Journal of Inequalities in Pure and Applied Mathematics

A CHARACTERIZATION OF λ-CONVEX FUNCTIONS

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ISSN (electronic): 1443-5756
060-04

volume 5, issue 3, article 71, 2004.

Received 18 March, 2004; accepted 19 April, 2004.

Communicated by: K. Nikodem

Abstract

The main result of this paper shows that λ-convex functions can be characterized in terms of a lower second-order generalized derivative.

2000 Mathematics Subject Classification: Primary 26A51, 39B62.
Key words: λ-convexity, Generalized 2nd-order derivative.

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1. Introduction

Let $I \subseteq \mathbb{R}$ be an open interval and $\lambda : I^2 \to (0,1)$ be a fixed function. A real-valued function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is called *λ-convex* if

$$\begin{aligned} (1.1)\quad & f(\lambda(x,y)x + (1-\lambda(x,y))y) \\ & \qquad \le \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) \quad \text{for} \quad x,y \in I. \end{aligned}$$

Such functions were introduced and discussed by Zs. Páles in [6], who obtained a Bernstein-Doetch type theorem for them. A Sierpiński-type result, stating that measurable λ-convex functions are convex, can be found in [2]. Recently K. Nikodem and Zs. Páles [5] proved that functions satisfying (1.1) with a constant λ can be characterized by use of a second-order generalized derivative. The main results of this paper show that λ-convexity, for λ not necessarily constant, can also be characterized in terms of a properly chosen lower second-order generalized derivative.

2. Divided Differences and Convexity Triplets

If $f : I \to \mathbb{R}$ is an arbitrary function then define the second-order divided difference of f for three pairwise distinct points x, y, z of I by

$$f[x,y,z] := \frac{f(x)}{(y-x)(z-x)} + \frac{f(y)}{(x-y)(z-y)} + \frac{f(z)}{(x-z)(y-z)}. \tag{2.1}$$

It is known (cf. e.g.[4], [7]) and easy to check that a function $f : I \to \mathbb{R}$ is convex if and only if

$$f[x,y,z] \geq 0$$

for every pairwise distinct points x, y, z of I. Motivated by this characterization of convexity, a triplet (x, y, z) in I^3 with pairwise distinct points x, y, z is called a *convexity triplet for a function* $f : I \to \mathbb{R}$ if $f[x, y, z] \geq 0$ and the set of all convexity triplets of f is denoted by $\mathcal{C}(f)$. Using this terminology, f is λ-convex if and only if

$$\big(x, \lambda(x,y)x + (1-\lambda(x,y))y, y\big) \in \mathcal{C}(f) \qquad \text{for} \quad x, y \in I \text{ with } x \neq y. \tag{2.2}$$

The following result obtained in [5] will be used in the proof of the main theorem.

Lemma 2.1. *(Chain Inequality) Let* $f : I \to \mathbb{R}$ *and* $x_0 < x_1 < \cdots < x_n$ $(n \geq 2)$ *be arbitrary points in* I*. Then, for all fixed* $0 < j < n$*,*

$$\min_{1 \leq i \leq n-1} f[x_{i-1}, x_i, x_{i+1}] \leq f[x_0, x_j, x_n] \leq \max_{1 \leq i \leq n-1} f[x_{i-1}, x_i, x_{i+1}]. \tag{2.3}$$

3. Main Results

Assume that $\lambda : I \to (0,1)$ is a fixed function and consider the *lower 2nd-order generalized λ-derivative* of a function $f : I \to \mathbb{R}$ at a point $\xi \in I$ defined by

$$\underline{\delta}_{\lambda}^{2} f(\xi) := \liminf_{\substack{(x,y)\to(\xi,\xi)\\ \xi\in \operatorname{co}\{x,y\}}} 2f[x, \lambda(x,y)x + (1-\lambda(x,y))y, y]. \tag{3.1}$$

One can easily show that if f is twice continuously differentiable at ξ then

$$\underline{\delta}_{\lambda}^{2} f(\xi) = f''(\xi).$$

Moreover, from (2.2) and (3.1), if a function $f : I \to \mathbb{R}$ is λ-convex, then $\underline{\delta}_{\lambda}^{2} f(\xi) \geq 0$ for every $\xi \in I$. The following example shows that the reverse implication is not true in general.

Example 3.1. *Define $\lambda : \mathbb{R}^2 \to (0,1)$ by the formula*

$$\lambda(x,y) = \begin{cases} \dfrac{1}{3} & \textit{if } x = y, \\[2ex] \dfrac{1}{2} & \textit{if } x \neq y, \end{cases}$$

and take the function $f : \mathbb{R} \to \mathbb{R}$;

$$f(x) = \begin{cases} 0 & \textit{if } x = 0, \\ 1 & \textit{if } x \neq 0. \end{cases}$$

It is easy to check that this function is not λ-convex, but $\underline{\delta}^2_\lambda f(\xi) \geq 0$ for every $\xi \in \mathbb{R}$.

Now, let $\lambda : I^2 \to (0,1)$ be a fixed function. Define

$$M(x,y) := \lambda(x,y)x + (1-\lambda(x,y))y$$

and write conditions

$$\inf_{x,y\in[x_0,y_0]} \lambda(x,y) > 0 \quad \text{and} \quad \sup_{x,y\in[x_0,y_0]} \lambda(x,y) < 1, \qquad \text{for all } x_0, y_0 \in I \text{ with } x_0 \leq y_0, \tag{3.2}$$

$$M(M(x, M(x,y)), M(y, M(x,y))) = M(x,y), \text{ for all } x,y \in I. \tag{3.3}$$

Of course, the above assumptions are satisfied for arbitrary constant λ. Moreover, observe that if M fulfils the bisymmetry equation (cf. [1], [3]) then it fulfils equation (3.3), too. Thus for each quasi-arithmetic mean M these conditions are also fulfilled.

Using a similar method as in [5] we can prove the following result.

Theorem 3.1. *(Mean Value Inequality for λ-convexity) Let $I \subseteq \mathbb{R}$ be an interval, $\lambda : I^2 \to (0,1)$ satisfies assumptions (3.2) – (3.3), $f : I \to \mathbb{R}$ and $x, y \in I$ with $x \neq y$. Then there exists a point $\xi \in \mathrm{co}\{x,y\}$ such that*

$$2f[x, \lambda(x,y)x + (1-\lambda(x,y))y, y] \geq \underline{\delta}^2_\lambda f(\xi). \tag{3.4}$$

Proof. In the sequel, a triplet $(x,u,y)\in I^3$ will be called a λ-*triplet* if

$$u=\lambda(x,y)x+(1-\lambda(x,y))y$$

or

$$u=\lambda(y,x)y+(1-\lambda(y,x))x.$$

Let x and y be distinct elements of I. Assume that $x<y$ (the proof in the case $x>y$ is similar). In what follows, we intend to construct a sequence of λ-triplets (x_n,u_n,y_n) such that

$$x_0\le x_1\le x_2\le\dots,\ \ y_0\ge y_1\ge y_2\ge\dots,\ \ x_n<u_n<y_n\ (n\in\mathbb{N}), \tag{3.5}$$

$$\begin{aligned} & y_n-x_n \\ & \le\left(\max\left\{1-\inf_{x,y\in[x_0,y_0]}\lambda(x,y),\sup_{x,y\in[x_0,y_0]}\lambda(x,y)\right\}\right)^n(y_0-x_0)\quad(n\in\mathbb{N}),\end{aligned} \tag{3.6}$$

and

$$f[x_0,u_0,y_0]\ge f[x_1,u_1,y_1]\ge f[x_2,u_2,y_2]\ge\cdots. \tag{3.7}$$

Define

$$(x_0,u_0,y_0):=(x,\lambda(x,y)x+(1-\lambda(x,y))y,y)$$

and assume that we have constructed (x_n,u_n,y_n). Now set

$$z_{n,0}:=x_n,\quad z_{n,1}:=\lambda(x_n,u_n)x_n+(1-\lambda(x_n,u_n))u_n,\quad z_{n,2}:=u_n,$$
$$z_{n,3}:=\lambda(y_n,u_n)y_n+(1-\lambda(y_n,u_n))u_n,\quad z_{n,4}:=y_n.$$

Then $(z_{n,i-1}, z_{n,i}, z_{n,i+1})$ are λ-triplets for $i \in \{1,2,3\}$ (for $i \in \{1,3\}$ immediately from the definition of λ-triplets and for $i = 2$ from condition (3.3)).

Using the Chain Inequality, we find that there exists an index $i \in \{1,2,3\}$ such that

$$f[x_n, u_n, y_n] \geq f[z_{n,i-1}, z_{n,i}, z_{n,i+1}].$$

Finally, define

$$(x_{n+1}, u_{n+1}, y_{n+1}) := (z_{n,i-1}, z_{n,i}, z_{n,i+1}).$$

The sequence so constructed clearly satisfies (3.5) and (3.7). We prove (3.6) by induction. It is obvious for $n = 0$. Assume that it holds for n and $u_n = \lambda(x_n, y_n)x_n + (1 - \lambda(x_n, y_n))y_n$ (if $u_n = \lambda(y_n, x_n)y_n + (1 - \lambda(y_n, x_n))x_n$ then the motivation is the same). Consider three cases.

(i)

$$(x_{n+1}, u_{n+1}, y_{n+1}) = (x_n, \lambda(x_n, u_n)x_n + (1 - \lambda(x_n, u_n))u_n, u_n)$$

then

$$\begin{aligned}
&y_{n+1} - x_{n+1} \\
&= u_n - x_n \\
&= \lambda(x_n, y_n)x_n + (1 - \lambda(x_n, y_n))y_n - x_n \\
&= (1 - \lambda(x_n, y_n))(y_n - x_n) \\
&\leq \max\left\{1 - \inf_{x,y\in[x_0,y_0]} \lambda(x, y), \sup_{x,y\in[x_0,y_0]} \lambda(x, y)\right\}(y_n - x_n)
\end{aligned}$$

$$\leq \left(\max\left\{1-\inf_{x,y\in[x_0,y_0]}\lambda(x,y),\ \sup_{x,y\in[x_0,y_0]}\lambda(x,y)\right\}\right)^{n+1}(y_0-x_0).$$

(ii)

$$\begin{aligned}&(x_{n+1},u_{n+1},y_{n+1})\\&=(\lambda(x_n,u_n)x_n+(1-\lambda(x_n,u_n))u_n,u_n,\lambda(y_n,u_n)y_n+(1-\lambda(y_n,u_n))u_n)\end{aligned}$$

then

$$\begin{aligned}&y_{n+1}-x_{n+1}\\&=\lambda(x_n,u_n)(u_n-x_n)+\lambda(y_n,u_n)(y_n-u_n)\\&=\lambda(x_n,u_n)(1-\lambda(x_n,y_n))(y_n-x_n)+\lambda(y_n,u_n)\lambda(x_n,y_n)(y_n-x_n)\\&\leq \max\left\{1-\inf_{x,y\in[x_0,y_0]}\lambda(x,y),\ \sup_{x,y\in[x_0,y_0]}\lambda(x,y)\right\}(1-\lambda(x_n,y_n))(y_n-x_n)\\&\qquad+\max\left\{1-\inf_{x,y\in[x_0,y_0]}\lambda(x,y),\ \sup_{x,y\in[x_0,y_0]}\lambda(x,y)\right\}\lambda(x_n,y_n)(y_n-x_n)\\&=\max\left\{1-\inf_{x,y\in[x_0,y_0]}\lambda(x,y),\ \sup_{x,y\in[x_0,y_0]}\lambda(x,y)\right\}(y_n-x_n)\\&\leq \left(\max\left\{1-\inf_{x,y\in[x_0,y_0]}\lambda(x,y),\ \sup_{x,y\in[x_0,y_0]}\lambda(x,y)\right\}\right)^{n+1}(y_0-x_0).\end{aligned}$$

(iii)
$$(x_{n+1}, u_{n+1}, y_{n+1}) = (u_n, \lambda(y_n, u_n)y_n + (1 - \lambda(y_n, u_n))u_n, y_n)$$

then

$$\begin{aligned} & y_{n+1} - x_{n+1} \\ & = y_n - u_n \\ & = y_n - (\lambda(x_n, y_n)x_n + (1 - \lambda(x_n, y_n))y_n) \\ & = \lambda(x_n, y_n)(y_n - x_n) \\ & \leq \max\left\{1 - \inf_{x,y\in[x_0,y_0]} \lambda(x,y), \sup_{x,y\in[x_0,y_0]} \lambda(x,y)\right\}(y_n - x_n) \\ & \leq \left(\max\left\{1 - \inf_{x,y\in[x_0,y_0]} \lambda(x,y), \sup_{x,y\in[x_0,y_0]} \lambda(x,y)\right\}\right)^{n+1} (y_0 - x_0). \end{aligned}$$

Thus (3.6) is also verified.

Due to the monotonicity properties of the sequences (x_n), (y_n) and also (3.2), (3.6), there exists a unique element $\xi \in [x, y]$ such that

$$\bigcap_{i=0}^{\infty} [x_n, y_n] = \{\xi\}.$$

Then, by (3.7) and symmetry of the second-order divided difference, we get that

$$\begin{aligned} f[x, \lambda(x,y)x + (1 - \lambda(x,y))y, y] &= f[x_0, u_0, y_0] \\ &\geq \liminf_{n\to\infty} f[x_n, u_n, y_n] \end{aligned}$$

$$\geq \liminf_{\substack{(v,w)\to(\xi,\xi)\\ \xi\in\mathrm{co}\{v,w\}}} f[v, \lambda(v,w)v + (1-\lambda(v,w))w, w]$$
$$= \frac{1}{2}\underline{\delta}^2_\lambda f(\xi),$$

which completes the proof. $\square$

As an immediate consequence of the above theorem, we get the following characterization of λ-convexity.

Theorem 3.2. *Let $\lambda : I^2 \to (0,1)$ be a fixed function satisfying assumptions* (3.2) – (3.3). *A function $f : I \to \mathbb{R}$ is λ-convex on I if and only if*

(3.8) $$\underline{\delta}^2_\lambda f(\xi) \geq 0, \text{ for all } \xi \in I.$$

Proof. If f is λ-convex, then, clearly $\underline{\delta}^2_\lambda f \geq 0$. Conversely, if $\underline{\delta}^2_\lambda f$ is nonnegative on I, then, by the previous theorem

$$f[x, \lambda(x,y)x + (1-\lambda(x,y))y, y] \geq 0$$

for all $x, y \in I$, i.e., f is λ-convex. $\square$

An obvious but interesting consequence of Theorem 3.2 is that the λ-convexity property is *localizable* in the following sense:

Corollary 3.3. *Let $\lambda : I^2 \to (0,1)$ be a fixed function satisfying assumptions* (3.2) – (3.3). *A function $f : I \to \mathbb{R}$ is λ-convex on I if and only if, for each point $\xi \in I$, there exists a neighborhood U of ξ such that f is λ-convex on $I \cap U$.*

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