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## SOME NEW INEQUALITIES SIMILAR TO HILBERT-PACHPATTE TYPE INEQUALITIES

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## Abstract

In this paper, some new inequalities similar to Hilbert-Pachpatte type inequalities are given.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Inequalities, Hilbert-Pachpatte inequalities, Hölder inequality.

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# 1. Introduction

In [1, Chap. 9], the well-known Hardy-Hilbert inequality is given as follows.

**Theorem 1.1.** Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $a_m, b_n \geq 0$ ,  $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ ,  $0 < \sum_{n=1}^{\infty} b_n^q < \infty$ . Then

$$(1.1) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} \leq \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}$$

where  $\frac{\pi}{\sin(\pi/p)}$  is best possible.

The integral analogue of the Hardy-Hilbert inequality can be stated as follows

**Theorem 1.2.** Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f(x), g(y) \geq 0$ ,  $0 < \int_0^\infty f^p(x) dx < \infty$ ,  $0 < \int_0^\infty g^q(y) dy < \infty$ . Then

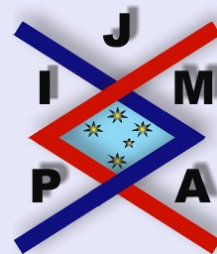
$$(1.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y) dy \right)^{\frac{1}{q}},$$

where  $\frac{\pi}{\sin(\pi/p)}$  is best possible.

In [1, Chap. 9] the following extension of Hardy-Hilbert's double-series theorem is given.

**Theorem 1.3.** Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} \geq 1$ ,  $0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} = \frac{1}{p} + \frac{1}{q} \leq 1$ . Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} \leq K \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},$$



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where  $K = K(p, q)$  depends on  $p$  and  $q$  only.

The following integral analogue of Theorem 1.3 is also given in [1, Chap. 9].

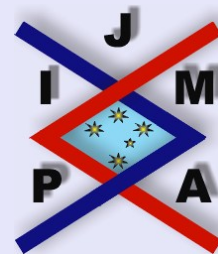
**Theorem 1.4.** *Under the same conditions as in Theorem 1.1 we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \leq K \left( \int_0^\infty f^p dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q dy \right)^{\frac{1}{q}},$$

where  $K = K(p, q)$  depends on  $p$  and  $q$  only.

The inequalities in Theorems 1.1 and 1.2 were studied by Yang and Kuang (see [2, 3]). In [4, 5], some new inequalities similar to the inequalities given in Theorems 1.1, 1.2, 1.3 and 1.4 were established.

In this paper, we establish some new inequalities similar to the Hilbert-Pachpatte inequality.



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## 2. Main Results

In what follows we denote by  $\mathbb{R}$  the set of real numbers. Let  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . We define the operator  $\nabla$  by  $\nabla u(t) = u(t) - u(t - 1)$  for any function  $u$  defined on  $N$ . For any function  $u(t) : [0, \infty) \rightarrow \mathbb{R}$ , we denote by  $u'$  the derivatives of  $u$ .

First we introduce some Lemmas.

**Lemma 2.1.** (see [2]). Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , define the weight function  $\omega_1(q, x)$  as

$$\omega_1(q, x) := \int_0^\infty \frac{1}{(x+y)^\lambda} \left(\frac{x}{y}\right)^{\frac{2-\lambda}{q}} dy, \quad x \in [0, \infty).$$

Then

$$(2.1) \quad \omega_1(q, x) = B\left(\frac{q + \lambda - 2}{q}, \frac{p + \lambda - 2}{p}\right) x^{1-\lambda},$$

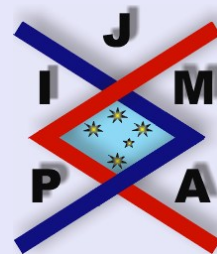
where  $B(p, q)$  is  $\beta$ -function.

**Lemma 2.2.** (see [3]). Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , define the weight function  $\omega_2(q, x)$  as

$$\omega_2(q, x) := \int_0^\infty \frac{1}{x^\lambda + y^\lambda} \left(\frac{x}{y}\right)^{\frac{2-\lambda}{q}} dy, \quad x \in [0, \infty).$$

Then

$$(2.2) \quad \omega_2(q, x) = \frac{1}{\lambda} B\left(\frac{q + \lambda - 2}{q\lambda}, \frac{p + \lambda - 2}{p\lambda}\right) x^{1-\lambda}.$$



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**Lemma 2.3.** Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , define the weight function  $\omega_3(q, m)$  as

$$\omega_3(q, m) := \sum_{n=1}^{\infty} \frac{1}{(m+n)^\lambda} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}, m \in \{1, 2, \dots\}.$$

Then

$$(2.3) \quad \omega_3(q, m) < B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) m^{1-\lambda},$$

where  $B(p, q)$  is  $\beta$ -function.

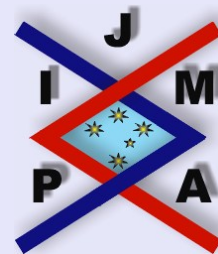
*Proof.* By Lemma 2.1, we have

$$\begin{aligned} \omega_3(q, m) &< \int_0^\infty \frac{1}{(m+y)^\lambda} \left(\frac{m}{y}\right)^{\frac{2-\lambda}{q}} dy \\ &= B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) m^{1-\lambda}. \end{aligned}$$

The proof is completed. □

**Lemma 2.4.** Let  $p > 1$ ,  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , define the weight function  $\omega_4(q, m)$  as

$$\omega_4(q, m) := \sum_{n=1}^{\infty} \frac{1}{m^\lambda + n^\lambda} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}, m \in \{1, 2, \dots\}.$$



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Then

$$(2.4) \quad \omega_4(q, m) < \frac{1}{\lambda} B \left( \frac{q + \lambda - 2}{q\lambda}, \frac{p + \lambda - 2}{p\lambda} \right) m^{1-\lambda}.$$

*Proof.* By Lemma 2.2, we have

$$\begin{aligned} \omega_4(q, m) &< \int_0^\infty \frac{1}{m^\lambda + y^\lambda} \left( \frac{m}{y} \right)^{\frac{2-\lambda}{q}} dy \\ &= \frac{1}{\lambda} B \left( \frac{q + \lambda - 2}{q\lambda}, \frac{p + \lambda - 2}{p\lambda} \right) m^{1-\lambda}. \end{aligned}$$

The proof is completed. □

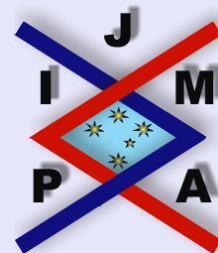
Our main result is given in the following theorem.

**Theorem 2.5.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $f(x), g(y)$  be real-valued continuous functions defined on  $[0, \infty)$ , respectively, and let  $f(0) = g(0) = 0$ , and

$$0 < \int_0^\infty \int_0^x |f'(\tau)|^p d\tau dx < \infty, \quad 0 < \int_0^\infty \int_0^y |g'(\delta)|^q d\delta dy < \infty.$$

Then

$$(2.5) \quad \int_0^\infty \int_0^\infty \frac{|f(x)| |g(y)|}{(qx^{p-1} + py^{q-1})(x+y)} dx dy \leq \frac{\pi}{\sin(\pi/p)pq} \left( \int_0^\infty \int_0^x |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}} \left( \int_0^\infty \int_0^y |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}.$$



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In particular, when  $p = q = 2$ , we have

$$(2.6) \quad \int_0^\infty \int_0^\infty \frac{|f(x)| |g(y)|}{(x+y)^2} dx dy \\ \leq \frac{\pi}{2} \left( \int_0^\infty \int_0^x |f'(\tau)|^2 d\tau dx \right)^{\frac{1}{2}} \left( \int_0^\infty \int_0^y |g'(\delta)|^2 d\delta dy \right)^{\frac{1}{2}}.$$

*Proof.* From the hypotheses, we have the following identities

$$(2.7) \quad f(x) = \int_0^x f'(\tau) d\tau,$$

and

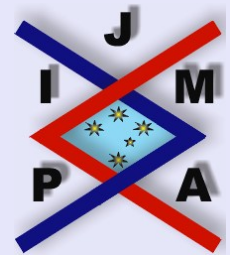
$$(2.8) \quad g(y) = \int_0^y g'(\delta) d\delta$$

for  $x, y \in (0, \infty)$ . From (2.7) and (2.8) and using Hölder's integral inequality, respectively, we have

$$(2.9) \quad |f(x)| \leq x^{\frac{1}{q}} \left( \int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}}$$

and

$$(2.10) \quad |g(y)| \leq y^{\frac{1}{p}} \left( \int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}}$$




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for  $x, y \in (0, \infty)$ . From (2.9) and (2.10) and using the elementary inequality

$$(2.11) \quad z_1 z_2 \leq \frac{z_1^p}{p} + \frac{z_2^q}{q}, \quad z_1 \geq 0, \quad z_2 \geq 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

we observe that

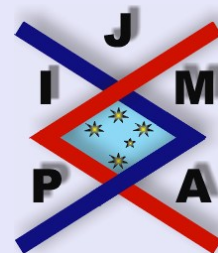
$$(2.12) \quad \begin{aligned} |f(x)||g(y)| &\leq x^{\frac{1}{q}} y^{\frac{1}{p}} \left( \int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}} \left( \int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}} \\ &\leq \left( \frac{x^{p-1}}{p} + \frac{y^{q-1}}{q} \right) \left( \int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}} \left( \int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}} \end{aligned}$$

for  $x, y \in (0, \infty)$ . From (2.12) we observe that

$$(2.13) \quad \frac{|f(x)||g(y)|}{qx^{p-1} + py^{q-1}} \leq \frac{1}{pq} \left( \int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}} \left( \int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}}.$$

Hence

$$(2.14) \quad \begin{aligned} \int_0^\infty \int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1} + py^{q-1})(x+y)} dx dy \\ \leq \frac{1}{pq} \int_0^\infty \int_0^\infty \frac{\left( \int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}} \left( \int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}}}{x+y} dx dy. \end{aligned}$$



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By Hölder's integral inequality and (2.1), we have

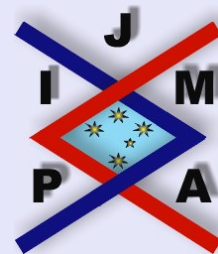
$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{(\int_0^x |f'(\tau)|^p d\tau)^{\frac{1}{p}} (\int_0^y |g'(\delta)|^q d\delta)^{\frac{1}{q}}}{x+y} dx dy \\
 &= \int_0^\infty \int_0^\infty \frac{(\int_0^x |f'(\tau)|^p d\tau)^{\frac{1}{p}}}{(x+y)^{\frac{1}{p}}} \left(\frac{x}{y}\right)^{\frac{1}{pq}} \frac{(\int_0^y |g'(\delta)|^q d\delta)^{\frac{1}{q}}}{(x+y)^{\frac{1}{q}}} \left(\frac{y}{x}\right)^{\frac{1}{pq}} dx dy \\
 &\leq \left( \int_0^\infty \int_0^\infty \frac{\int_0^x |f'(\tau)|^p d\tau}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{q}} dx dy \right)^{\frac{1}{p}} \\
 &\quad \times \left( \int_0^\infty \int_0^\infty \frac{\int_0^y |g'(\delta)|^q d\delta}{x+y} \left(\frac{y}{x}\right)^{\frac{1}{p}} dx dy \right)^{\frac{1}{q}} \\
 (2.15) \quad &\leq \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty \int_0^x |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}} \left( \int_0^\infty \int_0^y |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}
 \end{aligned}$$

by (2.14) and (2.15), we get (2.5). The proof of Theorem 2.5 is complete.  $\square$

In a similar way to the proof of Theorem 2.5, we can prove the following theorems.

**Theorem 2.6.** *Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , and  $f(x), g(y)$  be real-valued continuous functions defined on  $[0, \infty)$ , respectively, and let  $f(0) = g(0) = 0$ , and*

$$0 < \int_0^\infty \int_0^x x^{1-\lambda} |f'(\tau)|^p d\tau dx < \infty, \quad 0 < \int_0^\infty \int_0^y y^{1-\lambda} |g'(\delta)|^q d\delta dy < \infty,$$



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then

$$(2.16) \quad \int_0^\infty \int_0^\infty \frac{|f(x)| |g(y)|}{(qx^{p-1} + py^{q-1})(x+y)^\lambda} dx dy$$

$$\leq \frac{B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right)}{pq} \left( \int_0^\infty \int_0^x x^{1-\lambda} |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}}$$

$$\times \left( \int_0^\infty \int_0^y y^{1-\lambda} |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}.$$

In particular, when  $p = q = 2$ ,

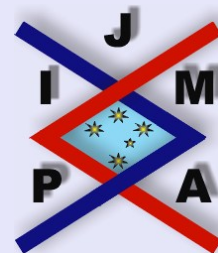
$$(2.17) \quad \int_0^\infty \int_0^\infty \frac{|f(x)| |g(y)|}{(x+y)^{1+\lambda}} dx dy$$

$$\leq \frac{B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)}{2} \left( \int_0^\infty \int_0^x x^{1-\lambda} |f'(\tau)|^2 d\tau dx \right)^{\frac{1}{2}}$$

$$\times \left( \int_0^\infty \int_0^y y^{1-\lambda} |g'(\delta)|^2 d\delta dy \right)^{\frac{1}{2}}.$$

**Theorem 2.7.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , and  $f(x), g(y)$  be real-valued continuous functions defined on  $[0, \infty)$ , respectively, and let  $f(0) = g(0) = 0$ , and

$$0 < \int_0^\infty \int_0^x x^{1-\lambda} |f'(\tau)|^p d\tau dx < \infty, \quad 0 < \int_0^\infty \int_0^y y^{1-\lambda} |g'(\delta)|^q d\delta dy < \infty.$$



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Then

$$(2.18) \quad \int_0^\infty \int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1} + py^{q-1})(x^\lambda + y^\lambda)} dx dy$$

$$\leq \frac{B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right)}{\lambda pq} \left( \int_0^\infty \int_0^x x^{1-\lambda} |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}}$$

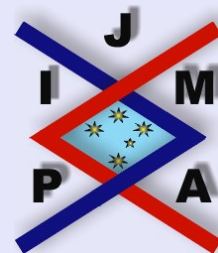
$$\times \left( \int_0^\infty \int_0^y y^{1-\lambda} |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}.$$

In particular, when  $p = q = 2$ ,

$$(2.19) \quad \int_0^\infty \int_0^\infty \frac{|f(x)||g(y)|}{(x^\lambda + y^\lambda)(x + y)} dx dy$$

$$\leq \frac{\pi}{2\lambda} \left( \int_0^\infty \int_0^x x^{1-\lambda} |f'(\tau)|^2 d\tau dx \right)^{\frac{1}{2}}$$

$$\times \left( \int_0^\infty \int_0^y y^{1-\lambda} |g'(\delta)|^2 d\delta dy \right)^{\frac{1}{2}}.$$



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### 3. Discrete Analogues

**Theorem 3.1.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $\{a(m)\}$  and  $\{b(n)\}$  be two sequences of real numbers where  $m, n \in \mathbb{N}_0$ , and  $a(0) = b(0) = 0$ , and  $0 < \sum_{m=1}^{\infty} \sum_{\tau=1}^m |\nabla a(\tau)|^p < \infty$ ,  $0 < \sum_{n=1}^{\infty} \sum_{\delta=1}^n |\nabla b(\delta)|^q < \infty$ , then

$$(3.1) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(qm^{p-1} + pn^{q-1})(m+n)} \leq \frac{\pi}{\sin(\pi/p)pq} \left( \sum_{m=1}^{\infty} \sum_{k=1}^m a_k^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \sum_{r=1}^n b_r^q \right)^{\frac{1}{q}}.$$

In particular, when  $p = q = 2$ , we have

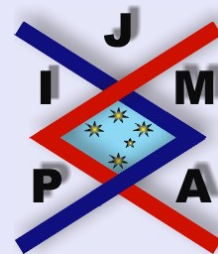
$$(3.2) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(m+n)^2} \leq \frac{\pi}{2} \left( \sum_{m=1}^{\infty} \sum_{k=1}^m a_k^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \sum_{r=1}^n b_r^2 \right)^{\frac{1}{2}}.$$

*Proof.* From the hypotheses, it is easy to observe that the following identities hold

$$(3.3) \quad a_m = \sum_{\tau=1}^m \nabla a(\tau),$$

and

$$(3.4) \quad b_n = \sum_{\delta=1}^n \nabla b(\delta)$$



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for  $m, n \in \mathbb{N}$ . From (3.3) and (3.4) and using Hölder's inequality, we have

$$(3.5) \quad |a_m| \leq m^{\frac{1}{q}} \left( \sum_{\tau=1}^m |\nabla a(\tau)|^p \right)^{\frac{1}{p}},$$

and

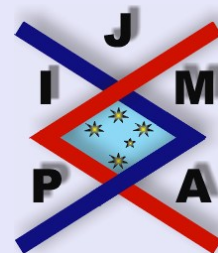
$$(3.6) \quad |b_n| \leq n^{\frac{1}{p}} \left( \sum_{\delta=1}^n |\nabla b(\delta)|^q \right)^{\frac{1}{q}}$$

for  $m, n \in \mathbb{N}$ . From (3.5) and (3.6) and using the elementary inequality (2.11), we observe that

$$(3.7) \quad \begin{aligned} |a_m| |b_n| &\leq m^{\frac{1}{q}} n^{\frac{1}{p}} \left( \sum_{\tau=1}^m |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left( \sum_{\delta=1}^n |\nabla b(\delta)|^q \right)^{\frac{1}{q}} \\ &\leq \left( \frac{m^{p-1}}{p} + \frac{n^{q-1}}{q} \right) \left( \sum_{\tau=1}^m |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left( \sum_{\delta=1}^n |\nabla b(\delta)|^q \right)^{\frac{1}{q}} \end{aligned}$$

for  $m, n \in \mathbb{N}$ . From (3.7), we observe that

$$(3.8) \quad \frac{|a_m| |b_n|}{qm^{p-1} + pn^{q-1}} \leq \frac{1}{pq} \left( \sum_{\tau=1}^m |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left( \sum_{\delta=1}^n |\nabla b(\delta)|^q \right)^{\frac{1}{q}}.$$




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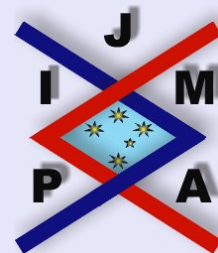
Hence

$$(3.9) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(qm^{p-1} + pn^{q-1})(m+n)} \\ \leq \frac{1}{pq} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\sum_{\tau=1}^m |\nabla a(\tau)|^p)^{\frac{1}{p}} (\sum_{\delta=1}^n |\nabla b(\delta)|^q)^{\frac{1}{q}}}{m+n}.$$

By the Hölder inequality and (2.3)

$$(3.10) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\sum_{\tau=1}^m |\nabla a(\tau)|^p)^{\frac{1}{p}} (\sum_{\delta=1}^n |\nabla b(\delta)|^q)^{\frac{1}{q}}}{m+n} \\ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\sum_{\tau=1}^m |\nabla a(\tau)|^p)^{\frac{1}{p}}}{(m+n)^{\frac{1}{p}}} \left(\frac{m}{n}\right)^{\frac{1}{pq}} \frac{(\sum_{\delta=1}^n |\nabla b(\delta)|^q)^{\frac{1}{q}}}{(m+n)^{\frac{1}{q}}} \left(\frac{n}{m}\right)^{\frac{1}{pq}} \\ \leq \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sum_{\tau=1}^m |\nabla a(\tau)|^p}{m+n} \left(\frac{m}{n}\right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \\ \times \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sum_{\delta=1}^n |\nabla b(\delta)|^q}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \\ < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} \sum_{\tau=1}^m |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \sum_{\delta=1}^n |\nabla b(\delta)|^q \right)^{\frac{1}{q}}$$

by (3.9) and (3.10), we get (3.1). The proof of Theorem 3.1 is complete.  $\square$



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In a similar manner to the proof of Theorem 3.1, we can prove the following theorems.

**Theorem 3.2.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , and  $\{a(m)\}$  and  $\{b(n)\}$  be two sequences of real numbers where  $m, n \in \mathbb{N}_0$ , and  $a(0) = b(0) = 0$ , and

$$0 < \sum_{m=1}^{\infty} \sum_{\tau=1}^m m^{1-\lambda} |\nabla a(\tau)|^p < \infty,$$

$$0 < \sum_{n=1}^{\infty} \sum_{\delta=1}^n n^{1-\lambda} |\nabla b(\delta)|^q < \infty,$$

then

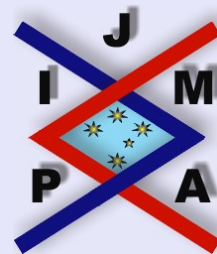
$$(3.11) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(qm^{p-1} + pn^{q-1})(m+n)^\lambda}$$

$$\leq \frac{B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right)}{pq} \left( \sum_{m=1}^{\infty} \sum_{\tau=1}^m m^{1-\lambda} |\nabla a(\tau)|^p \right)^{\frac{1}{p}}$$

$$\times \left( \sum_{n=1}^{\infty} \sum_{\delta=1}^n n^{1-\lambda} |\nabla b(\delta)|^q \right)^{\frac{1}{q}}.$$

In particular, when  $p = q = 2$ , we have

$$(3.12) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(m+n)^{1+\lambda}}$$



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$$\leq \frac{B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)}{2} \left( \sum_{m=1}^{\infty} \sum_{\tau=1}^m m^{1-\lambda} |\nabla a(\tau)|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \sum_{\delta=1}^n n^{1-\lambda} |\nabla b(\delta)|^2 \right)^{\frac{1}{2}}.$$

**Theorem 3.3.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda > 2 - \min\{p, q\}$ , and  $\{a(m)\}$  and  $\{b(n)\}$  be two sequences of real numbers where  $m, n \in \mathbb{N}_0$ , and  $a(0) = b(0) = 0$ , and

$$0 < \sum_{m=1}^{\infty} \sum_{\tau=1}^m m^{1-\lambda} |\nabla a(\tau)|^p < \infty,$$

$$0 < \sum_{n=1}^{\infty} \sum_{\delta=1}^n n^{1-\lambda} |\nabla b(\delta)|^q < \infty,$$

then

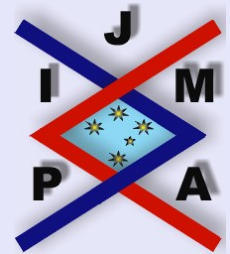
$$(3.13) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(qm^{p-1} + pn^{q-1})(m^\lambda + n^\lambda)}$$

$$\leq \frac{B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right)}{\lambda pq} \left( \sum_{m=1}^{\infty} \sum_{\tau=1}^m m^{1-\lambda} |\nabla a(\tau)|^p \right)^{\frac{1}{p}}$$

$$\times \left( \sum_{n=1}^{\infty} \sum_{\delta=1}^n n^{1-\lambda} |\nabla b(\delta)|^q \right)^{\frac{1}{q}}.$$

In particular, when  $p = q = 2$ , we have

$$(3.14) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m| |b_n|}{(m+n)(m^\lambda + n^\lambda)}$$



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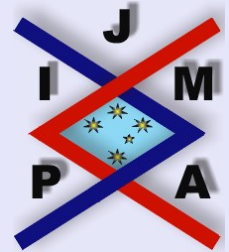
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$$\leq \frac{\pi}{2\lambda} \left( \sum_{m=1}^{\infty} \sum_{\tau=1}^m m^{1-\lambda} |\nabla a(\tau)|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \sum_{\delta=1}^n n^{1-\lambda} |\nabla b(\delta)|^2 \right)^{\frac{1}{2}} .$$




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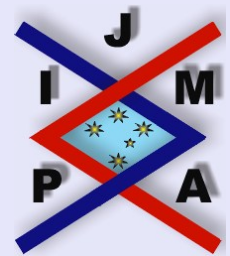
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