AN UPPER BOUND FOR THE DETERMINANT OF A MATRIX WITH GIVEN ENTRY SUM AND SQUARE SUM

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Abstract: By deducing characterisations of the matrices which have maximal determinant in the set of matrices with given entry sum and square sum, we prove the inequality $|\det M| \leq |\alpha|(\beta - \delta)^{(n-1)/2}$ for real $n \times n$-matrices M, where $n\alpha$ and $n\beta$ are the sum of the entries and the sum of the squared entries of M, respectively, and $\delta := (\alpha^2 - \beta)/(n-1)$, provided that $\alpha^2 \geq \beta$. This result is applied to find an upper bound for the determinant of a matrix whose entries are a permutation of an arithmetic progression.

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1. Introduction

Let $n \geq 2$ be a positive integer and $a = (a_1, \dots, a_{n^2})$ a vector of real numbers. What is the maximal determinant $D(a)$ of a matrix whose elements are a permutation of the entries of a? The answer is unknown even for the special case $a := (1, \dots, n^2)$ if $n > 6$, see [4]. By computational optimisation using algorithms like tabu search, we have found matrices with the following determinants, which thus are lower bounds for $D(1, \dots, n^2)$:

n	lower bound for $D(1, \dots, n^2)$
2	10
3	412
4	40 800
5	6 839 492
6	1 865 999 570
7	762 150 368 499
8	440 960 274 696 935
9	346 254 605 664 223 620
10	356 944 784 622 927 045 792

It would be nice to also have a good upper bound for $D(1, \dots, n^2)$. We will show how to find an upper bound by treating the problem of determining $D(a)$ as a continuous optimisation task. This is done by maximising the determinant under two equality contraints: by fixing the sum and the square sum of the matrix's entries.

Our result is a characterisation of the matrices with maximal determinant in the set of matrices with given entry sum and square sum, and a general inequality for the absolute value of the determinant of a matrix.

For the problem of finding $D(1, \dots, n^2)$, the upper bound derived in this way turns out to be quite sharp. So here we have an example where analytical optimisation gives valuable information about a combinatorial optimisation problem.

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2. Conventions

Throughout this article, let $n > 1$ be a natural number and $N := \{1, \ldots, n\}$. *Matrix* always means a real $n \times n$ matrix, the set of which we denote by $\mathbb{M}$.

For $M \in \mathbb{M}$ and $i, j \in N$ we denote by M_i the i-th row of M, by M^j the j-th column of M, and by $M_{i,j}$ the entry of M at position (i, j). If M is a matrix or a row or a column of a matrix, then by $s(M)$ we denote the sum of the entries of M and by $q(M)$ the sum of their squares.

The identity matrix is denoted by I. By J we name the matrix which has 1 at all of its fields, while e is the column vector in $\mathbb{R}^n$ with all entries being 1. Matrices of the structure $xI + yJ$ will play an important role, so we state some of their properties:

Lemma 2.1. *Let $x, y \in \mathbb{R}$ and $M := xI + yJ$. Then we have:*

1. $\det M = x^{n-1}(x + ny)$
2. *M is invertible if and only if $x \notin \{0, -ny\}$.*
3. *If M is invertible, then $M^{-1} = \frac{1}{x} I - \frac{y}{x(x+ny)} J$.*

Proof. Since $J = ee^T$, it holds that

$$Me = (xI + yee^T)e = (x + ye^T e)e = (x + ny)e \quad \text{and} \quad Mv = (xI + yee^T)v = xv$$

for all $v \in \mathbb{R}^n$ with $v \perp e$. Hence M has the eigenvalue x with multiplicity $n - 1$ and the simple eigenvalue $x + ny$. This shows (1). (2) is an immediate consequence of (1). (3) can be verified by a straight calculation. $\square$

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3. Main Theorem

Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$ and $\mathbb{M}_{\alpha,\beta} := \{M \in \mathbb{M} : s(M) = n\alpha,\ q(M) = n\beta\}$. Furthermore, let

$$\delta := \frac{\alpha^2 - \beta}{n-1}.$$

In the proof of the following lemma, matrices are specified whose determinants will later turn out to be the greatest possible:

Lemma 3.1.

1. *$\mathbb{M}_{\alpha,\beta} \neq \emptyset$ if and only if $\alpha^2 \leq n\beta$. If $\alpha^2 \leq n\beta$, then there exists an $M \in \mathbb{M}_{\alpha,\beta}$ with*
$$\det M = \alpha(\beta - \delta)^{\frac{n-1}{2}}.$$
2. *If $\alpha^2 \leq \beta$, then there exists an $M \in \mathbb{M}_{\alpha,\beta}$ with $\det M = \beta^{\frac{n}{2}}$.*
3. *There exists an $M \in \mathbb{M}_{\alpha,\beta}$ with $\det M \neq 0$ if and only if $\alpha^2 < n\beta$.*

Proof. (1) Suppose $\mathbb{M}_{\alpha,\beta} \neq \emptyset$, say $M \in \mathbb{M}_{\alpha,\beta}$. Reading M and J as elements of $\mathbb{R}^{n^2}$, the Cauchy inequality shows that

$$\begin{aligned}
\alpha^2 &= \frac{1}{n^2}\left(\sum_{i,j=1}^{n} M_{i,j}\right)^2 \\
&= \frac{1}{n^2}\langle M, J\rangle^2 \\
&\leq \frac{1}{n^2}\|M\|_2^2\,\|J\|_2^2 = \sum_{i,j=1}^{n} M_{i,j}^2 = n\beta.
\end{aligned}$$

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For the other implication suppose $\alpha^2 \leq n\beta$, i. e. $\beta \geq \delta$, and set $\gamma := (\beta - \delta)^{\frac{1}{2}}$ and $M := \gamma I + \frac{1}{n}(\alpha - \gamma)J$. Then $M \in \mathbb{M}_{\alpha,\beta}$, and by Lemma 2.1

$$\det M = \gamma^{n-1}\left(\gamma + n\tfrac{1}{n}(\alpha - \gamma)\right) = \gamma^{n-1}\alpha = \alpha(\beta - \delta)^{\frac{n-1}{2}}.$$

(2) Let $\alpha^2 \leq \beta$. First suppose $\alpha \geq 0$, so $\gamma := \frac{1}{2}\left(\frac{3\alpha}{\sqrt{\beta}} - 1\right)$ gives $\gamma^2 \leq 1$. Set

$$A := \begin{pmatrix} \alpha & \sqrt{\beta - \alpha^2} \\ -\sqrt{\beta - \alpha^2} & \alpha \end{pmatrix} \quad \text{and} \quad B := \sqrt{\beta}\begin{pmatrix} \gamma & \sqrt{1-\gamma^2} & 0 \\ -\sqrt{1-\gamma^2} & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $s(A) = 2\alpha$, $q(A) = 2\beta$, $\det A = \beta$, $s(B) = 3\alpha$, $q(B) = 3\beta$, $\det B = \beta^{\frac{3}{2}}$. In the case of $n = 2k$ with $k \in \mathbb{N}$, use k copies of A to build the block matrix

$$M := \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix},$$

which has the required properties. In the case of $n = 2k + 1$ with $k \in \mathbb{N}$, use $k - 1$ copies of A to build the block matrix

$$M := \begin{pmatrix} A & & & \\ & \ddots & & \\ & & A & \\ & & & B \end{pmatrix},$$

which again fulfills the requirements.

In the case of $\alpha < 0$, an $M' \in \mathbb{M}_{-\alpha,\beta}$ with $\det M' = \beta^{\frac{n}{2}}$ exists. For even n, the matrix $M := -M' \in \mathbb{M}_{\alpha,\beta}$ has the requested determinant, while for odd n swapping two rows of $-M'$ gives the desired matrix M.

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(3) If $\alpha^2 < n\beta$, then the existence of an $M \in \mathbb{M}_{\alpha,\beta}$ with $\det M \neq 0$ is proved by (1) in the case of $\alpha \neq 0$ and by (2) in the case of $\alpha = 0$. For $\alpha^2 = n\beta$ and $M \in \mathbb{M}_{\alpha,\beta}$, the calculation in (1) shows that $\langle M, J \rangle = \|M\|_2 \|J\|_2$. However, this equality holds only if M is a scalar multiple of J, so we have $\det M = 0$ because of $\det J = 0$. □

For $\alpha^2 \leq \beta$ we have given two types of matrices in Lemma 3.1, the first one having the determinant $\alpha(\beta - \delta)^{\frac{n-1}{2}}$, the second one with the determinant $\beta^{\frac{n}{2}}$. The proof of Theorem 3.3 below will use the fact that for $\alpha^2 < \beta$ the determinant of the first type is strictly smaller than that of the second type. Indeed, the following stronger statement holds:

Lemma 3.2. *Let $\alpha^2 \leq n\beta$. Then $|\alpha|(\beta - \delta)^{\frac{n-1}{2}} \leq \beta^{\frac{n}{2}}$ with equality if and only if $\alpha^2 = \beta$.*

Proof. This is obvious for $\alpha = 0$, so let $\alpha \neq 0$. With $f(x) := x\left(\frac{n-x}{n-1}\right)^{n-1}$ for $x \in [0, n]$ we have

$$|\alpha|(\beta - \delta)^{\frac{n-1}{2}} \beta^{-\frac{n}{2}} = \sqrt{f\left(\frac{\alpha^2}{\beta}\right)}.$$

The proof is completed by applying the AM-GM inequality to $f(x)^{1/n}$:

$$f(x)^{\frac{1}{n}} = \left(x \left(\frac{n-x}{n-1} \right)^{n-1} \right)^{\frac{1}{n}} \leq \frac{x + (n-1)\frac{n-x}{n-1}}{n} = 1$$

with equality if and only if $x = \frac{n-x}{n-1}$, i. e. if and only if $x = 1$. □

If $\alpha^2 < n\beta$, then by Lemma 3.1 there exists an $M \in \mathbb{M}_{\alpha,\beta}$ with $\det M \neq 0$, and, by possibly swapping two rows of M, $\det M > 0$ can be achieved. As $\mathbb{M}_{\alpha,\beta}$ is compact, the determinant function assumes a maximum value on $\mathbb{M}_{\alpha,\beta}$. The next

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theorem, which is essentially due to O. Gasper, shows that this maximum value is given by the determinants noted in Lemma 3.1:

Theorem 3.3. *Let $\alpha^2 < n\beta$ and $M \in \mathbb{M}_{\alpha,\beta}$ with maximal determinant. Then*

$$\text{if } \alpha^2 \leq \beta: \begin{cases} (1) & MM^T = \beta I \\ (2) & \det M = \beta^{\frac{n}{2}} \end{cases}$$

$$\text{if } \alpha^2 \geq \beta: \begin{cases} (3) & s(M_i) = s(M^j) = \alpha \text{ for all } i, j \in N \\ (4) & MM^T = (\beta - \delta)I + \delta J \\ (5) & \det M = |\alpha|(\beta - \delta)^{\frac{n-1}{2}} \end{cases}$$

Proof. From Lemma 3.1, we know that $\det M > 0$. The matrix M solves an extremum problem with equality contraints

$$\text{(P)} \qquad \begin{cases} \det X \longrightarrow \max \\ s(X) = n\alpha \\ q(X) = n\beta \end{cases} \qquad (X \in \mathbb{M}^*),$$

where $\mathbb{M}^*$ is the set of invertible matrices. The Lagrange function of (P) is given by

$$L(X, \lambda, \mu) = \det X - \lambda(s(X) - n\alpha) - \mu(q(X) - n\beta),$$

so there exist $\lambda, \mu \in \mathbb{R}$ with $\frac{d}{dM_{i,j}} L(M, \lambda, \mu) = 0$ for all $i, j \in N$. It is well known that

$$\left(\frac{d}{dM_{i,j}} \det M\right)_{i,j} = (\det M)\left(M^T\right)^{-1}$$

(see e. g. [1], 10.6), thus we get $(\det M)\left(M^T\right)^{-1} - \lambda M - 2\mu J = 0$, i. e.

$$(3.1) \qquad (\det M) I = \lambda M M^T + 2\mu J M^T.$$

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Suppose $\lambda = 0$. Then

$$(\det M)^n = \det(2\mu J M^T) = \det(2\mu J)\det M = 0 \det M = 0$$

by applying the determinant function to (3.1). This contradicts $\det M > 0$. Hence

$$\lambda \neq 0. \tag{3.2}$$

As MM^T has diagonal elements $q(M_1), \dots, q(M_n)$, and JM^T has diagonal elements $s(M_1), \dots, s(M_n)$, we get

$$n \det M = \lambda q(M) + 2\mu s(M) = \lambda n\beta + 2\mu n\alpha$$

by applying the trace function to (3.1), consequently

$$\det M = \lambda\beta + 2\mu\alpha. \tag{3.3}$$

The symmetry of $(\det M)I$ and the symmetry of λMM^T in (3.1) show that μJM^T is symmetric as well. As all rows of JM^T are identical, namely equal to $(s(M_1), \dots, s(M_n))$, we obtain

$$\mu s(M_1) = \cdots = \mu s(M_n). \tag{3.4}$$

In the following, we inspect the cases $\mu = 0$ and $\mu \neq 0$ and prove:

$$\begin{cases} \mu = 0 & \Longrightarrow \quad \alpha^2 \leq \beta \ \wedge \ (1) \ \wedge \ (2), \\ \mu \neq 0 & \Longrightarrow \quad \alpha^2 \geq \beta \ \wedge \ (3) \ \wedge \ (4) \ \wedge \ (5). \end{cases} \tag{3.5}$$

Case $\mu = 0$: Then (3.3) reads $\det M = \lambda\beta$, so taking (3.2) into account and dividing (3.1) by λ gives $\beta I = MM^T$, i. e. (1). Part (2) follows by applying the determinant

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function to (1). Using the Cauchy inequality and the fact that $(1/\sqrt{\beta})\,M$ is orthogonal and thus an isometry w.r.t. the euclidean norm $\|\cdot\|_2$, we get:

$$\begin{aligned}
\alpha^2 &= \frac{1}{n^2}\left(\sum_{i=1}^{n} s(M_i)\right)^2 \\
&\le \frac{1}{n^2}\, n \sum_{i=1}^{n} s(M_i)^2 \\
&= \frac{1}{n}\|Me\|_2^2 = \frac{1}{n}\beta\|e\|_2^2 = \frac{1}{n}\beta n = \beta.
\end{aligned} \tag{3.6}$$

Case $\mu \neq 0$: Then $s(M_1) = \cdots = s(M_n)$ by (3.4). The identity

$$s(M_1) + \cdots + s(M_n) = s(M) = n\alpha$$

shows that $s(M_i) = \alpha$ for all $i \in N$. Taking into account that the determinant is invariant against matrix transposition, this proves (3). Furthermore, $JM^T = \alpha J$, and (3.1) becomes

$$\lambda MM^T = (\det M)I - 2\mu\alpha J, \tag{3.7}$$

hence

$$q(M_i) = (MM^T)_{i,i} = \frac{1}{\lambda}(\det M - 2\mu\alpha)$$

for all $i \in N$, and $q(M_1) = \cdots = q(M_n)$. With

$$q(M_1) + \cdots + q(M_n) = q(M) = n\beta,$$

this shows that

$$(MM^T)_{i,i} = q(M_i) = \beta \quad \text{for all } i \in N. \tag{3.8}$$

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Let $i, j \in N$ with $i \neq j$. Equation (3.7) gives $(MM^T)_{i,k} = -\frac{1}{\lambda} 2\mu\alpha$ for all $k \in N \setminus \{i\}$, and we get

$$\begin{aligned}\beta + (n-1)(MM^T)_{i,j} &= (MM^T)_{i,i} + \sum_{k \neq i} (MM^T)_{i,k} \\ &= \sum_{k=1}^{n} (MM^T)_{i,k} \\ &= \sum_{k=1}^{n} \sum_{p=1}^{n} M_{i,p} M_{k,p} \\ &= \sum_{p=1}^{n} M_{i,p}\, s(M^p) \\ &= \sum_{p=1}^{n} M_{i,p}\, \alpha = s(M_i)\, \alpha = \alpha^2,\end{aligned}$$

so

$$(MM^T)_{i,j} = \frac{\alpha^2 - \beta}{n-1} = \delta. \tag{3.9}$$

Equations (3.8) and (3.9) together prove (4). With Lemma 2.1, this yields

$$(\det M)^2 = \det(MM^T) = (\beta - \delta)^{n-1}(\beta - \delta + n\delta) = \alpha^2(\beta - \delta)^{n-1},$$

and taking the square root gives (5). Suppose that $\alpha^2 < \beta$. Then by Lemma 3.1 there exists an $M' \in \mathbb{M}_{\alpha,\beta}$ with $\det M' = \beta^{\frac{n}{2}}$, and by Lemma 3.2,

$$\det M = |\alpha|(\beta - \delta)^{\frac{n-1}{2}} < \beta^{\frac{n}{2}} = \det M',$$

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which contradicts the maximality of $\det M$. Hence $\alpha^2 \geq \beta$.

We have now proved (3.5) and are ready to deduce the statements of the theorem: If $\alpha^2 < \beta$, then (3.5) shows that $\mu = 0$ and thus (1) and (2). If $\alpha^2 > \beta$, then (3.5) shows that $\mu \neq 0$ and thus (3), (4) and (5). Finally suppose that $\alpha^2 = \beta$. Then $\delta = 0$, hence (1) $\Longleftrightarrow$ (4) and (2) $\Longleftrightarrow$ (5). If $\mu \neq 0$, then (3.5) shows (3), (4) and (5), from which (1) and (2) follow. If $\mu = 0$, then (3.5) shows (1) and (2), from which (4) and (5) follow. It remains to prove (3) in the case of $\alpha^2 = \beta$ and $\mu = 0$. To this purpose, look at (3.6) again, where $\alpha^2 = \beta$ means equality in the Cauchy inequality, which tells us that $(s(M_1), \ldots, s(M_n))$ is a scalar multiple of e, hence $s(M_1) = \cdots = s(M_n)$, and (3) follows as in the case $\mu \neq 0$. $\square$

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4. Application

The following is a more application-oriented extract of Theorem 3.3:

Proposition 4.1. *Let $M \in \mathbb{M}$, $\alpha := \frac{1}{n}s(M)$, $\beta := \frac{1}{n}q(M)$ and $\delta := \frac{\alpha^2-\beta}{n-1}$. Then:*

$$\begin{aligned}
\alpha^2 < \beta \quad &\Longrightarrow \quad |\det M| \leq \beta^{\frac{n}{2}} \\
\alpha^2 = \beta \quad &\Longrightarrow \quad |\det M| \leq |\alpha|(\beta-\delta)^{\frac{n-1}{2}} = \beta^{\frac{n}{2}} \\
\alpha^2 > \beta \quad &\Longrightarrow \quad |\det M| \leq |\alpha|(\beta-\delta)^{\frac{n-1}{2}} < \beta^{\frac{n}{2}}
\end{aligned}$$

Proof. This is clear if $\det M = 0$. In the case of $\det M \neq 0$, we get $\alpha^2 < n\beta$ by Lemma 3.1, and the stated inequalities are true by Lemma 3.2 and Theorem 3.3. □

For $M \in \mathbb{M}$ with $|M_{i,j}| \leq 1$ for all $i,j \in N$, Proposition 4.1 tells us that

$$|\det M| \leq \beta^{\frac{n}{2}} = \left(\frac{1}{n}\sum_{i,j=1}^{n} M_{i,j}^2\right)^{\frac{n}{2}} \leq \left(\frac{1}{n}\sum_{i,j=1}^{n} 1\right)^{\frac{n}{2}} = n^{\frac{n}{2}}, \tag{4.1}$$

which is simply the determinant theorem of Hadamard [3]. If $M_{i,j} \in \{-1,1\}$ for all $i,j \in N$ and $|\det M| = n^{n/2}$, i. e. M is a Hadamard matrix, then Proposition 4.1 shows that $\alpha^2 \leq \beta$ must hold. For a Hadamard matrix M, the value $s(M)$ is called the *excess* of M. Since $q(M) = n^2$ in the case of $M_{i,j} \in \{-1,1\}$, Proposition 4.1 yields an upper bound for the excess, known as Best's inequality [2]:

$$M \text{ is a Hadamard matrix} \quad \Longrightarrow \quad s(M) \leq n^{\frac{3}{2}} \tag{4.2}$$

The results (4.1) and (4.2), which both can be proved more directly, are mentioned here just as by-products of Proposition 4.1. In the following, we are interested only

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in the case $\alpha^2 \geq \beta$, where the inequality

$$|\det M| \leq |\alpha|(\beta - \delta)^{\frac{n-1}{2}} =: g(M)$$

holds. Note that Lemma 3.2 states that $g(M) < \beta^{\frac{n}{2}}$ is true for $\alpha^2 < \beta$ also, but $|\det M|$ is not necessarily bounded by $g(M)$ in this situation:

$$M := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad |\det M| = 1\,, \quad g(M) = 0.$$

We are now going to apply Proposition 4.1 to the problem stated in the introduction. This problem is a special case of finding an upper bound for the determinant of matrices whose entries are a permutation of an arithmetic progression:

Proposition 4.2. *Let p, q be real numbers with $q > 0$ and M a matrix whose entries are a permutation of the numbers $p, p+q, \ldots, p+(n^2-1)q$. Set*

$$r := \frac{p}{q} + \frac{n^2-1}{2} \quad \textit{and} \quad \varrho := \frac{n^3+n^2+n+1}{12}.$$

Then

$$|\det M| \leq n^{\frac{n}{2}} q^n \left(r^2 + \frac{n^4-1}{12} \right)^{\frac{n}{2}}$$

and

$$r^2 > \varrho \quad \Longrightarrow \quad |\det M| \leq n^n q^n\, |r| \varrho^{\frac{n-1}{2}} < n^{\frac{n}{2}} q^n \left(r^2 + \frac{n^4-1}{12} \right)^{\frac{n}{2}}.$$

Proof. For $\alpha := \frac{1}{n}s(M)$ and $\beta := \frac{1}{n}q(M)$ a calculation shows that $\alpha^2 - \beta = n(n-1)q^2(r^2 - \varrho)$, hence $(\alpha^2 > \beta \iff r^2 > \varrho)$. The bounds noted in Proposition 4.1 yield the asserted inequalities for $|\det M|$. $\square$

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Corollary 4.3. *If M is a matrix whose entries are a permutation of $1, \dots, n^2$, then*

$$|\det M| \leq n^n \frac{n^2+1}{2} \left(\frac{n^3+n^2+n+1}{12} \right)^{\frac{n-1}{2}}.$$

Proof. Apply Proposition 4.2 to $(p, q) := (1, 1)$. For $r = (n^2+1)/2$ it is easy to see that $r^2 > \varrho$, which yields the stated bound. $\square$

Comparing the lower bounds for $D(1, \dots, n^2)$ noted in the introduction with the upper bounds resulting from rounding down the values given by Corollary 4.3 shows that the quality of these upper bounds is quite convincing:

n	determinant of best known matrix	upper bound given by Corollary 4.3
2	10	11
3	412	450
4	40 800	41 021
5	6 839 492	6 865 625
6	1 865 999 570	1 867 994 210
7	762 150 368 499	762 539 814 814
8	440 960 274 696 935	441 077 015 225 642
9	346 254 605 664 223 620	346 335 386 150 480 625
10	356 944 784 622 927 045 792	357 017 114 947 987 625 629

These are the record matrices $R(n)$ corresponding to the noted determinants:

$$R(2) = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}, \qquad R(3) = \begin{pmatrix} 9 & 3 & 5 \\ 4 & 8 & 1 \\ 2 & 6 & 7 \end{pmatrix}, \qquad R(4) = \begin{pmatrix} 12 & 13 & 6 & 2 \\ 3 & 8 & 16 & 7 \\ 14 & 1 & 9 & 10 \\ 5 & 11 & 4 & 15 \end{pmatrix},$$

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$$R(5)=\begin{pmatrix}25&15&9&11&4\\7&24&14&3&17\\6&12&23&20&5\\10&13&2&22&19\\16&1&18&8&21\end{pmatrix},\qquad R(6)=\begin{pmatrix}36&24&21&17&5&8\\3&35&25&15&23&11\\13&7&34&16&10&31\\14&22&2&33&12&28\\20&4&19&29&32&6\\26&18&9&1&30&27\end{pmatrix},$$

$$R(7)=\begin{pmatrix}46&42&15&2&27&24&18\\9&48&36&30&7&14&31\\39&11&44&34&13&29&5\\26&22&17&41&47&1&21\\20&8&40&6&33&23&45\\4&28&19&25&38&49&12\\32&16&3&37&10&35&43\end{pmatrix},\qquad R(8)=\begin{pmatrix}1&12&20&52&40&50&53&32\\44&35&3&14&43&15&45&61\\57&2&51&49&23&11&38&29\\28&22&55&4&64&41&18&27\\25&36&42&34&5&48&7&63\\19&60&33&56&46&6&16&24\\59&39&9&37&30&58&21&8\\26&54&47&13&10&31&62&17\end{pmatrix},$$

$$R(9)=\begin{pmatrix}68&7&12&62&73&26&29&58&34\\67&37&43&10&3&61&33&78&36\\30&20&79&53&49&71&40&25&2\\56&50&8&27&42&60&81&4&41\\23&14&54&63&11&18&72&44&70\\1&38&32&21&65&66&22&48&76\\45&74&31&80&17&46&5&24&47\\15&77&35&39&51&16&59&69&9\\64&52&75&13&57&6&28&19&55\end{pmatrix},$$

$$R(10)=\begin{pmatrix}1&2&61&84&81&82&39&54&41&60\\53&57&3&65&94&20&91&22&66&33\\46&63&47&4&45&78&83&28&13&98\\79&42&49&71&5&95&51&10&77&26\\17&75&87&58&30&6&38&27&86&80\\68&93&76&50&85&56&7&37&14&19\\100&16&31&35&62&34&8&64&67&88\\21&72&29&9&48&73&43&97&89&25\\69&15&99&32&44&24&90&74&40&18\\52&70&23&96&11&36&55&92&12&59\end{pmatrix}.$$

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Calculating the matrix MM^T for each record matrix M reveals that MM^T has roughly the structure $(\beta - \delta)I + \delta J$ that was noted in Theorem 3.3 for the optimal matrices of the corresponding real optimisation problem.

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