

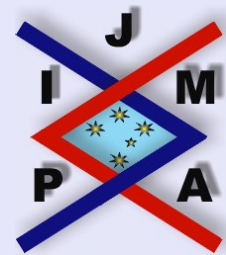
# Journal of Inequalities in Pure and Applied Mathematics

## A HÖLDER INEQUALITY FOR HOLOMORPHIC FUNCTIONS

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## Abstract

We prove a Hölder inequality for the  $L^p$ -spaces of analytic functions with respect to a complex Gaussian measure.

*2000 Mathematics Subject Classification:* 60H40

*Key words:* Complex hypercontractivity, Holomorphic, Hölder inequality.

We would like to thank Professors: Hall, Sontz, Janson, Gross, and Carlen for their suggestions and comments related to the result presented in this paper.

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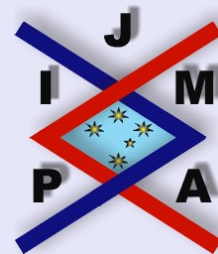
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# 1. Introduction

In this paper we will prove the following inequality: for any two entire analytic functions  $f, g : \mathbb{C}^n \rightarrow \mathbb{C}$  and any positive numbers  $p, q, r$ , and  $s$ , such that  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ , we have:

$$(1.1) \quad \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\sqrt{r}z)g(\sqrt{r}z)|^s e^{-|z|^2} dz \leq \left[ \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\sqrt{p}z)|^s e^{-|z|^2} dz \right] \left[ \frac{1}{\pi^n} \int_{\mathbb{C}^n} |g(\sqrt{q}z)|^s e^{-|z|^2} dz \right],$$

provided that the integrals from the right side are both finite. This inequality is motivated by the following facts from White Noise Analysis. The  $S$ -transform is known to be a unitary isomorphism from the space of square integrable functions defined on a white noise space onto the space  $\mathcal{H}L^2(E)$ , where  $E$  is a separable complex Hilbert space (see [4, p. 39] for the definition of the  $S$ -transform, and page 337 for the stated isomorphism). The space of generalized functions in White Noise Analysis is the union of an increasing family of weighted  $L^2$ -functions. The  $S$ -transform maps such a weighted  $L^2$ -space onto  $\Gamma(A)\mathcal{H}L^2(E)$ , where  $A$  is an operator on  $E$ , and  $\Gamma(A)\varphi(u) := \varphi(Au)$ . In White Noise Analysis there is a product between two generalized functions, called the Wick product. It is defined in such a way that the  $S$ -transform of a Wick product of two generalized functions is the product of the  $S$ -transforms of the two generalized functions. A natural question is the following: knowing the smallest weighted space in which a generalized function  $\varphi$  lives and the smallest weighted space in which another generalized function  $\psi$  lives, what is the smallest weighted space



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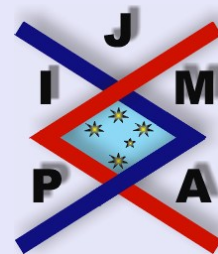
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in which the Wick product of  $\varphi$  and  $\psi$  lives? Applying the  $S$ -transform isomorphism, the question is reduced to the following question: If  $f \in \Gamma(A)\mathcal{HL}^2(\mathbb{C}^n)$  and  $g \in \Gamma(B)\mathcal{HL}^2(\mathbb{C}^n)$ , then what are the operators  $C$  having the minimal operatorial norm such that  $fg \in \Gamma(C)\mathcal{HL}^2(\mathbb{C}^n)$ ? This inequality, for  $s = 2$  only, was proven in [5] and called “a Young inequality for White Noise Analysis”. Although the inequality (1.1), for the space  $\mathcal{HL}^2(\mathbb{C}^n)$  only, gives a satisfactory answer to this question, from a mathematical point of view it is important and interesting to extend this sharp inequality to all the other  $\mathcal{HL}^p(\mathbb{C}^n)$  spaces. This is the purpose of this short paper and we do not know what applications it may have.




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## 2. A Complex Hölder Inequality

For any  $p \geq 1$ , let  $\mathcal{H}L^p(\mathbb{C}^n, \mu)$  denote the space of all holomorphic functions  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  such that:

$$\|f\|_p^p := \int_{\mathbb{C}^n} |f(z)|^p d\mu(z) < \infty,$$

where  $d\mu(z) = (1/\pi^n)e^{-|z|^2} dz$ . Here, if  $z = x + iy$ , then  $dz = dx dy$  is the Lebesgue measure on the space  $\mathbb{C}^n$  identified with  $\mathbb{R}^{2n}$ .

For any function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  and complex number  $a \in \mathbb{C}$ , we define a new function  $\Gamma(a)f : \mathbb{C}^n \rightarrow \mathbb{C}$ , by  $\Gamma(a)f(z) := f(az)$ . Observe that if  $f$  is holomorphic, then  $\Gamma(a)f$  is also holomorphic. The following hypercontractivity result gives us a relation between the spaces  $\mathcal{H}L^p(\mathbb{C}^n, \mu)$ , when  $1 \leq p < \infty$ .

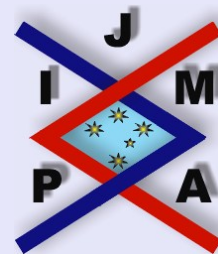
**Theorem 2.1.** *For any  $1 \leq p < q < \infty$  and any holomorphic function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ , the following inequality holds provided that the right hand side is finite:*

$$(2.1) \quad \left\| \Gamma\left(\frac{1}{\sqrt{q}}\right) f \right\|_q \leq \left\| \Gamma\left(\frac{1}{\sqrt{p}}\right) f \right\|_p.$$

This theorem was first proven by Janson in [2]. Later Carlen in [1] and Zhou in [6] simultaneously proved the cases of equality. Using this theorem we will prove the following:

**Theorem 2.2.** *Let  $p, q$ , and  $r$  be strictly positive numbers (not necessarily larger than or equal to 1) such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$



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Let  $s \geq 1$ . If  $f$  and  $g$  are holomorphic functions such that  $\Gamma(\sqrt{p})f \in \mathcal{HL}^s(\mathbb{C}^n, \mu)$  and  $\Gamma(\sqrt{q})g \in \mathcal{HL}^s(\mathbb{C}^n, \mu)$ , then  $\Gamma(\sqrt{r})(fg) \in \mathcal{HL}^s(\mathbb{C}^n, \mu)$  and

$$(2.2) \quad \|\Gamma(\sqrt{r})(fg)\|_s \leq \|\Gamma(\sqrt{p})f\|_s \cdot \|\Gamma(\sqrt{q})g\|_s.$$

The equality holds if and only if one of the functions  $f$  and  $g$  is identically equal to zero, or

$$f(z) = c_1 e^{\frac{1}{p} \sum_{j=1}^n a_j z_j}$$

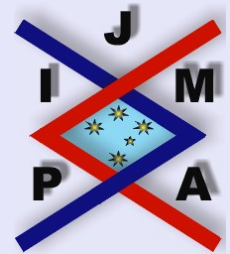
$$g(z) = c_2 e^{\frac{1}{q} \sum_{j=1}^n a_j z_j},$$

where  $c_1, c_2, a_1, a_2, \dots, a_n$  are arbitrary complex numbers.

*Proof.* Using Hölder's inequality  $\left(\frac{r}{p} + \frac{r}{q} = 1\right)$  we obtain:

$$\begin{aligned} & \|\Gamma(\sqrt{r})(fg)\|_s \\ &= \left[ \int_{\mathbb{C}^n} |f(\sqrt{r}z)|^s |g(\sqrt{r}z)|^s d\mu(z) \right]^{\frac{1}{s}} \\ &\leq \left\{ \left[ \int_{\mathbb{C}^n} |f(\sqrt{r}z)|^{s \cdot \frac{p}{r}} d\mu(z) \right]^{\frac{r}{p}} \left[ \int_{\mathbb{C}^n} |g(\sqrt{r}z)|^{s \cdot \frac{q}{r}} d\mu(z) \right]^{\frac{r}{q}} \right\}^{\frac{1}{s}} \\ &= \left[ \int_{\mathbb{C}^n} \left| f \left( \frac{\sqrt{r}}{\sqrt{sp}} (\sqrt{sp}z) \right) \right|^{\frac{sp}{r}} d\mu(z) \right]^{\frac{r}{sp}} \left[ \int_{\mathbb{C}^n} \left| g \left( \frac{\sqrt{r}}{\sqrt{sq}} (\sqrt{sq}z) \right) \right|^{\frac{sq}{r}} d\mu(z) \right]^{\frac{r}{sq}}. \end{aligned}$$

Observe that  $\frac{sp}{r} > s \geq 1$  and  $\frac{sq}{r} > s \geq 1$  and thus applying the “complex hypercontractivity” inequality (2.1) (which says that for any holomorphic functions  $h$  and any  $1 \leq u < v < \infty$ , we have  $\|\Gamma\left(\frac{1}{\sqrt{v}}\right) f\|_v \leq \|\Gamma\left(\frac{1}{\sqrt{u}}\right) f\|_u$ ) to



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the holomorphic functions:  $f(\sqrt{sp}z)$  with  $u = s$  and  $v = \frac{sp}{r}$ , and  $g(\sqrt{sq}z)$  with  $u = s$  and  $v = \frac{sq}{r}$  respectively, we obtain:

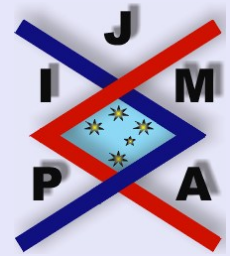
$$\begin{aligned} & \|\Gamma(\sqrt{r})(fg)\|_s \\ & \leq \left[ \int_{\mathbb{C}^n} \left| f\left(\frac{\sqrt{r}}{\sqrt{sp}}(\sqrt{sp}z)\right) \right|^{\frac{sp}{r}} d\mu(z) \right]^{\frac{r}{sp}} \left[ \int_{\mathbb{C}^n} \left| g\left(\frac{\sqrt{r}}{\sqrt{sq}}(\sqrt{sq}z)\right) \right|^{\frac{sq}{r}} d\mu(z) \right]^{\frac{r}{sq}} \\ & \leq \left[ \int_{\mathbb{C}^n} \left| f\left(\frac{1}{\sqrt{s}}(\sqrt{sp}z)\right) \right|^s d\mu(z) \right]^{\frac{1}{s}} \left[ \int_{\mathbb{C}^n} \left| g\left(\frac{1}{\sqrt{s}}(\sqrt{sq}z)\right) \right|^s d\mu(z) \right]^{\frac{1}{s}} \\ & = \left[ \int_{\mathbb{C}^n} |f(\sqrt{p}z)|^s d\mu(z) \right]^{\frac{1}{s}} \left[ \int_{\mathbb{C}^n} |g(\sqrt{q}z)|^s d\mu(z) \right]^{\frac{1}{s}} \\ & = \|\Gamma(\sqrt{p})(f)\|_s \cdot \|\Gamma(\sqrt{q})(g)\|_s. \end{aligned}$$

It is clear that if one of the functions  $f$  or  $g$  is identically equal to zero, then our inequality becomes an equality. Let us assume that both functions  $f$  and  $g$  are different from the zero functions.

From [1] and [6], we know that, in order to have equality in the “complex hypercontractivity” inequality,  $f$  and  $g$  must be functions of the form:

$$f(z) = c_1 e^{\sum_{j=1}^n \alpha_j z_j} \quad \text{and} \quad g(z) = c_2 e^{\sum_{j=1}^n \beta_j z_j},$$

where  $c_1, c_2, \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$  are arbitrary complex numbers. To have equality in Hölder’s inequality, there must be a constant  $k$  such that, for all  $z \in \mathbb{C}^n$ ,  $|f(z)|^{p/r} = k|g(z)|^{q/r}$ . Since  $f$  and  $g$  are holomorphic we obtain the condition that, for all  $1 \leq j \leq n$ ,  $\frac{p\alpha_j}{r} = \frac{q\beta_j}{r}$ . Denoting by  $a_j$  the common value



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of  $p\alpha_j$  and  $q\beta_j$ , we obtain that the equality holds in our inequality only for a pair of functions of the form:

$$f(z) = c_1 e^{\frac{1}{p} \sum_{j=1}^n a_j z_j}$$

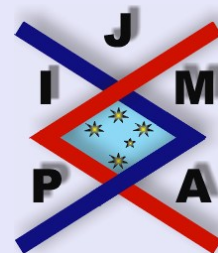
$$g(z) = c_2 e^{\frac{1}{q} \sum_{j=1}^n a_j z_j},$$

where  $c_1, c_2, a_1, a_2, \dots, a_n$  are arbitrary complex numbers. □

We are thankful to Professor Svante Janson for adding the following:

**Remark 2.1.** *The inequality (2.2) holds even for  $0 < s < 1$ .*

This is true since in [2] the “complex hypercontractivity” inequality (2.1) is proved not only for  $1 \leq s < \infty$ , but also for any  $0 < s < 1$ . The equality, for the case  $0 < s < 1$ , holds only for functions of the same form as above. This is true since the equality case in inequality (2.1) occurs only for exponential functions, even in the case  $0 < s < 1$ . This was proven in [1].




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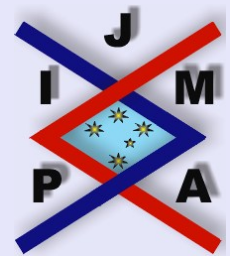
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