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INEQUALITIES RELATED TO REARRANGEMENTS OF POWERS AND SYMMETRIC POLYNOMIALS

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ABSTRACT. In [2] the second author proposed to find a description (or examples) of real-valued n-variable functions satisfying the following two inequalities:

$$\textit{if } x_i \leq y_i, i = 1, \dots, n, \textit{ then } F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n),$$

with strict inequality if there is an index i such that $x_i < y_i$; and for $0 < x_1 < x_2 < \cdots < x_n$, then,

$$F(x_1^{x_2}, x_2^{x_3}, \dots, x_n^{x_1}) \leq F(x_1^{x_1}, x_2^{x_2}, \cdots, x_n^{x_n}).$$

In this short note we extend in a direction a result of [2] and we prove a theorem that provides a large class of examples satisfying the two inequalities, with F replaced by any symmetric polynomial with positive coefficients. Moreover, we find that the inequalities are not specific to expressions of the form x^y, rather they hold for any function $g(x, y)$ that satisfies some conditions. A simple consequence of this result is a theorem of Hardy, Littlewood and Polya [1].

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1. Introduction

In [2], the following problem was proposed: *find examples of functions* $F : \mathbb{R}_+^n \to \mathbb{R}$ *with the properties*

$$\text{if } x_i \le y_i, i = 1, \dots, n, \text{ then } F(x_1, \dots, x_n) \le F(y_1, \dots, y_n), \\ \text{with strict inequality if there is an index } i \text{ such that } x_i < y_i, \tag{1.1}$$

and

$$\text{for } 0 < x_1 < x_2 < \cdots < x_n, \text{then}, F(x_1^{x_2}, x_2^{x_3}, \dots, x_n^{x_1}) \le F(x_1^{x_1}, x_2^{x_2}, \cdots, x_n^{x_n}). \tag{1.2}$$

In [2], the following result was proved.

Theorem 1.1. *Assume that the permutation σ can be written as a product of disjoint circular cycles $C_1 \times C_2 \times \cdots \times C_r$, where each C_i is a cyclic permutation, that is $C_i(j) = j + t_i$, for some fixed t_i. For any increasing sequence $0 < x_1 < \cdots < x_n$, we have*

$$\sum_{i=1}^n a_i x_i^{x_{\sigma(i)}} \le \sum_{i=1}^n a_i x_i^{x_i}, \text{ and} \\ \prod_{i=1}^n a_i x_i^{x_{\sigma(i)}} \le \prod_{i=1}^n a_i x_i^{x_i}, \tag{1.3}$$

where $a_i \ge 0$ is increasing on the cycles C_i of σ.

(The condition on a_i was inadvertently omitted in the final version of [2].)

In this short note we extend in a direction the previous result of [2] to any permutation, not only the permutations which are products of circular cycles, by proving (1.1) and (1.2) for symmetric polynomials with positive coefficients. Finally, we prove that these inequalities are not specific only to rearrangements of powers, that is, we find other classes of functions of 2-variables with real values, say $g(x, y)$, such that, for any $\sigma \in S_n$ (the group of permutations), we have

$$F(g(x_1, x_{\sigma(1)}), \dots, g(x_n, x_{\sigma(n)})) \le F(g(x_1, x_1), \dots, g(x_n, x_n)), \tag{1.4}$$

where F is any symmetric polynomial with positive coefficients.

2. The Results

Lemma 2.1. *If $f \in \mathbb{R}[X_1, X_2]$ is a symmetric polynomial with positive coefficients and $(x_1, x_2) \in \mathbb{R}_+^2$ and $(y_1, y_2) \in \mathbb{R}_+^2$ are such that $x_1 x_2 \le y_1 y_2$ and $x_1^n + x_2^n \le y_1^n + y_2^n$, $\forall\, n \in \mathbb{N}$, then $f(x_1, x_2) \le f(y_1, y_2)$.*

Proof. We have

$$f(X_1, X_2) = \sum a_{ij} X_1^i X_2^j = \sum_{i<j} \left(a_{ij} X_1^i X_2^j + a_{ji} X_1^j X_2^i \right) + \sum a_{ii} X_1^i X_2^i,$$

where $a_{ij} \in \mathbb{R}_+$. Since f is symmetric $a_{ij} = a_{ji}$, and therefore

$$\begin{aligned} f(X_1, X_2) &= \sum_{i<j} a_{ij} \left(X_1^i X_2^j + X_1^j X_2^i \right) + \sum a_{ii} X_1^i X_2^i \\ &= \sum_{i<j} a_{ij} X_1^i X_2^i \left(X_1^{j-i} + X_2^{j-i} \right) + \sum a_{ii} X_1^i X_2^i. \end{aligned}$$

It is clear now that the two conditions imposed on (x_1, x_2) and (y_1, y_2) imply that $f(x_1, x_2) \le f(y_1, y_2)$. □

We will consider $A \subset \mathbb{R}$ and a function $g : A \times A \to [0, \infty)$ with the following property: for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that $x_1 \le x_2$ and $y_1 \le y_2$ the following two inequalities are satisfied:

$$g(x_1, y_2)g(x_2, y_1) \le g(x_1, y_1)g(x_2, y_2) \tag{2.1}$$

$$[g(x_1, y_2)]^n + [g(x_2, y_1)]^n \le [g(x_1, y_1)]^n + [g(x_2, y_2)]^n,\ \forall\, n \in \mathbb{N}. \tag{2.2}$$

Theorem 2.2. *Let $F(X_1, X_2, \dots, X_n)$ be a symmetric polynomial with positive coefficients and g as above. Then for any $\sigma \in S_n$ and any $x_1, x_2, \dots, x_n \in A$ we have:*

$$F(g(x_1, x_{\sigma(1)}), g(x_2, x_{\sigma(2)}), \dots, g(x_n, x_{\sigma(n)})) \le F(g(x_1, x_1), g(x_2, x_2), \dots, g(x_n, x_n)).$$

Proof. Consider $x_1, x_2, ..., x_n \in A$ arbitrary and fixed. Without loss of generality we may assume that $x_1 \le x_2 \le \dots \le x_n$. Let

$$m = \max\{F\left(g(x_1, x_{\sigma(1)}), \dots, g(x_n, x_{\sigma(n)})\right) \mid \sigma \in S_n\}$$

and let

$$P = \{\sigma \in S_n \mid F\left(g(x_1, x_{\sigma(1)}), \dots, g(x_n, x_{\sigma(n)})\right) = m\}.$$

We would like to prove that $e \in P$ where e is the identity. Let $\tau \in P$ the permutation that has the minimum number of inversions among all elements of P and suppose that $\tau \neq e$. Since e is the only increasing permutation it follows that there exists $i \in \{1, 2, \dots, n-1\}$ such that $\tau(i) > \tau(i+1)$. Without loss of generality we may assume that $i = 1$. Consider $\tau' \in S_n$ defined as follows: $\tau'(1) = \tau(2)$, $\tau'(2) = \tau(1)$ and $\tau'(j) = \tau(j)$ if $j \ge 3$. Then τ' has fewer inversions than τ and therefore $\tau' \notin P$, which implies that:

$$F\left(g(x_1, x_{\tau'(1)}), \dots, g(x_n, x_{\tau'(n)})\right) < F\left(g(x_1, x_{\tau(1)}), \dots, g(x_n, x_{\tau(n)})\right). \tag{2.3}$$

Consider $f(X_1, X_2) = F(X_1, X_2, g(x_3, x_{\tau(3)}), \dots, g(x_n, x_{\tau(n)}))$. It follows that f is symmetric and has positive coefficients. If we set $y_1 = x_{\tau'(1)} = x_{\tau(2)}$ and $y_2 = x_{\tau'(2)} = x_{\tau(1)}$ it follows that $y_1 \le y_2$. Using the two properties of g and Lemma 2.1 we deduce that $f(g(x_1, y_2), g(x_2, y_1)) \le f(g(x_1, y_1), g(x_2, y_2))$ and therefore

$$F\left(g(x_1, x_{\tau(1)}), \dots, g(x_n, x_{\tau(n)})\right) \le F\left(g(x_1, x_{\tau'(1)}), \dots, g(x_n, x_{\tau'(n)})\right),$$

which contradicts (2.3). □

If $g(x, y) = x^y$, then the conditions imposed on g are

$$x_1^{y_2} x_2^{y_1} \le x_1^{y_1} x_2^{y_2},$$
$$x_1^{ny_2} + x_2^{ny_1} \le x_1^{ny_1} + x_2^{ny_2},$$

which are equivalent to

$$x_1^{y_2 - y_1} \le x_2^{y_2 - y_1},$$
$$x_1^{ny_1}(x_1^{n(y_2 - y_1)} - 1) \le x_2^{ny_1}(x_2^{n(y_2 - y_1)} - 1).$$

The first inequality is certainly true as $x_1 \le x_2$ and $y_1 \le y_2$. The second inequality is true if $1 \le x_1 \le x_2$ and $y_1 \le y_2$. Therefore

Corollary 2.3. *The inequalities* (1.1) *and* (1.2) *are satisfied for all n-variable symmetric polynomials with positive coefficients, defined on $[1, \infty)^n$.*

If $F(x_1, \dots, x_n) := x_1 + \dots + x_n$, we can prove a result similar to the one of Theorem 2.2 even if we significantly weaken the assumption on g.

Theorem 2.4. *Let $A \subset \mathbb{R}$ and $g : A \times A \to \mathbb{R}$ be a function such that $h_{a,b}(y) = g(a, y) - g(b, y)$ ($a > b$) is increasing. Then for any $x_1, x_2, \dots, x_n \in A$ and any $\sigma \in S_n$ we have:*

$$F\left(g(x_1, x_{\sigma(1)}), g(x_2, x_{\sigma(2)}), \dots, g(x_n, x_{\sigma(n)})\right) \le F\left(g(x_1, x_1), g(x_2, x_2), \dots, g(x_n, x_n)\right).$$

Proof. We follow the proof of Theorem 2.2 and the only thing we have to check is that

$$F\left(g(x_1, x_{\tau(1)}), \dots, g(x_n, x_{\tau(n)})\right) \le F\left(g(x_1, x_{\tau'(1)}), \dots, g(x_n, x_{\tau'(n)})\right).$$

But this inequality is equivalent to

$$g(x_1, x_{\tau(1)}) + g(x_2, x_{\tau(2)}) \le g(x_1, x_{\tau'(1)}) + g(x_2, x_{\tau'(2)}).$$

If we set $y_1 = x_{\tau'(1)} = x_{\tau(2)}$ and $y_2 = x_{\tau'(2)} = x_{\tau(1)}$, it follows that $y_1 \le y_2$ and the previous inequality can be written as

$$g(x_1, y_2) + g(x_2, y_1) \le g(x_1, y_1) + g(x_2, y_2),$$

which is equivalent to

$$h_{x_2,x_1}(y_1) \le h_{x_2,x_1}(y_2).$$

This inequality is satisfied because $y_1 \le y_2$ and h_{x_2,x_1} is increasing. □

Corollary 2.5. *Let u, v be increasing functions on $\mathbb{R}$ with values in $[1, \infty)$. The following inequalities are true for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$*

$$\sum_{i=1}^{n} u(x_i)v(x_{\sigma(i)}) \le \sum_{i=1}^{n} u(x_i)v(x_i), \tag{2.4}$$

$$\sum_{i=1}^{n} u(x_i)^{v(x_{\sigma(i)})} \le \sum_{i=1}^{n} u(x_i)^{v(x_i)}, \tag{2.5}$$

$$\prod_{i=1}^{n} u(x_i)^{v(x_{\sigma(i)})} \le \prod_{i=1}^{n} u(x_i)^{v(x_i)}. \tag{2.6}$$

Proof. It suffices to prove that the following functions $g(x, y) = u(x)v(y)$, $g(x, y) = u(x)^{v(y)}$, or $g(x, y) = u(y)^{v(x)}$ have the associated h's increasing.

Let $g(x, y) = u(x)v(y)$. Then $h(y) = u(a)v(y) - u(b)v(y) = (u(a) - u(b))v(y)$ which is increasing since $u(a) \ge u(b)$ and $v(y)$ is increasing.

Let $g(x, y) = u(x)^{v(y)}$. Then $h(y) = u(a)^{v(y)} - u(b)^{v(y)}$. Since $u(a) \ge u(b) \ge 1$, and $v(y)$ is increasing, by writing

$$h(y) = u(b)^{v(y)} \left(\left(\frac{u(a)}{u(b)} \right)^{v(y)} - 1 \right),$$

we see that h is increasing.

We remark that to prove (2.4) we only needed u, v to have positive values. Using the previous remark, to show the last inequality, apply (2.4) with $w = \log(u)$ and v (which are both increasing). □

Corollary 2.6. *If the function h is decreasing on A, then all the inequalities are reversed.*

Remark 2.7. We see that Theorem 368 of [1] follows from (2.4) and Corollary 2.6.

REFERENCES

[1] G. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Univ. Press, 2001.

[2] P. STĂNICĂ, *Inequalities on linear functions and circular powers*, *J. Ineq. in Pure and Applied Math.*, **3**(3) (2002), Art.43. [ONLINE: `http://jipam.vu.edu.au/v3n3/015_02.html`]