{n-2}^*(L,X)\langle X\rangle_L\right)^p\right] \leq E\left[\left(I_{n-2}^*(L,X)\right)^{\frac{np}{n-2}}\right]^{\frac{n-2}{n}} E\left[\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{2}{n}}$$

$$\leq c_{n,p} E\left[\left(1 + \log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right].$$

Clearly, the inequality above is also true for $n = 2$.

It follows from (3.16) that for $n \geq 2$

$$\left(\frac{1}{\sqrt{n!}}\right)^p E\left[\left(1 + \log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$

$$\leq E\left[\left(1 + \log^{\frac{np}{2}}\frac{1}{J_L}\right)\left(\sqrt{n-1}\langle I_n(X)\rangle_L^{\frac{1}{2}}\right.\right.$$

$$\left.\left. + \sqrt{n}\left(I_n^*(L,X)I_{n-2}^*(L,X)\langle X\rangle_L\right)^{\frac{1}{2}}\right)^p\right]$$

$$\leq \hat{c}_{n,p} E\left[\left(1 + \log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]$$

$$+ c_{n,p} E\left[\left(1 + \log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]^{\frac{1}{2}} E\left[\left(I_{n-2}^*(L,X)\langle X\rangle_L\right)^p\right]^{\frac{1}{2}}$$

$$\leq \hat{c}_{n,p} E\left[\left(1 + \log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]$$
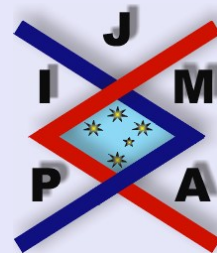
$$+ c_{n,p} E\left[\left(1 + \log^{np}\frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]^{\frac{1}{2}} E\left[\left(1 + \log^{\frac{np}{2}}\frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{2}}.$$

Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

Combining this with (3.12), we get the quadratic inequality as follows

$$E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$

$$\leq \hat{c}_{n,p}E\left[\left(1 + \log^{np} \frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]$$

$$+ c_{n,p}E\left[\left(1 + \log^{np} \frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right]^{\frac{1}{2}}$$

$$\times E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]^{\frac{1}{2}}.$$

Solving the above quadratic inequality leads to the inequality

$$(3.20) \quad E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right] \leq c_{n,p}E\left[\left(1 + \log^{np} \frac{1}{J_L}\right)\left(I_n^*(L,X)\right)^p\right].$$
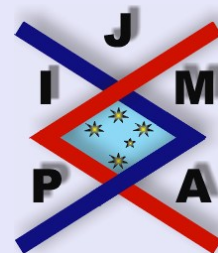
Consequently, by Lemma 3.3

$$(3.21) \quad E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right] \leq c_{n,p}E\left[\left(1 + \log^{(n+\frac{1}{2})p} \frac{1}{J_L}\right)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right],$$

and so by (2.5) and (2.9)

$$(3.22) \quad E\left[\left(1 + \log^{\frac{np}{2}} \frac{1}{J_L}\right)\langle X\rangle_L^{\frac{np}{2}}\right]$$

$$\leq c_{n,p}E\left[\left(1 + \log^{(2n+1)p} \frac{1}{J_L}\right)\left(\mathcal{L}_L^*(n,X)\right)^p\right].$$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Now, the inequalities (3.17) – (3.19) are consequences of (3.20) – (3.22) by Lemma 2.1 and Lemma 2.2. This completes the proof. □

Remark 3.1. Let $0 < p < \infty$ and $L \in \mathbb{L}$. As some special cases of the inequalities in Theorem 3.4, we can show that the inequalities

$$(3.23) \qquad E\left[\langle X \rangle_L^p\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)\left(I_2^*(L, X)\right)^p\right],$$

$$(3.24) \qquad E\left[\langle X \rangle_L^p\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)\langle I_2(X) \rangle_L^{\frac{p}{2}}\right],$$

$$(3.25) \qquad E\left[\left(X_L^*\right)^{2p}\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)\left(I_2^*(L, X)\right)^p\right],$$

$$(3.26) \qquad E\left[\left(X_L^*\right)^{2p}\right] \leq c_p E\left[\left(1 + \log^{\frac{p}{2}} \frac{1}{J_L}\right)\langle I_2(X) \rangle_L^{\frac{p}{2}}\right],$$
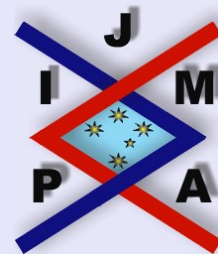
$$(3.27) \qquad E\left[\left(X_L^*\right)^{2p}\right] \leq c_p E\left[\left(1 + \log^p \frac{1}{J_L}\right)\left(\mathcal{L}_L^*(2, X)\right)^p\right],$$

$$(3.28) \qquad E\left[\langle X \rangle_L^p\right] \leq c_p E\left[\left(1 + \log^p \frac{1}{J_L}\right)\left(\mathcal{L}_L^*(2, X)\right)^p\right],$$

$$(3.29) \qquad E\left[\left(\mathcal{L}_L^*(X)\right)^{2p}\right] \leq c_p E\left[\left(1 + \log^{2p} \frac{1}{J_L}\right)\left(I_2^*(L, X)\right)^p\right],$$

$$(3.30) \qquad E\left[\left(\mathcal{L}_L^*(X)\right)^{2p}\right] \leq c_p E\left[\left(1 + \log^{\frac{3p}{2}} \frac{1}{J_L}\right)\langle I_2(X) \rangle_L^{\frac{p}{2}}\right],$$

$$(3.31) \qquad E\left[\left(\mathcal{L}_L^*(X)\right)^{2p}\right] \leq c_p E\left[\left(1 + \log^{3p} \frac{1}{J_L}\right)\left(\mathcal{L}_L^*(2, X)\right)^p\right]$$

Page 22 of 27

J. Ineq. Pure and Appl. Math. 3(4) Art. 62, 2002

http://jipam.vu.edu.au

hold for all continuous local martingales X with $X_0 = 0$. In fact, from (3.16) we have

$$E\Big[\langle X\rangle_L^p\Big] \leq \hat{c}_p E\Big[\langle I_2(X)\rangle_L^{\frac{p}{2}}\Big] + c_p E\Big[\big(I_2^*(L,X)\langle X\rangle_L\big)^{\frac{p}{2}}\Big]$$

for $0 < p < \infty$ and so

$$E\Big[\langle X\rangle_L^p\Big] \leq \hat{c}_p E\Big[\langle I_2(X)\rangle_L^{\frac{p}{2}}\Big] + c_p E\Big[\big(I_2^*(L,X)\big)^p\Big]^{\frac{1}{2}} E\Big[\langle X\rangle_L^p\Big]^{\frac{1}{2}}.$$

Combining this with Lemma 2.1, we find

$$E\Big[\langle X\rangle_L^p\Big] \leq \hat{c}_p E\left[\Big(1 + \log^{\frac{p}{2}}\frac{1}{I_L}\Big)\big(I_2^*(L,X)\big)^p\right]$$
$$+ c_p E\left[\Big(1 + \log^{\frac{p}{2}}\frac{1}{I_L}\Big)\big(I_2^*(L,X)\big)^p\right]^{\frac{1}{2}} E\Big[\langle X\rangle_L^p\Big]^{\frac{1}{2}}$$
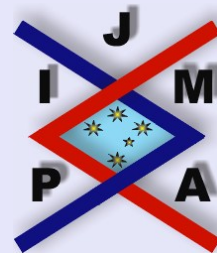
and

$$E\Big[\langle X\rangle_L^p\Big] \leq \hat{c}_p E\left[\Big(1 + \log^{\frac{p}{2}}\frac{1}{I_L}\Big)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]$$
$$+ c_p E\left[\Big(1 + \log^{\frac{p}{2}}\frac{1}{I_L}\Big)\langle I_n(X)\rangle_L^{\frac{p}{2}}\right]^{\frac{1}{2}} E\Big[\langle X\rangle_L^p\Big]^{\frac{1}{2}}.$$

The above quadratic inequalities lead to (3.23) and (3.24).

Next, observe that from (3.6)

$$(X_L^*)^2 \leq 2I_2^*(L,X) + \langle X\rangle_L,$$

we obtain the inequalities (3.25) – (3.28).

Finally, combining (3.16) with Lemma 3.3, we get

$$
E\left[\left(1 + \log^p \frac{1}{J_L}\right)\langle X\rangle_L^p\right]
$$

$$
\leq E\left[\left(1 + \log^p \frac{1}{J_L}\right)\left(\sqrt{2}\langle I_2(X)\rangle_L^{\frac{1}{2}} + 2\big(I_2^*(L,X)\langle X\rangle_L\big)^{\frac{1}{2}}\right)^p\right]
$$

$$
\leq \hat{c}_p E\left[\left(1 + \log^p \frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]
$$

$$
\quad + c_p E\left[\left(1 + \log^p \frac{1}{J_L}\right)\big(I_2^*(L,X)\big)^p\right]^{\frac{1}{2}} E\left[\left(1 + \log^p \frac{1}{J_L}\right)\langle X\rangle_L^p\right]^{\frac{1}{2}}.
$$

$$
\leq \hat{c}_p E\left[\left(1 + \log^{\frac{3p}{2}} \frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]
$$

$$
\quad + c_p E\left[\left(1 + \log^{\frac{3p}{2}} \frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]^{\frac{1}{2}} E\left[\left(1 + \log^p \frac{1}{J_L}\right)\langle X\rangle_L^p\right]^{\frac{1}{2}},
$$

which gives a quadratic inequality

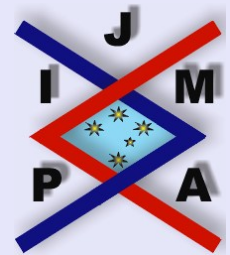$$
x^2 - \hat{c}_p y^2 - c_p xy \leq 0 \qquad (\hat{c}_p, c_p \geq 0)
$$

with

$$
x = E\left[\left(1 + \log^p \frac{1}{J_L}\right)\langle X\rangle_L^p\right]^{\frac{1}{2}} \quad \text{and} \quad y = E\left[\left(1 + \log^{\frac{3p}{2}} \frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right]^{\frac{1}{2}}.
$$

Solving the quadratic inequality leads to

$$
E\left[\left(1 + \log^p \frac{1}{J_L}\right)\langle X\rangle_L^p\right] \leq c_p E\left[\left(1 + \log^{\frac{3p}{2}} \frac{1}{J_L}\right)\langle I_2(X)\rangle_L^{\frac{p}{2}}\right],
$$
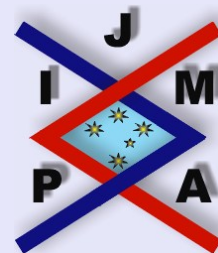
which gives (3.30) and (3.31).

Thus, we obtain the inequalities (3.23) – (3.31).

On Some Inequalities of Local Times of Iterated Stochastic Integrals
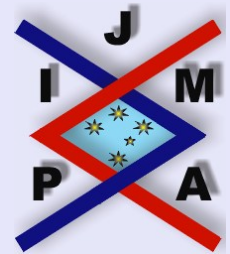
Litan Yan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 25 of 27

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

http://jipam.vu.edu.au

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Title Page

Contents

◀◀ ▶▶

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Go Back

Close

Quit

Page 26 of 27

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

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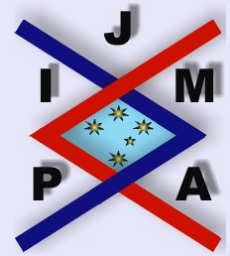
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On Some Inequalities of Local Times of Iterated Stochastic Integrals

Litan Yan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 27 of 27

J. Ineq. Pure and Appl. Math. **3**(4) Art. 62, 2002

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