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HADAMARD TYPE INEQUALITIES FOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish several Hadamard type inequalities for differentiable m-convex and (α, m)-convex functions. We also establish Hadamard type inequalities for products of two m-convex or (α, m)-convex functions. Our results generalize some results of B.G. Pachpatte as well as some results of C.E.M. Pearce and J. Pečarić.

Key words and phrases: m-convex functions, (α, m)-convex functions, Hadamard's inequalities.

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1. INTRODUCTION

The following definitions are well known in literature.

Let $[0, b]$, where b is greater than 0, be an interval of the real line $\mathbb{R}$, and let $K(b)$ denote the class of all functions $f : [0, b] \to \mathbb{R}$ which are continuous and nonnegative on $[0, b]$ and such that $f(0) = 0$.

We say that the function f is *convex* on $[0, b]$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

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holds for all $x, y \in [0, b]$ and $t \in [0, 1]$. Let $K_C(b)$ denote the class of all functions $f \in K(b)$ convex on $[0, b]$, and let $K_F(b)$ be the class of all functions $f \in K(b)$ *convex in mean* on $[0, b]$, that is, the class of all functions $f \in K(b)$ for which $F \in K_C(b)$, where the mean function F of the function $f \in K(b)$ is defined by

$$F(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, & x \in (0, b]; \\ 0, & x = 0. \end{cases}$$

Let $K_S(b)$ denote the class of all functions $f \in K(b)$ which are *starshaped* with respect to the origin on $[0, b]$, that is, the class of all functions f with the property that

$$f(tx) \le tf(x)$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$. In [1] Bruckner and Ostrow, among others, proved that

$$K_C(b) \subset K_F(b) \subset K_S(b).$$

In [9] G. Toader defined m*-convexity*: another intermediate between the usual convexity and starshaped convexity.

Definition 1.1. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.
Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ for which $f(0) \le 0$.

Obviously, for $m = 1$ Definition 1.1 recaptures the concept of standard convex functions on $[0, b]$, and for $m = 0$ the concept of starshaped functions.

The following lemmas hold (see [10]).

Lemma A. *If f is in the class $K_m(b)$, then it is starshaped.*

Lemma B. *If f is in the class $K_m(b)$ and $0 < n < m \le 1$, then f is in the class $K_n(b)$.*

From Lemma A and Lemma B it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \le 0$, that is, $K_1(b)$ is a proper subclass of the class of convex functions on $[0, b]$.

It is interesting to point out that for any $m \in (0, 1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense (see [11]).

In [3] S.S. Dragomir and G. Toader proved the following Hadamard type inequality for m-convex functions.

Theorem A. *Let $f : [0, \infty) \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. If $0 \le a < b < \infty$ and $f \in L^1([a, b])$ then*

$$\text{(1.1)} \qquad \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \min\left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}.$$

Some generalizations of this result can be found in [4].

The notion of m-convexity has been further generalized in [5] as it is stated in the following definition:

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.
Denote by $K_m^{\alpha}(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that for $(\alpha, m) \in \{(0,0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex functions respectively. Note that in the class $K_1^1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is $K_1^1(b)$ is a proper subclass of the class of all convex functions on $[0, b]$. The interested reader can find more about partial ordering of convexity in [8, p. 8, 280].

In [2] in order to prove some inequalities related to Hadamard's inequality S. S. Dragomir and R. P. Agarwal used the following lemma.

Lemma C. *Let $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, be a differentiable mapping on $\mathring{I}$, and $a, b \in I$, where $a < b$. If $f' \in L^1([a, b])$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta + (1-t)b)\mathrm{d}t. \tag{1.2}$$

Here $\mathring{I}$ denotes the interior of I.

In [7], using the same Lemma C, C.E.M. Pearce and J. Pečarić proved the following theorem.

Theorem B. *Let $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, be a differentiable mapping on I^0, and $a, b \in I$, where $a < b$. If $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In [6] B. G. Pachpatte established two new Hadamard type inequalities for products of convex functions. They are given in the next theorem.

Theorem C. *Let $f, g : [a, b] \to [0, \infty)$ be convex functions on $[a, b] \subset \mathbb{R}$, where $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\, g(x)\, \mathrm{d}x \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b), \tag{1.3}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

The main purpose of this paper is to establish new inequalities like those given in Theorems A, B and C, but now for the classes of m-convex functions (Section 2) and (α, m)-convex functions (Section 3).

2. INEQUALITIES FOR m-CONVEX FUNCTIONS

Theorem 2.1. *Let I be an open real interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L^1([a,b])$, where $0 \le a < b < \infty$. If $|f'|^q$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q \in [1,\infty)$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}},\left(\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Proof. Suppose that $q=1$. From Lemma C we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,\mathrm{d}t. \tag{2.1}$$

Since $|f'|$ is m-convex on $[a,b]$ we know that for any $t\in[0,1]$

$$\begin{aligned}|f'(ta+(1-t)b)| &= \left|f'\left(ta+m(1-t)\frac{b}{m}\right)\right| \\ &\le t\,|f'(a)|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|,\end{aligned}$$

hence

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\le \frac{b-a}{2}\int_0^1|1-2t|\left[t\,|f'(a)|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t \\ &= \frac{b-a}{2}\int_0^1\left[t\,|1-2t|\,|f'(a)|+m(1-t)\,|1-2t|\left|f'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t \\ &= \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}}\left[t(1-2t)\,|f'(a)|+m(1-t)(1-2t)\left|f'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t\right. \\ &\qquad\left.+\int_{\frac{1}{2}}^1\left[t(2t-1)\,|f'(a)|+m(1-t)(2t-1)\left|f'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t\right\} \\ &= \frac{b-a}{8}\left(|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right).\end{aligned}$$

Analogously we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{8}\left(m\left|f'\left(\frac{a}{m}\right)\right|+|f'(b)|\right),$$

which completes the proof for this case.

Suppose now that $q > 1$. Using the well known Hölder inequality for q and $p = q/(q-1)$ we obtain

$$\begin{aligned}
&\int_0^1 |1-2t|\,|f'(ta+(1-t)\,b)|\,\mathrm{d}t \\
&= \int_0^1 |1-2t|^{1-\frac{1}{q}}\,|1-2t|^{\frac{1}{q}}\,|f'(ta+(1-t)\,b)|\,\mathrm{d}t \\
&\le \left(\int_0^1 |1-2t|\,\mathrm{d}t\right)^{\frac{q-1}{q}} \left(\int_0^1 |1-2t|\,|f'(ta+(1-t)\,b)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}. \qquad (2.2)
\end{aligned}$$

Since $|f'|^q$ is m-convex on $[a,b]$ we know that for every $t \in [0,1]$

$$|f'(ta+(1-t)\,b)|^q \le t\,|f'(a)|^q + m\,(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q, \qquad (2.3)$$

hence from (2.1), (2.2) and (2.3) we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{2}\left(\int_0^1 |1-2t|\,\mathrm{d}t\right)^{\frac{q-1}{q}} \left(\int_0^1 |1-2t|\left|f'\left(ta+m\,(1-t)\,\frac{b}{m}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{b-a}{2}\left(\int_0^1 |1-2t|\,\mathrm{d}t\right)^{\frac{q-1}{q}} \left[\frac{1}{4}\left(|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q\right)\right]^{\frac{1}{q}} \\
&= \frac{b-a}{4}\left(\frac{m\,|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

and analogously

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

which completes the proof. □

Theorem 2.2. *Suppose that all the assumptions of Theorem 2.1 are satisfied. Then*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\qquad \le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \left(\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Our starting point here is the identity (see [7, Theorem 2])

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{1}{b-a}\int_a^b S(x)\,f'(x)\mathrm{d}x,$$

where

$$S(x) = \begin{cases} x-a, & x \in \left[a, \frac{a+b}{2}\right); \\ x-b, & x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

We have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(x-a)\left|f'(x)\right|\mathrm{d}x+\int_{\frac{a+b}{2}}^b (b-x)\left|f'(x)\right|\mathrm{d}x\right] \\
&=(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)\,b)\right|\mathrm{d}t+\int_{\frac{1}{2}}^1 (1-t)\left|f'(ta+(1-t)\,b)\right|\mathrm{d}t\right] \\
&\leq (b-a)\left[\int_0^{\frac{1}{2}} t\left(t\left|f'(a)\right|+m\,(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)\mathrm{d}t\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1 (1-t)\left(t\left|f'(a)\right|+m\,(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)\mathrm{d}t\right] \\
&=\frac{b-a}{8}\left(\left|f'\left(a\right)\right|+m\left|f'\left(\frac{b}{m}\right)\right|\right),
\end{aligned}$$

and analogously

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\leq\frac{b-a}{8}\left(m\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(b\right)\right|\right).$$

This completes the proof for the case $q=1$.

An argument similar to the one used in the proof of Theorem 2.1 gives the proof for the case $q\in(1,\infty)$. □

As a special case of Theorem 2.1 for $m=1$, that is for $|f'|^q$ convex on $[a,b]$, we obtain the first inequality in Theorem B. Similarly, as a special case of Theorem 2.2 we obtain the second inequality in Theorem B.

Theorem 2.3. *Let I be an open real interval such that $[0,\infty)\subset I$. Let $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L^1([a,b])$, where $0\leq a<b<\infty$. If $|f'|^q$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$ and $q\in(1,\infty)$, then*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| &\leq\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right) \\
&\leq\frac{b-a}{4}\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right),
\end{aligned}\tag{2.4}$$

where

$$\mu_1=\min\left\{\frac{\left|f'\left(a\right)\right|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q}{2},\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right\},$$

$$\mu_2=\min\left\{\frac{\left|f'\left(b\right)\right|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q}{2},\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right\}.$$

Proof. If $|f'|^q$ is m-convex from Theorem A we have

$$2\int_{\frac{1}{2}}^{1} |f'(ta+(1-t)\,b)|^q \,\mathrm{d}t \le \mu_1,$$

$$2\int_{0}^{\frac{1}{2}} |f'(ta+(1-t)\,b)|^q \,\mathrm{d}t \le \mu_2,$$

hence

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t))|\,\mathrm{d}t \\
&= \frac{b-a}{2}\left[\int_0^{\frac{1}{2}} (1-2t)\,|f'(ta+(1-t)\,b)|\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (2t-1)\,|f'(ta+(1-t)\,b)|\,\mathrm{d}t\right].
\end{aligned}$$

Using Hölder's inequality for $q\in(1,\infty)$ and $p = q/\,(q-1)$ we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{2}\left[\left(\int_0^{\frac{1}{2}} (1-2t)^{\frac{q}{q-1}}\,\mathrm{d}t\right)^{\frac{q-1}{q}}\left(\int_0^{\frac{1}{2}} |f'(ta+(1-t)\,b)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\int_{\frac{1}{2}}^1 (2t-1)^{\frac{q}{q-1}}\,\mathrm{d}t\right)^{\frac{q-1}{q}}\left(\int_{\frac{1}{2}}^1 |f'(ta+(1-t)\,b)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&\le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}}\right),
\end{aligned}$$

since

$$\int_0^{\frac{1}{2}} (1-2t)^{\frac{q}{q-1}}\,\mathrm{d}t = \int_{\frac{1}{2}}^1 (2t-1)^{\frac{q}{q-1}}\,\mathrm{d}t = \frac{q-1}{2\,(2q-1)}.$$

This completes the proof of the first inequality in (2.4). The second inequality in (2.4) follows from the fact

$$\frac{1}{2} < \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} < 1, \qquad q\in(1,\infty).$$

□

Theorem 2.4. *Let $f, g : [0,\infty) \to [0,\infty)$ be such that fg is in $L^1([a,b])$, where $0 \le a < b < \infty$. If f is m_1-convex and g is m_2-convex on $[a,b]$ for some fixed $m_1, m_2 \in (0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le \min\{M_1, M_2\},$$

where

$$\begin{aligned}
M_1 = \frac{1}{3}&\left[f(a)\,g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right] \\
&\qquad + \frac{1}{6}\left[m_2 f(a)\,g\left(\frac{b}{m_2}\right) + m_1 f\left(\frac{b}{m_1}\right) g(a)\right],
\end{aligned}$$

$$M_2 = \frac{1}{3}\left[f(b)\,g(b) + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[m_2 f(b)\,g\left(\frac{a}{m_2}\right) + m_1 f\left(\frac{a}{m_1}\right) g(b)\right].$$

Proof. We have

$$f\left(ta + m_1(1-t)\frac{b}{m_1}\right) \le tf(a) + m_1(1-t)\,f\left(\frac{b}{m_1}\right),$$
$$g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \le tg(a) + m_2(1-t)\,g\left(\frac{b}{m_2}\right),$$

for all $t \in [0,1]$. f and g are nonnegative, hence

$$f\left(ta + m_1(1-t)\frac{b}{m_1}\right) g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \le t^2 f(a)\,g(a) + m_2 t(1-t)\,f(a)\,g\left(\frac{b}{m_2}\right) + m_1 t(1-t)\,f\left(\frac{b}{m_1}\right) g(a) + m_1 m_2 (1-t)^2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right).$$

Integrating both sides of the above inequality over $[0,1]$ we obtain

$$\int_0^1 f(ta+(1-t)\,b)g\,(ta+(1-t)\,b)\,\mathrm{d}t = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le \frac{1}{3}\left(f(a)\,g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right) + \frac{1}{6}\left(m_2 f(a)\,g\left(\frac{b}{m_2}\right) + m_1 f\left(\frac{b}{m_1}\right) g(a)\right).$$

Analogously we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le \frac{1}{3}\left(f(b)\,g(b) + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\right) + \frac{1}{6}\left(m_2 f(b)\,g\left(\frac{a}{m_2}\right) + m_1 f\left(\frac{a}{m_1}\right) g(b)\right),$$

hence

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le \min\{M_1, M_2\}.$$

□

Remark 1. If in Theorem 2.4 we choose a 1-convex (convex) function $g : [0,\infty) \to [0,\infty)$ defined by $g(x) = 1$ for all $x \in [0,\infty)$, we obtain

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\},$$

which is (1.1). If the functions f and g are 1-convex we obtain (1.3).

3. INEQUALITIES FOR (α, m)-CONVEX FUNCTIONS

In this section on two examples we ilustrate how the same inequalities as in Section 2 can be obtained for the class of (α, m)-convex functions.

Theorem 3.1. *Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L^1([a, b])$, where $0 \le a < b < \infty$. If $|f'|^q$ is (α, m)-convex on $[a, b]$ for some fixed $\alpha, m \in (0, 1]$ and $q \in [1, \infty)$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \cdot \min\left\{\left(\nu_1 |f'(a)|^q + \nu_2 m \left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}, \left(\nu_1 |f'(b)|^q + \nu_2 m \left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\},$$

where

$$\nu_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left[\alpha + \left(\frac{1}{2}\right)^{\alpha}\right],$$

$$\nu_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{\alpha^2+\alpha+2}{2} - \left(\frac{1}{2}\right)^{\alpha}\right].$$

Proof. Suppose that $q = 1$. From Lemma A we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\int_0^1 |1-2t|\, |f'(ta+(1-t)b)|\, \mathrm{d}t. \tag{3.1}$$

Since $|f'|$ is (α, m)-convex on $[a, b]$ we know that for any $t \in [0, 1]$

$$\left|f'\left(ta + m(1-t)\frac{b}{m}\right)\right| \le t^{\alpha}|f'(a)| + m(1-t^{\alpha})\left|f'\left(\frac{b}{m}\right)\right|,$$

thus we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{2}\int_0^1 |1-2t|\left[t^{\alpha}|f'(a)| + m(1-t^{\alpha})\left|f'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t \\
&= \frac{b-a}{2}\int_0^1 \left[t^{\alpha}|1-2t|\,|f'(a)| + m(1-t^{\alpha})|1-2t|\left|f'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t.
\end{aligned}$$

We have

$$\int_0^1 t^{\alpha}|1-2t|\,\mathrm{d}t = \frac{1}{(\alpha+1)(\alpha+2)}\left[\alpha + \left(\frac{1}{2}\right)^{\alpha}\right] = \nu_1,$$

$$\int_0^1 (1-t^{\alpha})|1-2t|\,\mathrm{d}t = \frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{\alpha^2+\alpha+2}{2} - \left(\frac{1}{2}\right)^{\alpha}\right] = \nu_2,$$

hence

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2}\left(\nu_1|f'(a)| + \nu_2 m\left|f'\left(\frac{b}{m}\right)\right|\right).$$

Analogously we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{b-a}{2}\left(\nu_1\left|f'(b)\right|+\nu_2 m\left|f'\left(\frac{a}{m}\right)\right|\right),$$

which completes the proof for this case.

Suppose now that $q\in(1,\infty)$. Similarly to Theorem 2.1 we have

$$\int_0^1 |1-2t|\left|f'(ta+(1-t)\,b)\right|\mathrm{d}t \leq \left(\int_0^1 |1-2t|\,\mathrm{d}t\right)^{\frac{q-1}{q}}\left(\int_0^1 |1-2t|\left|f'(ta+(1-t)\,b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}. \tag{3.2}$$

Since $|f'|^q$ is (α,m)-convex on $[a,b]$ we know that for every $t\in[0,1]$

$$\left|f'\left(ta+m\,(1-t)\,\frac{b}{m}\right)\right|^q \leq t^\alpha\left|f'(a)\right|^q+m\,(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|^q, \tag{3.3}$$

hence from (3.1) , (3.2) and (3.3) we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\,\mathrm{d}t\right)^{\frac{q-1}{q}}\left(\int_0^1 |1-2t|\left|f'\left(ta+m\,(1-t)\,\frac{b}{m}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\ &\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\nu_1\left|f'(a)\right|^q+\nu_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\end{aligned}$$

and analogously

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\nu_1\left|f'(b)\right|^q+\nu_2 m\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}},$$

which completes the proof. □

Observe that if in Theorem 3.1 we have $\alpha=1$ the statement of Theorem 3.1 becomes the statement of Theorem 2.1.

Theorem 3.2. *Let* $f,g:[0,\infty)\to[0,\infty)$ *be such that* fg *is in* $L^1([a,b])$, *where* $0\leq a<b<\infty$. *If* f *is* (α_1,m_1)*-convex and* g *is* (α_2,m_2)*-convex on* $[a,b]$ *for some fixed* $\alpha_1,m_1,\alpha_2,m_2\in(0,1]$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g\,(x)\,\mathrm{d}x\leq\min\{N_1,N_2\},$$

where

$$\begin{aligned}N_1=\frac{f\,(a)\,g\,(a)}{\alpha_1+\alpha_2+1}&+m_2\left[\frac{1}{\alpha_1+1}-\frac{1}{\alpha_1+\alpha_2+1}\right]f\,(a)\,g\left(\frac{b}{m_2}\right)\\ &+m_1\left[\frac{1}{\alpha_2+1}-\frac{1}{\alpha_1+\alpha_2+1}\right]f\left(\frac{b}{m_1}\right)g(a)\\ &+m_1m_2\left[1-\frac{1}{\alpha_1+1}-\frac{1}{\alpha_2+1}+\frac{1}{\alpha_1+\alpha_2+1}\right]f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right),\end{aligned}$$

and

$$N_2 = \frac{f(b)\,g(b)}{\alpha_1+\alpha_2+1} + m_2\left[\frac{1}{\alpha_1+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f(b)\, g\left(\frac{a}{m_2}\right) + m_1\left[\frac{1}{\alpha_2+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{a}{m_1}\right) g(b) + m_1 m_2\left[1 - \frac{1}{\alpha_1+1} - \frac{1}{\alpha_2+1} + \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).$$

Proof. Since f is (α_1, m_1)-convex and g is (α_2, m_2)-convex on $[a, b]$ we have

$$f\left(ta + m_1(1-t)\frac{b}{m_1}\right) \le t^{\alpha_1} f(a) + m_1(1-t^{\alpha_1}) f\left(\frac{b}{m_1}\right),$$

$$g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \le t^{\alpha_2} g(a) + m_2(1-t^{\alpha_2}) g\left(\frac{b}{m_2}\right),$$

for all $t \in [0,1]$. The functions f and g are nonnegative, hence

$$f(ta+(1-t)b)g(ta+(1-t)b) \le t^{\alpha_1+\alpha_2} f(a)\,g(a) + m_2 t^{\alpha_1}(1-t^{\alpha_2}) f(a)\, g\left(\frac{b}{m_2}\right) + m_1 t^{\alpha_2}(1-t^{\alpha_1}) f\left(\frac{b}{m_1}\right) g(a) + m_1 m_2 (1-t^{\alpha_1})(1-t^{\alpha_2}) f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right).$$

Integrating both sides of the above inequality over $[0,1]$ we obtain

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\,\mathrm{d}t \\
&= \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \\
&\le \frac{f(a)\,g(a)}{\alpha_1+\alpha_2+1} + m_2\left[\frac{1}{\alpha_1+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f(a)\, g\left(\frac{b}{m_2}\right) \\
&\qquad + m_1\left[\frac{1}{\alpha_2+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{b}{m_1}\right) g(a) \\
&\qquad + m_1 m_2\left[1 - \frac{1}{\alpha_1+1} - \frac{1}{\alpha_2+1} + \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right).
\end{aligned}$$

Analogously we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \\
&\le \frac{f(b)\,g(b)}{\alpha_1+\alpha_2+1} + m_2\left[\frac{1}{\alpha_1+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f(b)\, g\left(\frac{a}{m_2}\right) \\
&\qquad + m_1\left[\frac{1}{\alpha_2+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{a}{m_1}\right) g(b) \\
&\qquad + m_1 m_2\left[1 - \frac{1}{\alpha_1+1} - \frac{1}{\alpha_2+1} + \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right),
\end{aligned}$$

which completes the proof. □

If in Theorem 3.2 we have $\alpha_1 = \alpha_2 = 1$, the statement of Theorem 3.2 becomes the statement of Theorem 2.4.

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