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# ON BEST EXTENSIONS OF HARDY-HILBERT'S INEQUALITY WITH TWO PARAMETERS

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ABSTRACT. This paper deals with some extensions of Hardy-Hilbert's inequality with the best constant factors by introducing two parameters $\lambda$ and $\alpha$ and using the Beta function. The equivalent form and some reversions are considered.

*Key words and phrases:* Hardy-Hilbert's inequality; Beta function; Hölder's inequality.

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## 1. INTRODUCTION

If $a_{n,}b_n \geq 0$ satisfy

$$0 < \sum_{n=1}^{\infty} a_n^2 < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} b_n^2 < \infty,$$

then one has two equivalent inequalities as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{\frac{1}{2}} \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2, \tag{1.2}$$

where the constant factors $\pi$ and $\pi^2$ are the best possible. Inequality (1.1) is well known as Hilbert's inequality (cf. Hardy et al. [1]). In 1925, Hardy [2] gave some extensions of (1.1) and

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(1.2) by introducing the $(p,q)-$parameter as: If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n,b_n\geq 0$ satisfy

$$0<\sum_{n=1}^{\infty}a_n^p<\infty \quad \text{and} \quad 0<\sum_{n=1}^{\infty}b_n^q<\infty,$$

then one has the following two equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}} \tag{1.3}$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p<\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p\sum_{n=1}^{\infty}a_n^p, \tag{1.4}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are the best possible. Inequality (1.3) is called Hardy-Hilbert's inequality, and is important in analysis and its applications (cf. Mitrinović et al. [3]).

In 1997-1998, by estimating the weight coefficient and introducing the Euler constant $\gamma$, Yang and Gao [4, 5] gave a strengthened version of (1.3) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{1-\gamma}{n^{\frac{1}{p}}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{1-\gamma}{n^{\frac{1}{q}}}\right]b_n^q\right\}^{\frac{1}{q}}, \tag{1.5}$$

where $1-\gamma=0.42278433^{+}$ is the best value. In 1998, Yang [6] first introduced an independent parameter $\lambda$ and the Beta function to build an extension of Hilbert's integral inequality. Recently, by introducing a parameter $\lambda$, Yang [7] and Yang et al. [8] gave some extensions of (1.3) and (1.4) as: If $2-\min\{p,q\}<\lambda\leq 2$, $a_n,b_n\geq 0$ satisfy

$$0<\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p<\infty \quad \text{and} \quad 0<\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q<\infty,$$

then one has the following two equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<k_{\lambda}(p)\left\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \tag{1.6}$$

and

$$\sum_{n=1}^{\infty}n^{(p-1)(\lambda-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^{\lambda}}\right]^p dy<[k_{\lambda}(p)]^p\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p, \tag{1.7}$$

where the constant factors $k_{\lambda}(p)=B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ and $[k_{\lambda}(p)]^p$ are the best possible ($B(u,v)$ is the $\beta$ function). For $\lambda=1$, inequalities (1.6) and (1.7) reduce respectively to (1.3) and (1.4). By introducing a parameter $\alpha$, Kuang [9] gave an extension of (1.3), and Yang [10] gave an improvement of [9] as: If $0<\alpha\leq\min\{p,q\}$, $a_n,b_n\geq 0$ satisfy

$$0<\sum_{n=1}^{\infty}n^{(p-1)(1-\alpha)}a_n^p<\infty \quad \text{and} \quad 0<\sum_{n=1}^{\infty}n^{(q-1)(1-\alpha)}b_n^q<\infty,$$

then one has two equivalent inequalities as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\alpha+n^\alpha}<\frac{\pi}{\alpha\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}n^{(p-1)(1-\alpha)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\alpha)}b_n^q\right\}^{\frac{1}{q}} \tag{1.8}$$

and

$$\sum_{n=1}^{\infty}n^{\alpha-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{m^\alpha+n^\alpha}\right]^p dy<\left[\frac{\pi}{\alpha\sin\left(\frac{\pi}{p}\right)}\right]^p\sum_{n=1}^{\infty}n^{(p-1)(1-\alpha)}a_n^p, \tag{1.9}$$

where the constant factors $\frac{\pi}{\alpha\sin(\pi/p)}$ and $\left[\frac{\pi}{\alpha\sin(\pi/p)}\right]^p$ are the best possible. For $\alpha=1$, inequalities (1.8) and (1.9) reduce respectively to (1.3) and (1.4). Recently, Hong [11] gave an extension of (1.3) by introducing two parameters $\lambda$ and $\alpha$ as: If $\alpha\geq 1$, $1-\frac{1}{\alpha r}<\lambda\leq 1$ $(r=p,q)$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}<H_{\lambda,\alpha}(p)\left\{\sum_{n=1}^{\infty}n^{\alpha(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\alpha(1-\lambda)}b_n^p\right\}^{\frac{1}{q}}, \tag{1.10}$$

where

$$H_{\lambda,\alpha}(p)=\left[B\left(1-\frac{1}{\alpha q},\lambda+\frac{1}{\alpha q}-1\right)\right]^{\frac{1}{p}}\left[B\left(1-\frac{1}{\alpha p},\lambda+\frac{1}{\alpha p}-1\right)\right]^{\frac{1}{q}}.$$

For $\lambda=\alpha=1$, (1.10) reduces to (1.3). However, it is obvious that (1.10) is not an extension of (1.6) or (1.8).

In 2003, Yang et al. [12] provided an extensive account of the above results. More recently, Yang [13] gave some extensions of (1.1) and (1.2) as: If $0<\lambda\leq\min\{p,q\}$, satisfy

$$0<\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p<\infty \quad\text{and}\quad 0<\sum_{n=1}^{\infty}n^{q-1-\lambda}a_n^p<\infty,$$

then one has the following two equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda}<K_\lambda(p)\left\{\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-1-\lambda}b_n^p\right\}^{\frac{1}{q}} \tag{1.11}$$

and

$$\sum_{n=1}^{\infty}n^{(p-1)\lambda-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^\lambda}\right]^p dy<[K_\lambda(p)]^p\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p, \tag{1.12}$$

where the constants $K_\lambda(p)=B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ and $[K_\lambda(p)]^p$ are the best possible. For $\lambda=1$, (1.11) and (1.12) reduce to the following two equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}n^{p-2}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-2}b_n^q\right\}^{\frac{1}{q}} \tag{1.13}$$

and

$$\sum_{n=1}^{\infty}n^{p-2}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p<\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p\sum_{n=1}^{\infty}n^{p-2}a_n^p. \tag{1.14}$$

For $p = q = 2$, inequalities (1.13) and (1.14) reduce respectively to (1.1) and (1.2). We find that inequalities (1.3) and (1.13) are different, although both of them are the best extensions of (1.1) with the $(p, q)-$parameter.

The main objective of this paper is to obtain some extensions of (1.3) with the best constant factors, by introducing two parameters $\lambda$ and $\alpha$ and using the Beta function, related to the double series as $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}$ $(\lambda, \alpha > 0)$, so that inequality (1.10) can be improved. The equivalent form and some reversions are considered.

## 2. SOME LEMMAS

First, we need the form of the Beta function as (cf. Wang et al. [14]):

$$B(u, v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt = B(v, u) \qquad (u, v > 0). \tag{2.1}$$

**Lemma 2.1.** *If $p > 0$ $(p \neq 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $\phi_r = \phi_r(\lambda, \alpha) > 0\,(r = p, q)$, satisfy $\phi_p + \phi_q = \lambda\alpha$, define the weight function $\omega_r(x)$ as*

$$\omega_r(x) := \int_0^\infty \frac{x^{\lambda\alpha-\phi_r}}{(x^\alpha + y^\alpha)^\lambda}\left(\frac{1}{y}\right)^{1-\phi_r} dy \qquad (x > 0;\ r = p, q). \tag{2.2}$$

*Then for $x > 0$, each $\omega_r(x)$ is constant, that is*

$$\omega_r(x) = \frac{1}{\alpha} B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right) \qquad (x > 0; r = p, q). \tag{2.3}$$

*Proof.* Setting $u = \left(\frac{y}{x}\right)^\alpha$ in the integral (2.2), one has $dy = \frac{x}{\alpha} u^{\frac{1}{\alpha}-1} du$ and

$$\begin{aligned}\omega_r(x) &= x^{\lambda\alpha-\phi_r}\int_0^\infty \frac{1}{(x^\alpha + x^\alpha u)^\lambda}\left(\frac{1}{xu^{1/\alpha}}\right)^{1-\phi_r}\frac{x}{\alpha}u^{\frac{1}{\alpha}-1}du \\ &= \frac{1}{\alpha}\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\phi_r}{\alpha}-1} du \qquad (r = p, q).\end{aligned}$$

By (2.1), since $\phi_p + \phi_q = \lambda\alpha$, one has (2.3). The lemma is proved. □

**Lemma 2.2.** *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $\phi_r > 0$ $(r = p, q)$, satisfy $\phi_p + \phi_q = \lambda\alpha$, and $0 < \varepsilon < q\phi_p$, then one has*

$$\begin{aligned}I_1 &:= \int_1^\infty \int_1^\infty \frac{x^{-1+\phi_q-\frac{\varepsilon}{p}}}{(x^\alpha + y^\alpha)^\lambda} y^{-1+\phi_p-\frac{\varepsilon}{q}} dx dy \\ &> \frac{1}{\varepsilon\alpha} B\left(\frac{\phi_p}{\alpha} - \frac{\varepsilon}{q\alpha}, \frac{\phi_q}{\alpha} + \frac{\varepsilon}{q\alpha}\right) - O(1).\end{aligned} \tag{2.4}$$

*If $0 < p < 1$ and $0 < \varepsilon < -q\phi_q$, with the above assumption, then one has*

$$\begin{aligned}I_2 &:= \sum_{m=1}^\infty \int_0^\infty \frac{m^{-1+\phi_q-\frac{\varepsilon}{p}}}{(m^\alpha + y^\alpha)^\lambda} y^{-1+\phi_p-\frac{\varepsilon}{q}} dy \\ &= \frac{1}{\alpha} B\left(\frac{\phi_p}{\alpha} - \frac{\varepsilon}{q\alpha}, \frac{\phi_q}{\alpha} + \frac{\varepsilon}{q\alpha}\right)\sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}}.\end{aligned} \tag{2.5}$$

*Proof.* Setting $u = \left(\frac{y}{x}\right)^{\alpha}$ in the integral $I_1$, one has

$$\begin{aligned} I_1 &= \int_1^{\infty} x^{-1+\phi_q-\frac{\varepsilon}{p}} \left[\int_1^{\infty} \frac{1}{(x^{\alpha}+y^{\alpha})^{\lambda}} y^{-1+\phi_p-\frac{\varepsilon}{q}} dy\right] dx \\ &= \frac{1}{\alpha}\int_1^{\infty} x^{-1-\varepsilon} \int_{\frac{1}{x^{\alpha}}}^{\infty} \frac{1}{(1+u)^{\lambda}} u^{\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha}-1} du dx \\ &= \frac{1}{\varepsilon\alpha}\int_0^{\infty} \frac{u^{\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha}-1}}{(1+u)^{\lambda}} du - \frac{1}{\alpha}\int_1^{\infty} x^{-1-\varepsilon}\int_0^{\frac{1}{x^{\alpha}}} \frac{u^{\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha}-1}}{(1+u)^{\lambda}} du dx \\ &> \frac{1}{\varepsilon\alpha}\int_0^{\infty} \frac{u^{\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha}-1}}{(1+u)^{\lambda}} du - \frac{1}{\alpha}\int_1^{\infty} x^{-1}\int_0^{\frac{1}{x^{\alpha}}} u^{\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha}-1} du dx \\ &= \frac{1}{\varepsilon\alpha}\int_0^{\infty} \frac{u^{\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha}-1}}{(1+u)^{\lambda}} du - \left(\phi_p - \frac{\varepsilon}{q}\right)^{-2}. \end{aligned} \tag{2.6}$$

By (2.1), it follows that (2.4) is valid. For $0 < p < 1$, setting $u = (\frac{y}{m})^{\alpha}$ in the integral of $I_2$, in the same manner, one has (2.5). The lemma is thus proved. □

## 3. MAIN RESULTS

**Theorem 3.1.** *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $0 < \phi_r \le 1\,(r = p, q)$, $\phi_p + \phi_q = \lambda\alpha$ *and* $a_n, b_n \ge 0$ *satisfy*

$$0 < \sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q < \infty,$$

*then one has*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^{\alpha}+n^{\alpha})^{\lambda}} < \frac{1}{\alpha} B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right) \left\{\sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q\right\}^{\frac{1}{q}}, \tag{3.1}$$

*where the constant factor* $\frac{1}{\alpha} B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$ *is the best possible.*

*Proof.* By Hölder's inequality with weight (see [15]), one has

$$\begin{aligned} H(a_m, b_n) &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^{\alpha}+n^{\alpha})^{\lambda}} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(m^{\alpha}+n^{\alpha})^{\lambda}} \left[\frac{m^{(1-\phi_q)/q}}{n^{(1-\phi_p)/p}} a_m\right] \left[\frac{n^{(1-\phi_p)/p}}{m^{(1-\phi_q)/q}} b_n\right] \\ &\le \left\{\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} \frac{m^{\lambda\alpha-\phi_p}}{(m^{\alpha}+n^{\alpha})^{\lambda}} \cdot \frac{1}{n^{1-\phi_p}}\right] m^{p(1-\phi_q)-1} a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{n^{\lambda\alpha-\phi_q}}{(m^{\alpha}+n^{\alpha})^{\lambda}} \cdot \frac{1}{m^{1-\phi_q}}\right] n^{q(1-\phi_p)-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{3.2}$$

Since $\lambda, \alpha > 0$, and $1 - \phi_r \ge 0\,(r = p, q)$, in view of (2.2), we rewrite (3.2) as

$$H(a_m, b_n) < \left\{\sum_{m=1}^{\infty} \omega_p(m) m^{p(1-\phi_q)-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \omega_q(n) n^{q(1-\phi_p)-1} b_n^q\right\}^{\frac{1}{q}},$$

and then by (2.3), one has (3.1). For $0 < \varepsilon < q\phi_p$, setting $a'_n$and $b'_n$ as: $a'_n = n^{-1+\phi_q-\frac{\varepsilon}{p}}$, $b'_n = n^{-1+\phi_p-\frac{\varepsilon}{q}}$, $n \in \mathbb{N}$, then we find

$$\text{(3.3)} \qquad \left\{\sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n'^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n'^q\right\}^{\frac{1}{q}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}} < 1 + \int_1^{\infty} \frac{1}{t^{1+\varepsilon}} dt = \frac{1}{\varepsilon}(1+\varepsilon).$$

If the constant factor $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$ in (3.1) is not the best possible, then there exists a positive constant $k$ (with $k < \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$), such that (3.1) is still valid if one replaces $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$ by $k$. In particular, by (2.4) and (3.3),

$$\begin{aligned} \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha} - \frac{\varepsilon}{q\alpha}, \frac{\phi_q}{\alpha} + \frac{\varepsilon}{q\alpha}\right) - \varepsilon O(1) &< \varepsilon I_1 \\ &< \varepsilon H(a'_m, b'_n) \\ &< \varepsilon k \left\{\sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n'^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n'^q\right\}^{\frac{1}{q}} \\ &= k(1+\varepsilon), \end{aligned}$$

and then $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right) \le k$ $(\varepsilon \to 0^+)$. This contradicts the fact that $k < \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$. Hence the constant factor $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$ in (3.1) is the best possible. The theorem is proved. □

**Theorem 3.2.** *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $0 < \phi_r \le 1$ $(r = p, q)$, $\phi_p + \phi_q = \lambda\alpha$ and $a_n \ge 0$ satisfy*

$$0 < \sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n^p < \infty,$$

*then one has*

$$\text{(3.4)} \qquad \sum_{n=1}^{\infty} n^{p\phi_p-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\alpha)^\lambda}\right]^p < \left[\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)\right]^p \sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n^p,$$

*where the constant factor $\left[\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)\right]^p$ is the best possible. Inequality (3.4) is equivalent to (3.1).*

*Proof.* Set

$$b_n := n^{p\phi_p-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\alpha)^\lambda}\right]^{p-1},$$

and use (3.1) to obtain

$$\begin{aligned}0 &< \sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q \\ &= \sum_{n=1}^{\infty} n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} \\ &\le \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)\left\{\sum_{n=1}^{\infty} n^{p(1-\phi_q)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}0 &< \left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q\right\}^{\frac{1}{p}} \\ &= \left\{\sum_{n=1}^{\infty} n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p\right\}^{\frac{1}{p}} \\ &\le \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)\left\{\sum_{n=1}^{\infty} n^{p(1-\phi_q)-1}a_n^p\right\}^{\frac{1}{p}} < \infty.\end{aligned} \tag{3.6}$$

It follows that (3.5) takes the form of strict inequality by using (3.1); so does (3.6). Hence, one has (3.4).

On the other hand, if (3.4) is valid, by Hölder's inequality, one has

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} &= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{n^{\frac{1}{q}-1+\phi_p}a_m}{(m^\alpha+n^\alpha)^\lambda}\right]\left[n^{1-\phi_p-\frac{1}{q}}b_n\right] \\ &\le \left\{\sum_{n=1}^{\infty} n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{3.7}$$

By (3.4), one has (3.1). It follows that inequalities (3.4) and (3.1) are equivalent. If the constant factor in (3.4) is not the best possible, one can obtain a contradiction that the constant factor in (3.1) is not the best possible by using (3.7). Hence the constant factor in (3.4) is still the best possible. Thus the theorem is proved. □

**Theorem 3.3.** *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$A = \{(\lambda,\alpha);\ \lambda,\alpha>0,\ 0<\phi_r\le 1\,(r=p,q),\ \phi_p+\phi_q=\lambda\alpha\}\neq\Phi,$$

*and* $a_n, b_n \ge 0$ *satisfy*

$$0<\sum_{n=1}^{\infty} n^{p(1-\phi_q)-1}a_n^p<\infty \quad \textit{and} \quad 0<\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q<\infty,$$

*then for* $(\lambda, \alpha) \in A$, *one has*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} > \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)\left\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.8}$$

*where* $0 < \theta_p(n) = O\left(\frac{1}{n^{\phi_p}}\right) < 1$; *the constant* $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha}\right)$ *is the best possible.*

*Proof.* By the reverse of Hölder's inequality (see [15]), following the method of proof in Theorem 3.1, since $0 < p < 1$ and $q < 0$, one has

$$H(a_m, b_n) > \left\{\sum_{n=1}^{\infty}\varpi_p(n)n^{p(1-\phi_q)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega_q(n)n^{q(1-\phi_p)-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.9}$$

where $\omega_q(n)$ is defined as in (2.2) and

$$\varpi_p(n) := \sum_{k=1}^{\infty}\frac{n^{\lambda\alpha-\phi_p}}{(n^\alpha + k^\alpha)^\lambda}\left(\frac{1}{k}\right)^{1-\phi_p} \qquad (n \in \mathbb{N}). \tag{3.10}$$

Define $\theta_p(n)$ as

$$\theta_p(n) := \frac{n^{\lambda\alpha-\phi_p}}{\omega_p(n)}\int_0^1\frac{1}{(n^\alpha + y^\alpha)^\lambda}\left(\frac{1}{y}\right)^{1-\phi_p}dy \qquad (n \in \mathbb{N}). \tag{3.11}$$

Since

$$\omega_p(n) > \int_0^1\frac{n^{\lambda\alpha-\phi_p}}{(n^\alpha + y^\alpha)^\lambda}\left(\frac{1}{y}\right)^{1-\phi_p}dy,$$

then we find $0 < \theta_p(n) < 1$, and

$$\varpi_p(n) > \int_1^{\infty}\frac{n^{\lambda\alpha-\phi_p}}{(n^\alpha + y^\alpha)^\lambda}\left(\frac{1}{y}\right)^{1-\phi_p}dy = \omega_p(n)\,\left[1-\theta_p(n)\right]. \tag{3.12}$$

By (3.12), (2.3) and (3.9), one has (3.8). Since

$$0 < \theta_p(n) < \frac{n^{\lambda\alpha-\phi_p}}{\omega_p(n)}\int_0^1\frac{1}{n^{\lambda\alpha}}\left(\frac{1}{y}\right)^{1-\phi_p}dy = \frac{1}{\omega_p(n)\phi_p}\cdot\frac{1}{n^{\phi_p}}, \tag{3.13}$$

and $\omega_p(n)$ is a constant, we have $\theta_p(n) = O\left(\frac{1}{n^{\phi_p}}\right)\ (n \to \infty)$.

For $0 < \varepsilon < \min\{q(\phi_p - 1), -q\phi_q\}$, setting $a_n'$ and $b_n'$ as: $a_n' = n^{-1+\phi_q-\frac{\varepsilon}{p}}$, $b_n' = n^{-1+\phi_p-\frac{\varepsilon}{q}}$, $n \in \mathbb{N}$, since $\phi_p > 0$, then

$$\sum_{n=1}^{\infty}O\left(\frac{1}{n^{\phi_{p+1+\varepsilon}}}\right) = O(1)(\varepsilon \to 0^+),$$

and

$$\left\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n'^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n'^q\right\}^{\frac{1}{q}} \tag{3.14}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\frac{\sum_{n=1}^{\infty}O\left(\frac{1}{n^{\phi_p+1+\varepsilon}}\right)}{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}}\right\}^{\frac{1}{p}}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}(1-o(1))^{\frac{1}{p}}.$$

If the constant $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)$ in (3.8) is not the best possible, then there exists a positive number $K$ (with $K>\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)$), such that (3.8) is still valid if one replaces $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)$ by $K$. In particular, by (3.14) and (2.5), one has

$$K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\{1-o(1)\}^{\frac{1}{p}}$$
$$=K\left\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n'^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n'^q\right\}^{\frac{1}{q}}$$
$$<H(a_m',b_n')$$
$$<I_2=\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha}-\frac{\varepsilon}{q\alpha},\frac{\phi_q}{\alpha}+\frac{\varepsilon}{q\alpha}\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}},$$

and then $K\le\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)$ $(\varepsilon\to 0^+)$. By this contradiction we can conclude that the constant $\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)$ in (3.8) is the best possible. Thus the theorem is proved. □

**Theorem 3.4.** *If* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$A=\{(\lambda,\alpha);\lambda,\alpha>0,\ 0<\phi_r\le 1\,(r=p,q),\ \phi_p+\phi_q=\lambda\alpha\}\neq\Phi,$$

*and* $a_n,b_n\ge 0$ *satisfy*

$$0<\sum_{n=1}^{\infty}n^{p(1-\phi_q)-1}a_n^p<\infty,$$

*for* $(\lambda,\alpha)\in A$, *one has*

$$\sum_{n=1}^{\infty}n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p>\left[\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)\right]^p\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n^p, \tag{3.15}$$

*where* $0<\theta_p(n)=O\left(\frac{1}{n^{\phi_p}}\right)<1$, *and the constant factor* $\left[\frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)\right]^p$ *is the best possible. Inequality (3.15) is equivalent to (3.8).*

*Proof.* Still setting

$$b_n:=n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^{p-1},$$

by (3.8), one has

$$\begin{aligned}(3.16)\qquad 0 &< \sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q \\ &= \sum_{n=1}^{\infty} n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} \\ &\geq \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)\left\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q\right\}^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}0 &< \left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q\right\}^{\frac{1}{p}} \\ &= \left\{\sum_{n=1}^{\infty} n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p\right\}^{\frac{1}{p}} \\ (3.17)\qquad &\geq \frac{1}{\alpha}B\left(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha}\right)\left\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n^p\right\}^{\frac{1}{p}}.\end{aligned}$$

If $\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q < \infty$, by using (3.8), (3.16) takes the form of strict inequality; so does (3.17). If $\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q = \infty$, (3.17) takes naturally strict inequality. Hence we have (3.15).

On the other hand, if (3.15) is valid, by the reverse of Hölder's inequality,

$$\begin{aligned}(3.18)\quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} &= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{n^{\frac{1}{q}-1+\phi_p}a_m}{(m^\alpha+n^\alpha)^\lambda}\right][n^{1-\phi_p-\frac{1}{q}}b_n] \\ &\geq \left\{\sum_{n=1}^{\infty} n^{p\phi_p-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q\right\}^{\frac{1}{q}}.\end{aligned}$$

Hence by (3.15), one has (3.8). If the constant factor in (3.15) is not the best possible, we can conclude that the constant factor in (3.8) is not the best possible by using (3.18). The theorem is proved. □

**Note:** In view of (3.1), if $\phi_r = \phi_r(\lambda,\alpha)$ $(r = p, q)$ satisfy

$$B(\phi_p(1,1),\phi_q(1,1)) = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)},$$

and $r\phi_r(1,1) = 1$ $(r = p, q)$, one can get a best extension of (1.3); if $B(\phi_p(1,1),\phi_q(1,1)) = \frac{\pi}{\sin(\pi/p)}$ (or $\frac{\pi}{2}$), and $r\phi_r(1,1) \neq 1$ $(r = p, q)$, one can get a best extension of (1.1) but not a best extension of (1.3). For example, setting $\phi_r = \left[\frac{1}{r}(\alpha-2)+1\right]\lambda$ $(r = p, q)$, then $r\phi_r(1,1) = r-1 \neq 1$, by Theorems 3.1 – 3.4, one can get a best extension of (1.13) and (1.1) as follows:

**Corollary 3.5.** *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 0$, $\alpha > 2-\min\{p, q\}$, $\left[\frac{1}{r}(\alpha - 2) + 1\right] \lambda \le 1$ $(r = p, q)$, $a_n, b_n \ge 0$*, satisfy*

$$0 < \sum_{n=1}^{\infty} n^{p[1-\lambda(\alpha-1)]+(\alpha-2)\lambda-1} a_n^p < \infty$$

*and*

$$0 < \sum_{n=1}^{\infty} n^{q[1-\lambda(\alpha-1)]+(\alpha-2)\lambda-1} b_n^q < \infty,$$

*one has equivalent inequalities as:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} < K_{\lambda,\alpha}(p) \times \left\{\sum_{n=1}^{\infty} n^{p[1-\lambda(\alpha-1)]+(\alpha-2)\lambda-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q[1-\lambda(\alpha-1)]+(\alpha-2)\lambda-1} b_n^q\right\}^{\frac{1}{q}} \tag{3.19}$$

*and*

$$\sum_{n=1}^{\infty} n^{(p+\alpha-2)\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\alpha)^\lambda}\right]^p < [K_{\lambda,\alpha}(p)]^p \sum_{n=1}^{\infty} n^{p[1-\lambda(\alpha-1)]+(\alpha-2)\lambda-1} a_n^p, \tag{3.20}$$

*where*

$$K_{\lambda,\alpha}(p) = \frac{1}{\alpha} B\left(\lambda\frac{p+\alpha-2}{\alpha p}, \lambda\frac{q+\alpha-2}{\alpha q}\right)$$

*and* $[K_{\lambda,\alpha}(p)]^p$ *are the best possible. In particular,*

(i) *for* $\alpha = 1$, *one has* $0 < \lambda \le \min\{p, q\}$ *and (1.11);*
(ii) *for* $\lambda = 1$, *one has* $2 - \min\{p, q\} < \alpha \le 2$ *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^\alpha + n^\alpha} < K_{1,\alpha}(p) \left\{\sum_{n=1}^{\infty} n^{(p-1)(2-\alpha)-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(2-\alpha)-1} b_n^q\right\}^{\frac{1}{q}} \tag{3.21}$$

*and*

$$\sum_{n=1}^{\infty} n^{p+\alpha-3} \left[\sum_{m=1}^{\infty} \frac{a_m}{m^\alpha + n^\alpha}\right]^p < [K_{1,\alpha}(p)]^p \sum_{n=1}^{\infty} n^{(p-1)(2-\alpha)-1} a_n^p. \tag{3.22}$$

*If* $0 < p < 1$, *for* $(\lambda, \alpha) = (1, 2) \in A(\neq \Phi)$, *by (3.8), (3.15) and (3.13), one can obtain two equivalent reversions as*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^2 + n^2} > \frac{\pi}{2} \left\{\sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n}\right) \frac{a_n^p}{n}\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \frac{b_n^q}{n}\right\}^{\frac{1}{q}} \tag{3.23}$$

*and*

$$\sum_{n=1}^{\infty} n^{p-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2}\right]^p > \left(\frac{\pi}{2}\right)^p \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n}\right) \frac{a_n^p}{n}, \tag{3.24}$$

*where the constant factors in the above inequalities are the best possible.*

## 4. SOME BEST EXTENSIONS OF (1.3)

Setting $\phi_r = \frac{\lambda\alpha}{r}\ (r = p, q)$, by Theorems 3.1 – 3.4, one has

**Corollary 4.1.** *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $\lambda\alpha \leq \min\{p, q\}$, $a_n, b_n \geq 0$*, satisfy*

$$0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda\alpha)} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda\alpha)} b_n^q < \infty,$$

*then one has the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} < \frac{K_\lambda(p)}{\alpha} \left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda\alpha)} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda\alpha)} b_n^q\right\}^{\frac{1}{q}} \tag{4.1}$$

*and*

$$\sum_{n=1}^{\infty} n^{\lambda\alpha-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\alpha)^\lambda}\right]^p < \left[\frac{K_\lambda(p)}{\alpha}\right]^p \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda\alpha)} a_n^p, \tag{4.2}$$

*where* $K_\lambda(p) = B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$*. In particular,*

(i) *for* $\alpha = 1$*, one has* $0 < \lambda \leq \min\{p, q\}$ *and the following two equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < K_\lambda(p) \left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\right\}^{\frac{1}{q}} \tag{4.3}$$

*and*

$$\sum_{n=1}^{\infty} n^{\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda}\right]^p < [K_\lambda(p)]^p \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p; \tag{4.4}$$

(ii) *for* $\lambda = 1$, $0 < \alpha \leq \min\{p, q\}$ *one has two equivalent inequalities as:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^\alpha + n^\alpha} < \frac{\pi}{\alpha \sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\alpha)} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\alpha)} b_n^q\right\}^{\frac{1}{q}} \tag{4.5}$$

*and*

$$\sum_{n=1}^{\infty} n^{\alpha-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{m^\alpha + n^\alpha}\right]^p < \left[\frac{\pi}{\alpha \sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=1}^{\infty} n^{(p-1)(1-\alpha)} a_n^p, \tag{4.6}$$

*where the constant factors in the above inequalities are the best possible.*

**Note:** Since for $0 < p < 1$, $\phi_q = \frac{\lambda\alpha}{q} < 0$, then $A = \Phi$. It follows that both (4.1) and (4.2) do not possess reversions. Setting $\phi_r = \frac{\lambda\alpha-1}{2} + \frac{1}{r}\ (r = p, q)$, by Theorems 3.1 – 3.4, one has

**Corollary 4.2.** *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $1 - 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\} < \lambda\alpha \leq 1 + 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $a_n, b_n \geq 0$*, satisfy*

$$0 < \sum_{n=1}^{\infty} n^{\frac{p}{2}(1-\lambda\alpha)} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^{\infty} n^{\frac{q}{2}(1-\lambda\alpha)} b_n^q < \infty,$$

*then one has the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} < \frac{1}{\alpha}B\left(\frac{p\lambda\alpha-p+2}{2p\alpha},\frac{q\lambda\alpha-q+2}{2q\alpha}\right)\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}(1-\lambda\alpha)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}(1-\lambda\alpha)}b_n^q\right\}^{\frac{1}{q}} \tag{4.7}$$

*and*

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}(\lambda\alpha-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^p dy < \left[\frac{1}{\alpha}B\left(\frac{p\lambda\alpha-p+2}{2p\alpha},\frac{q\lambda\alpha-q+2}{2q\alpha}\right)\right]^p\sum_{n=1}^{\infty}n^{\frac{p}{2}(1-\lambda\alpha)}a_n^p. \tag{4.8}$$

*In particular,*

(i) *for* $\alpha=1$, *one has* $1-2\min\left\{\frac{1}{p},\frac{1}{q}\right\}<\lambda\leq 1+2\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, *and the following two equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{p\lambda-p+2}{2p},\frac{q\lambda-q+2}{2q}\right)\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}(1-\lambda)}b_n^q\right\}^{\frac{1}{q}} \tag{4.9}$$

*and*

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}(\lambda-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^\lambda}\right]^p dy < \left[B\left(\frac{p\lambda-p+2}{2p},\frac{q\lambda-q+2}{2q}\right)\right]^p\sum_{n=1}^{\infty}n^{\frac{p}{2}(1-\lambda\alpha)}a_n^p; \tag{4.10}$$

(ii) *for* $\lambda=1$, *one has* $1-2\min\left\{\frac{1}{p},\frac{1}{q}\right\}<\alpha\leq 1+2\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ *and two equivalent inequalities as:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\alpha+n^\alpha} < \frac{1}{\alpha}B\left(\frac{p\alpha-p+2}{2p\alpha},\frac{q\alpha-q+2}{2q\alpha}\right)\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}(1-\alpha)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}(1-\alpha)}b_n^q\right\}^{\frac{1}{q}} \tag{4.11}$$

*and*

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}(\alpha-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{m^\alpha+n^\alpha}\right]^p dy < \left[\frac{1}{\alpha}B\left(\frac{p\alpha-p+2}{2p\alpha},\frac{q\alpha-q+2}{2q\alpha}\right)\right]^p\sum_{n=1}^{\infty}n^{\frac{p}{2}(1-\alpha)}a_n^p, \tag{4.12}$$

*where the constant factors in the above inequalities are the best possible.*

**Note:** Since for $0 < p < 1$, we find

$$A = \left\{ (\lambda, \alpha);\ \lambda, \alpha > 0,\ 0 < \frac{\lambda\alpha - 1}{2} + \frac{1}{r} \le 1 \, (r = p, q) \right\} = \Phi,$$

it follows that both (4.7) and (4.8) do not possess reversions. Setting $\phi_r = \left(1 - \frac{1}{r}\right)(\lambda\alpha - 2) + 1$ $(r = p, q)$, by Theorems 3.1 – 3.4, one has

**Corollary 4.3.** *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \alpha > 0$, $2 - \min\{p, q\} < \lambda\alpha \le 2$, $a_n, b_n \ge 0$, *satisfy*

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda\alpha} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^{\infty} n^{1-\lambda\alpha} b_n^q < \infty,$$

*then one has the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} < \frac{1}{\alpha} B\left( \frac{p + \lambda\alpha - 2}{p\alpha}, \frac{q + \lambda\alpha - 2}{q\alpha} \right) \left\{ \sum_{n=1}^{\infty} n^{1-\lambda\alpha} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda\alpha} b_n^q \right\}^{\frac{1}{q}} \tag{4.13}$$

*and*

$$\sum_{n=1}^{\infty} n^{(p-1)(\lambda\alpha-1)} \left[ \sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\alpha)^\lambda} \right]^p < \left[ \frac{1}{\alpha} B\left( \frac{p + \lambda\alpha - 2}{p\alpha}, \frac{q + \lambda\alpha - 2}{q\alpha} \right) \right]^p \sum_{n=1}^{\infty} n^{1-\lambda\alpha} a_n^p, \tag{4.14}$$

*where the constant factors in the above inequalities are the best possible.*

*If* $0 < p < 1$, *one has* $(\alpha/2, \alpha) \in A$ *and two equivalent reversions as:*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\alpha)^{2/\alpha}} > k_\alpha \left\{ \sum_{n=1}^{\infty} \left(1 - \frac{1}{k_\alpha n}\right) \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} b_n^q \right\}^{\frac{1}{q}} \tag{4.15}$$

*and*

$$\sum_{n=1}^{\infty} n^{p-1} \left[ \sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\alpha)^{2/\alpha}} \right]^p > k_\alpha^p \sum_{n=1}^{\infty} \left(1 - \frac{1}{k_\alpha n}\right) \frac{1}{n} a_n^p, \tag{4.16}$$

*where* $k_\alpha = \frac{1}{\alpha} B\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right)$ *, and the constant factors are the best possible.*

*Proof.* For $0 < p < 1$,

$$\phi_r = \left(1 - \frac{1}{r}\right)(\lambda\alpha - 2) + 1 \le 1 \qquad (r = p, q),$$

we obtain $\lambda\alpha = 2$ and $\phi_r = 1$. By (3.13), we find

$$0 < \theta_p(n) < \frac{1}{\omega_p(n)\phi_p} \cdot \frac{1}{n^{\phi_p}} = \frac{1}{k_\alpha n}.$$

Hence by (3.8) and (3.15), one has (4.15) and (4.16). The corollary is proved. □

**Remark 4.4.**

(i) For $\alpha = 1,$ by (4.13) and (4.14), one has (1.6) and (1.7); for $\lambda = 1,$ by (4.13) and (4.14), one has (1.8) and (1.9). It follows that (4.13) is an extension of (1.6) and (1.8), which is an improvement of (1.10), and (4.14) is an extension of (1.7) and (1.9). (4.11) and (4.12) are extensions of (3.23) and (3.24).

(ii) Inequalities (1.6), (4.3) and (4.9) are different extensions of (1.3) with a parameter $\lambda$; inequalities (1.8), (4.5) and (4.11) are different extensions of (1.3) with a parameter $\alpha$ and inequalities (4.1), (4.7) and (4.13) are different extensions of (1.3) with two parameters $\lambda$ and $\alpha.$

(iii) Inequalities (1.7), (4.4) and (4.10) are different extensions of (1.4) with a parameter $\lambda$; inequalities (1.9), (4.6) and (4.12) are different extensions of (1.4) with a parameter $\alpha$ and inequalities (4.2), (4.8) and (4.14) are different extensions of (1.4) with two parameters $\lambda$ and $\alpha$. Since the above inequalities and some reversions are all with the best constant factors, one gives some new results.

## REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[2] G.H. HARDY, Note on a theorem of Hilbert concerning series of positive terms, *Proc. Math. Soc.*, **23**(2) (1925), Records of Proc. XLV-XLVI.

[3] D.S. MINTRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivertives*, Kluwer Academic Publishers, Boston, 1991.

[4] BICHENG YANG AND MINGZHE GAO, On a best value of hardy-Hilbert's inequality, *Advances in Math.*, **26**(2) (1997), 159–164.

[5] MINGZHE GAO AND BICHENG YANG, On the extended Hilbert's inequality, *Proc. Amer. Math. Soc.*, **126**(3) (1998), 751–759.

[6] BICHENG YANG, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220** (1998), 778–785.

[7] BICHENG YANG AND L. DEBNATH, On the extended Hardy-Hilbert's inequality, *J. Math. Anal. Appl.*, **272** (2002), 187–199.

[8] BICHENG YANG, On generalizations of Hardy-Hilbert's inequality and their equivalent forms, *J. of Math.*, **24**(1) (2004), 24–30.

[9] JICHANG KUANG, On a new extension of Hilbert's integral inequality, *J. Math. Anal. Appl.*, **235** (1999), 608–614.

[10] BICHENG YANG, On an extension of Hardy-Hilbert's inequality, *Chinese Annals of Math.*, **23A**(2) (2002), 247–254.

[11] YONG HONG, A extension and improvement of Hardy-Hilbert's double series inequality, *Mathematics in Practice and Theory*, **32**(5) (2002), 850–854.

[12] BICHENG YANG AND Th. M. RASSIAS, On the way of weight coefficient and research for the Hilbert-type inequalities, *Math. Ineq. Appl.*, **6**(4) (2003), 625–658.

[13] BICHENG YANG, On new extensions of Hilbert's inequality, *Acta Math. Hungar.*, **104**(3) (2004), 293–301.

[14] ZHUXI WANG AND DUNRIN GUO, *An Introduction to Special Functions*, Science Press, Beijing, 1979.

[15] JICHANG KUANG, *Applied Inequalities*, Hunan Education Press, Changsha, 2004.