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SOME INEQUALITIES ASSOCIATED WITH A LINEAR OPERATOR DEFINED FOR A CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper, we give a sufficient condition on a linear operator $L_p(a,c)g(z)$ which can guarantee that for α a complex number with $\mathrm{Re}(\alpha) > 0$,

$$\mathrm{Re}\left\{(1-\alpha)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \alpha\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \rho, \quad \rho < 1,$$

in the unit disk E, implies

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \rho' > \rho, \quad z \in E.$$

Some interesting applications of this result are also given.

Key words and phrases: Analytic functions, Differential subordination, Ruscheweyh derivatives, Linear operator.

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1. INTRODUCTION

Let $A(p,n)$ denote the class functions f normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \qquad (p,n \in \mathbb{N} = \{1,2,3,...\}), \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : z \in C, |z| < 1\}$.

In particular, we set $A(p,1) = A_p$ and $A(1,1) = A_1 = A$.

The Hadamard product $(f * g)(z)$ of two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \qquad (p,n \in \mathbb{N}),$$

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is defined, as usual, by

$$(f*g)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k = (g*f)(z).$$

The Ruscheweyh derivative of $f(z)$ of order $\delta + p - 1$ is defined by

$$D^{\delta+p-1} f(z) = \frac{z^p}{(1-z)^{\delta+p}} * f(z) \qquad (f \in A(p,n); \delta \in \mathbb{R} \setminus (-\infty, -p]) \tag{1.2}$$

or, equivalently, by

$$D^{\delta+p-1} f(z) = z^p + \sum_{k=p+n}^{\infty} \binom{\delta+k-1}{k-p} a_k z^k, \tag{1.3}$$

where $f(z) \in A(p,n)$ and $\delta \in \mathbb{R} \setminus (-\infty, -p]$. In particular, if $\delta = l \in \mathbb{N} \bigcup \{0\}$, we find from (1.2) or (1.3) that

$$D^{l+p-1} f(z) = \frac{z^p}{(l+p-1)!} \frac{d^{l+k-1}}{dz^{l+p-1}} \left\{ z^{l-1} f(z) \right\}.$$

The author has proved the following result in [4].

Theorem A. *Let* α *be a complex number satisfying* $\mathrm{Re}(\alpha) > 0$ *and* $\rho < 1$. *Let* $\delta > -p, f, g \in A_p$ *and*

$$\mathrm{Re}\left\{ \alpha \frac{D^{\delta+p-1} g(z)}{D^{\delta+p} g(z)} \right\} > \gamma, \quad 0 \le \gamma < \mathrm{Re}(\alpha),\ z \in E.$$

Then

$$\mathrm{Re}\left\{ \frac{D^{\delta+p-1} f(z)}{D^{\delta+p-1} g(z)} \right\} > \frac{2\rho(\delta+p)+\gamma}{2(\delta+p)+\gamma}, \qquad z \in E,$$

whenever

$$\mathrm{Re}\left\{ (1-\alpha) \frac{D^{\delta+p-1} f(z)}{D^{\delta+p-1} g(z)} + \alpha \frac{D^{\delta+p} f(z)}{D^{\delta+p} g(z)} \right\} > \rho, z \in E.$$

The Pochhammer symbol $(\lambda)_k$ or the shifted factorial is given by $(\lambda)_0 = 1$ and $(\lambda)_k = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1)$, $k \in \mathbb{N}$. In terms of $(\lambda)_k$, we now define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p}, \qquad z \in E,$$

where $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus z_0^-$; $z_0^- = \{0, -1, -2, \dots\}$.

Saitoh [3] introduced a linear operator $L_P(a, c)$, which is defined by

$$L_p(a, c) f(z) = \phi_p(a, c, ; z) * f(z), \qquad z \in E, \tag{1.4}$$

or, equivalently by

$$L_p(a, c) f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p} z^{k+p}, \qquad z \in E, \tag{1.5}$$

where $f(z) \in A_p, a \in \mathbb{R}, c \in \mathbb{R} \setminus z_0^-$.

For $f(z) \in A(p,n)$ and $\delta \in \mathbb{R} \setminus (-\infty, -p]$, we obtain

$$L_p(\delta + p, 1) f(z) = D^{\delta+p-1} f(z), \tag{1.6}$$

which can easily be verified by comparing the definitions (1.3) and (1.5).

The main object of this paper is to present an extension of Theorem A to hold true for a linear operator $L_P(a, c)$ associated with the class $A(p, n)$.

The basic tool in proving our result is the following lemma.

Lemma 1.1 (cf. Miller and Mocanu [2, p. 35, Theorem 2.3 i(i)]). *Let Ω be a set in the complex plane C. Suppose that the function $\Psi : C^2 \times E \longrightarrow C$ satisfies the condition $\Psi(ix_2, y_1; z) \notin \Omega$ for all $z \in E$ and for all real x_2 and y_1 such that*

$$y_1 \leq -\frac{1}{2}n(1+x_2^2). \tag{1.7}$$

If $p(z) = 1 + c_n z^n + \cdots$ is analytic in E and for $z \in E$, $\Psi(p(z), zp'(z); z) \subset \Omega$, then $\mathrm{Re}(p(z)) > 0$ in E.

2. MAIN RESULTS

Theorem 2.1. *Let α be a complex number satisfying $\mathrm{Re}(\alpha) > 0$ and $\rho < 1$. Let $a > 0, f, g \in A(p, n)$ and*

$$\mathrm{Re}\left\{\alpha \frac{L_p(a,c)g(z)}{L_p(a+1,c)g(z)}\right\} > \gamma, \qquad 0 \leq \gamma < \mathrm{Re}(\alpha), \quad z \in E. \tag{2.1}$$

Then

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \frac{2a\rho + n\gamma}{2a+n\gamma}, \quad z \in E,$$

whenever

$$\mathrm{Re}\left\{(1-\alpha)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \alpha \frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \rho, \quad z \in E. \tag{2.2}$$

Proof. Let $\tau = (2a\rho + n\gamma)/(2a + n\gamma)$ and define the function $p(z)$ by

$$p(z) = (1-\tau)^{-1}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} - \tau\right\}. \tag{2.3}$$

Then, clearly, $p(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \cdots$ and is analytic in E. We set $u(z) = \alpha L_p(a,c)g(z)/L_p(a+1,c)g(z)$ and observe from (2.1) that $\mathrm{Re}(u(z)) > \gamma$, $z \in E$. Making use of the familiar identity

$$z(L_p(a,c)f(z))' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z),$$

we find from (2.3) that

$$(1-\alpha)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \alpha\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} = \tau + (1-\tau)\left[p(z) + \frac{u(z)}{a}zp'(z)\right]. \tag{2.4}$$

If we define $\Psi(x, y; z)$ by

$$\Psi(x,y;z) = \tau + (1-\tau)\left(x + \frac{u(z)}{a}y\right), \tag{2.5}$$

then, we obtain from (2.2) and (2.4) that

$$\{\Psi(p(z), zp'(z); z) : |z| < 1\} \subset \Omega = \{w \in C : \mathrm{Re}(w) > \rho\}.$$

Now for all $z \in E$ and for all real x_2 and y_1 constrained by the inequality (1.7), we find from (2.5) that

$$\begin{aligned}\mathrm{Re}\{\Psi(ix_2, y_1; z)\} &= \tau + \frac{(1-\tau)}{a}y_1 \,\mathrm{Re}(u(z)) \\ &\leq \tau - \frac{(1-\tau)n\gamma}{2a} \equiv \rho.\end{aligned}$$

Hence $\Psi(ix_2, y_1; z) \notin \Omega$. Thus by Lemma 1.1, $\mathrm{Re}(p(z)) > 0$ and hence $\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \tau$ in E. This proves our theorem. □

Remark 2.2. Theorem A is a special case of Theorem 2.1 obtained by taking $a = \delta + p$ and $c = n = 1$, which reduces to Theorem 2.1 of [1], when $p = 1$.

Corollary 2.3. *Let α be a real number with $\alpha \geq 1$ and $\rho < 1$. Let $a > 0$, $f, g \in A(p,n)$ and*

$$\mathrm{Re}\left\{\frac{L_p(a,c)g(z)}{L_p(a+1,c)g(z)}\right\} > \gamma, \quad 0 \leq \gamma < 1, \; z \in E.$$

Then

$$\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \frac{\alpha(2a\rho + n\gamma) - (1-\rho)n\gamma}{\alpha(2a+n\gamma)}, \quad z \in E,$$

whenever

$$\mathrm{Re}\left\{(1-\alpha)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \alpha\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \rho, \quad z \in E.$$

Proof. Proof follows from Theorem 2.1 (Since $\alpha \geq 1$). □

In its special case when $\alpha = 1$, Theorem 2.1 yields:

Corollary 2.4. *Let $a > 0, f, g \in A(p,n)$ and $\mathrm{Re}\left\{\frac{L_p(a,c)g(z)}{L_p(a+1,c)g(z)}\right\} > \gamma, 0 \leq \gamma < 1$, then for $\rho < 1$,*

$$\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \rho, \quad z \in E,$$

implies

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \frac{2a\rho + n\gamma}{2a+n\gamma}, \quad z \in E.$$

If we set

$$v(z) = \frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} - \left(\frac{1}{\alpha} - 1\right)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)},$$

then for $a > 0, \alpha > 0$ and $\rho = 0$, Theorem 2.1 reduces to

$$\mathrm{Re}(v(z)) > 0, \quad z \in E$$

implies

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \frac{n\alpha\gamma}{2a+n\alpha\gamma}, \qquad z \in E, \tag{2.6}$$

whenever $\mathrm{Re}(L_p(a,c)g(z)/L_p(a+1,c)g(z)) > \gamma$, $0 \leq \gamma < 1$. Let $\alpha \to \infty$.

Then (2.6) is equivalent to

$$\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} - \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > 0 \text{ in } E$$

implies

$$\mathrm{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > 1 \text{ in } E,$$

whenever $\mathrm{Re}(L_p(a,c)g(z)/L_p(a+1,c)g(z)) > \gamma$, $0 \leq \gamma < 1$.

In the following theorem we shall extend the above result, the proof of which is similar to that of Theorem 2.1.

Theorem 2.5. *Let* $a > 0$, $\rho < 1$, $f, g \in A(p,n)$ *and* $\operatorname{Re}\left\{\frac{l_p(a,c)g(z)}{L_p(a+1,c)g(z))}\right\} > \gamma$, $0 \le \gamma < 1$.
If

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} - \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > -\frac{n\gamma(1-\rho)}{2a}, \qquad z \in E,$$

then

$$\operatorname{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \rho, \qquad z \in E,$$

and

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \frac{\rho(2a+n\gamma)-n\gamma}{2a}, \qquad z \in E.$$

Using Theorem 2.1 and Theorem 2.5, we can generalize and improve several other interesting results available in the literature by taking $g(z) = z^p$. We illustrate a few in the following theorem.

Theorem 2.6. *Let* $a > 0$, $\rho < 1$ *and* $f(z) \in A(p,n)$. *Then*

(a) *for* α *a complex number satisfying* $\operatorname{Re}(\alpha) > 0$, *we have*

$$\operatorname{Re}\left\{(1-\alpha)\frac{L_p(a,c)f(z)}{z^p} + \alpha\frac{L_p(a+1,c)f(z)}{z^p}\right\} > \rho, \qquad z \in E,$$

implies

$$\operatorname{Re}\left\{\frac{L_p(a,c)f(z)}{z^p}\right\} > \frac{2a\rho + n\operatorname{Re}(\alpha)}{2a + n\operatorname{Re}(\alpha)}, \qquad z \in E.$$

(b) *for* α *real and* $\alpha \ge 1$, *we have*

$$\operatorname{Re}\left\{(1-\alpha)\frac{L_p(a,c)f(z)}{z^p} + \alpha\frac{L_p(a+1,c)f(z)}{z^p}\right\} > \rho, \qquad \textit{in } E$$

implies

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p}\right\} > \frac{(2a+n)\rho + n(\alpha-1)}{2a+n\alpha} \qquad \textit{in } E$$

(c) *for* $z \in E$,

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p} - \frac{L_p(a,c)f(z)}{z^p}\right\} > -\frac{n(1-\rho)}{2a}$$

implies

$$\operatorname{Re}\left\{\frac{L_p(a+1,c)f(z)}{z^p}\right\} > \frac{(2a+n)\rho - n}{2a}.$$

Remark 2.7. By taking $a = \delta + p, c = n = 1$ in Theorem 2.6 we obtain Theorem 1.6 of the author [4], which when $p = 1$ reduces to Theorem 2.3 of Bhoosnurmath and Swamy [1].

In a manner similar to Theorem 2.1, we can easily prove the following, which when $r = 1$ reduces to part (a) of Theorem 2.6.

Theorem 2.8. *Let* $a > 0$, $r > 0$, $\rho < 1$ *and* $f(z) \in A(p, n)$.*Then for* α *a complex number with* $\mathrm{Re}(\alpha) > 0$, *we have*

$$\mathrm{Re}\left\{\left(\frac{L_p(a,c)f(z)}{z^p}\right)^r\right\} > \frac{2a\rho r + n\,\mathrm{Re}(\alpha)}{2ar + n\,\mathrm{Re}(\alpha)}, \qquad z \in E,$$

whenever

$$\mathrm{Re}\left\{(1-\alpha)\left(\frac{L_p(a,c)f(z)}{z^p}\right)^r + \alpha\left(\frac{L_p(a+1,c)f(z)}{z^p}\right)\left(\frac{L_p(a,c)f(z)}{z^p}\right)^{r-1}\right\} > \rho,$$

$z \in E$.

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