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CHARACTERIZATIONS OF CONVEX VECTOR FUNCTIONS AND OPTIMIZATION

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ABSTRACT. In this paper we characterize nonsmooth convex vector functions by first and second order generalized derivatives. We also prove optimality conditions for convex vector problems involving nonsmooth data.

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1. INTRODUCTION

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a given vector function and $C \subset \mathbb{R}^m$ be a pointed closed convex cone. We say that f is C-convex if

$$f(tx + (1-t)y) - tf(x) - (1-t)f(y) \in C$$

for all $x, y \in \mathbb{R}^n$ and $t \in (0,1)$. The notion of C-convexity has been studied by many authors because this plays a crucial role in vector optimization (see [4, 11, 13, 14] and the references therein). In this paper we prove first and second order characterizations of nonsmooth C-convex functions by first and second order generalized derivatives and we use these results in order to obtain optimality criteria for vector problems.

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The notions of local minimum point and local weak minimum point are recalled in the following definition.

Definition 1.1. A point $x_0 \in \mathbb{R}^n$ is called a *local minimum point* (*local weak minimum point*) of (VO) if there exists a neighbourhood U of x_0 such that no $x \in U \cap X$ satisfies $f(x_0) - f(x) \in C\backslash\{0\}$ ($f(x_0) - f(x) \in \operatorname{int} C$).

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally Lipschitz at $x_0 \in \mathbb{R}^n$ if there exist a constant K_{x_0} and a neighbourhood U of x_0 such that $\|f(x_1) - f(x_2)\| \leq K_{x_0}\|x_1 - x_2\|$, $\forall x_1, x_2 \in U$. By Rademacher's theorem, a locally Lipschitz function is differentiable almost everywhere (in the sense of Lebesgue measure). Then the generalized Jacobian of f at x_0, denoted by $\partial f(x_0)$, exists and is given by

$$\partial f(x_0) := \operatorname{cl}\operatorname{conv}\{\lim \ \nabla f(x_k) : x_k \to x_0, \nabla f(x_k) \text{ exists}\}$$

where $\operatorname{cl}\operatorname{conv}\{\dots\}$ stands for the closed convex hull of the set under the parentheses. Now assume that f is a differentiable vector function from $\mathbb{R}^m$ to $\mathbb{R}^n$; if ∇f is locally Lipschitz at x_0, the generalized Hessian of f at x_0, denoted by $\partial^2 f(x_0)$, is defined as

$$\partial^2 f(x_0) := \operatorname{cl}\operatorname{conv}\{\lim \ \nabla^2 f(x_k) : x_k \to x_0, \nabla^2 f(x_k) \text{ exists}\}.$$

Thus $\partial^2 f(x_0)$ is a subset of the finite dimensional space $L(\mathbb{R}^m; L(\mathbb{R}^m; \mathbb{R}^n))$ of linear operators from $\mathbb{R}^m$ to the space $L(\mathbb{R}^m; \mathbb{R}^n)$ of linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$. The elements of $\partial^2 f(x_0)$ can therefore be viewed as bilinear function on $\mathbb{R}^m \times \mathbb{R}^m$ with values in $\mathbb{R}^n$. For the case $n = 1$, the terminology "generalized Hessian matrix" was used in [10] to denote the set $\partial^2 f(x_0)$. By the previous construction, the second order subdifferential enjoys all properties of the generalized Jacobian. For instance, $\partial^2 f(x_0)$ is a nonempty convex compact set of the space $L(\mathbb{R}^m; L(\mathbb{R}^m; \mathbb{R}^n))$ and the set valued map $x \mapsto \partial^2 f(x)$ is upper semicontinuous. Let $u \in \mathbb{R}^m$; in the following we will denote by Lu the value of a linear operator $L : \mathbb{R}^m \to \mathbb{R}^n$ at the point $u \in \mathbb{R}^m$ and by $H(u, v)$ the value of a bilinear operator $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ at the point $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$. So we will set

$$\partial f(x_0)(u) = \{Lu : L \in \partial f(x_0)\}$$

and

$$\partial^2 f(x_0)(u, v) = \{H(u, v) : H \in \partial^2 f(x_0)\}.$$

Some important properties are listed in the following ([9]).

- Mean value theorem. Let f be a locally Lipschitz function and $a, b \in \mathbb{R}^m$.Then
$$f(b) - f(a) \in \operatorname{cl}\operatorname{conv}\{\partial f(x)(b - a) : x \in [a, b]\}$$
where $[a, b] = \operatorname{conv}\{a, b\}$.
- Taylor expansion. Let f be a differentiable function. If ∇f is locally Lipschitz and $a, b \in \mathbb{R}^m$ then
$$f(b) - f(a) \in \nabla f(a)(b - a) + \frac{1}{2}\operatorname{cl}\operatorname{conv}\{\partial^2 f(x)(b - a, b - a) : x \in [a, b]\}.$$

2. A First Order Generalized Derivative for Vector Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a given function and $x_0 \in \mathbb{R}^n$. For such a function, the definition of Dini generalized derivative $\overline{f}'_D$ at x_0 in the direction $u \in \mathbb{R}^n$ is

$$\overline{f}'_D(x_0; u) = \limsup_{s \downarrow 0} \frac{f(x_0 + su) - f(x_0)}{s}.$$

Now let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a vector function and $x_0\in\mathbb{R}^n$. We can define a generalized derivative at $x_0\in\mathbb{R}^n$ in the sense of Dini as follows

$$f'_D(x_0;u)=\left\{l=\lim_{k\to+\infty}\frac{f(x_0+s_ku)-f(x_0)}{s_k},s_k\downarrow 0\right\}.$$

The previous set can be empty; however, if f is locally Lipschitz at x_0 then $f'(x_0;u)$ is a nonempty compact subset of $\mathbb{R}^m$. The following lemma states the relations between the scalar and the vector case.

Remark 2.1. If $f(x)=(f_1(x),\dots,f_m(x))$ then from the previous definition it is not difficult to prove that

$$f'_D(x_0;u)\subset(f_1)'_D(x_0;u)\times\cdots\times(f_m)'(x_0;u).$$

We now show that this inclusion may be strict.

Let us consider the function $f(x)=(x\sin(x^{-1}),x\cos(x^{-1}))$; for it we have

$$f'_D(0;1)\subset\{d\in\mathbb{R}^2:\|d\|=1\}$$

while

$$(f_1)'_D(0;1)=(f_2)'_D(0;1)=[-1,1].$$

Lemma 2.2. *Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a given locally Lipschitz vector function at $x_0\in\mathbb{R}^n$. Then, $\forall\xi\in\mathbb{R}^m$, we have $\overline{\xi f}'_D(x_0;u)\in\xi f'_D(x_0;u)$.*

Proof. There exists a sequence $s_k\downarrow 0$ such that the following holds

$$\overline{\xi f}'_D(x_0;u)=\limsup_{s\downarrow 0}\frac{(\xi f)(x_0+su)-(\xi f)(x_0)}{s}=\lim_{k\to+\infty}\frac{(\xi f)(x_0+s_ku)-(\xi f)(x_0)}{s_k}.$$

By trivial calculations and eventually by extracting subsequences, we obtain

$$=\sum_{i=1}^m\xi_i\lim_{k\to+\infty}\frac{f_i(x_0+s_ku)-f_i(x_0)}{s_k}=\sum_{i=1}^m\xi_il=\xi l$$

with $l\in f'_D(x_0;u)$ and then $\overline{\xi f}'_D(x_0;u)\in\xi f'_D(x_0;u)$. ☐

Corollary 2.3. *Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a differentiable function at $x_0\in\mathbb{R}^n$. Then $f'_D(x_0;u)=\nabla f(x_0)u$, $\forall u\in\mathbb{R}^n$.*

We now prove a generalized mean value theorem for f'_D.

Lemma 2.4. [6] *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz function. Then $\forall a,b\in\mathbb{R}^n$, $\exists\alpha\in[a,b]$ such that*

$$f(b)-f(a)\le\overline{f}'_D(\alpha;b-a).$$

Theorem 2.5. *Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a locally Lipschitz vector function. Then the following generalized mean value theorem holds*

$$0\in f(b)-f(a)-\operatorname{cl}\operatorname{conv}\left\{f'_D(x;b-a):x\in[a,b]\right\}.$$

Proof. For each $\xi\in\mathbb{R}^m$ we have

$$(\xi f)(b)-(\xi f)(a)\le\overline{\xi f}'_D(\alpha;b-a)=\xi l_\xi,\ \ l_\xi\in f'_D(\alpha;b-a),$$

where $\alpha\in[a,b]$ and then

$$\xi(f(b)-f(a)-l_\xi)\le 0,\ \ l_\xi\in f'_D(\alpha;b-a)$$

$$\xi\left(f(b)-f(a)-\operatorname{cl}\operatorname{conv}\left\{f'_D(x;b-a):x\in[a,b]\right\}\right)\cap\mathbb{R}_-\neq\emptyset,\ \forall\xi\in\mathbb{R}^m$$

and a classical separation separation theorem implies

$$0 \in f(b) - f(a) - \operatorname{cl}\operatorname{conv}\left\{f'_D(x; b-a) : x \in [a,b]\right\}.$$

□

Theorem 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz vector function at x_0. Then $f'_D(x_0; u) \subset \partial f(x_0)(u)$.*

Proof. Let $l \in f'_D(x_0; u)$. Then there exists a sequence $s_k \downarrow 0$ such that

$$l = \lim_{k\to+\infty} \frac{f(x_0 + s_k u) - f(x_0)}{s_k}.$$

So, by the upper semicontinuity of ∂f, we have

$$\begin{aligned}\frac{f(x_0 + s_k u) - f(x_0)}{s_k} &\in \operatorname{cl}\operatorname{conv}\left\{\partial f(x)(u); x \in [x_0, x_0 + s_k u]\right\}\\ &\subset \partial f(x_0)(u) + \epsilon B,\end{aligned}$$

where B is the unit ball of $\mathbb{R}^m$, $\forall n \geq n_0(\epsilon)$. So $l \in \partial f(x_0)u + \epsilon B$. Taking the limit when $\epsilon \to 0$, we obtain $l \in \partial f(x_0)(u)$. □

Example 2.1. Let $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^2 \sin(x^{-1}) + x^2, x^2)$. f is locally Lipschitz at $x_0 = 0$ and $f'_D(0; 1) = (0, 0) \in \partial f(0)(1) = [-1, 1] \times \{0\}$.

3. A PARABOLIC SECOND ORDER GENERALIZED DERIVATIVE FOR VECTOR FUNCTIONS

In this section we introduce a second order generalized derivative for differentiable functions. We consider a very different kind of approach, relying on the Kuratowski limit. It can be considered somehow a global one, since set-valued directional derivatives of vector-valued functions are introduced without relying on components. Unlike the first order case, there is not a common agreement on which is the most appropriate second order incremental ratio; in this section the choice goes to the second order parabolic ratio

$$h_f^2(x, t, w, d) = 2t^{-2}[f(x + td + 2^{-1}t^2 w) - f(x) - t\nabla f(x) \cdot d]$$

introduced in [1]. In fact, if f is twice differentiable at x_0, then

$$h_f^2(x, t_k, w, d) \to \nabla f(x) \cdot w + \nabla^2 f(x)(d, d)$$

for any sequence $t_k \downarrow 0$. Just supposing that f is differentiable at x_0, we can introduce the following second order set–valued directional derivative in the same fashion as the first order one.

Definition 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable vector function at $x_0 \in \mathbb{R}^n$. The second order parabolic set valued derivative of f at the point x_0 in the directions $d, w \in \mathbb{R}^n$ is defined as

$$D^2 f(x_0)(d, w) = \left\{ l : l = \lim_{k\to+\infty} 2\frac{f(x_0 + t_k d + \frac{t_k^2}{2} w) - f(x_0) - t_k \nabla f(x_0) d}{t_k^2}, t_k \downarrow 0 \right\}.$$

This notion generalizes to the vector case the notion of parabolic derivative introduced by Ben-Tal and Zowe in [1]. The following result states some properties of the parabolic derivative.

Proposition 3.1. *Suppose $f = (\phi_1, \phi_2)$ with $\phi_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$, $m_1 + m_2 = m$.*

- $D^2 f(x_0)(w, d) \subseteq D^2 \phi_1(x_0)(w, d) \times D^2 \phi_2(x_0)(w, d)$.

- *If ϕ_2 is twice differentiable at x_0, then*

$$D^2 f(x_0)(w,d) = D^2\phi_1(x_0)(w,d) \times \{\nabla\phi_2(x_0)\cdot w + \nabla^2\phi_2(x_0)(d,d)\}.$$

Proof. Trivial. □

The following example shows that the inclusion in (i) can be strict.

Example 3.1. Consider the function $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (\phi_1(x), \phi_2(x))$,

$$\phi_1(x) = \begin{cases} x^2 \sin\ln|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\phi_2(x) = \begin{cases} -x^2 \sin^3\ln|x|(\cos x - 2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

It is easy to check that $\nabla f(0) = (0,0)$ and

$$D^2(\phi_1,\phi_2)(0)(d,w) \subset \{l = (l_1,l_2), l_1 l_2 \le 0\} \cap -\operatorname{int}\mathbb{R}^2_+ = \emptyset,$$

$$D^2\phi_1(0)(d,w) = D^2\phi_2(0)(d,w) = [-2d^2, 2d^2]$$

and this shows that $D^2(\phi_1,\phi_2)(0)(d,w) \neq D^2 f_1(0)(w,d) \times D^2 f_2(0)(d,w)$.

Proposition 3.2. *Suppose f is differentiable in a neighbourhood of $x_0 \in \mathbb{R}^n$. Then, the equality*

$$D^2 f(x_0)(w,d) = \nabla f(x_0)\cdot w + \partial^2_* f(x_0)(d,d)$$

holds for any $d, w \in \mathbb{R}^n$, where $\partial^2_* f(x_0)(d,d)$ denotes the set of all cluster points of the sequences $\{2t_k^{-2}[f(x_0 + t_k d) - f(x_0) - t_k\nabla f(x_0)\cdot d]\}$ such that $t_k \downarrow 0$.

Proof. Trivial. □

Proposition 3.3. $D^2 f(x_0)(w,d) \subseteq \nabla f(x_0)\cdot w + \partial^2 f(x_0)(d,d)$.

Proof. Let $z \in D^2 f(x_0)(w,d)$. Then, we have $h^2_f(x_0,t_k,w,d) \to z$ for some suitable $t_k \downarrow 0$. Let us introduce the two sequences

$$a_k = 2t_k^{-2}[f(x_0 + t_k d + 2^{-1}t_k^2 w) - f(x_0 + t_k d)]$$

and

$$b_k = 2t_k^{-2}[f(x_0 + t_k d) - f(x_0) - t_k\nabla f(x_0)\cdot d]$$

such that $h^2_f(x_0,t_k,w,d) = a_k + b_k$. Since f is differentiable near x_0, then a_k converges to $\nabla f(x_0)\cdot w$ and thus b_k converges to $z_1 = z - \nabla f(x_0)\cdot w$. Therefore, the thesis follows if $z_1 \in \partial^2 f(x_0)(d,d)$. Given any $\theta \in \mathbb{R}^m$, let us introduce the functions

$$\phi_1(t) = 2t^{-2}[(\theta\cdot f)(x_0 + td) - (\theta\cdot f)(x_0) - t\nabla(\theta\cdot f)(x_0)\cdot d], \quad \phi_2(t) = t^2,$$

where $(\theta\cdot f)(x) = \theta\cdot f(x)$. Thus, we have

$$\theta\cdot b_k = \frac{[\phi_1(t_k) - \phi_1(0)]}{[\phi_2(t_k) - \phi_2(0)]} = \frac{\phi_1'(\xi_k)}{\phi_2'(\xi_k)}$$

for some $\xi_k \in [0,t_k]$. Since this sequence converges to $\theta\cdot z_1$, we also have

$$\lim_{k\to+\infty}\frac{\phi_1'(\xi_k)}{\phi_2'(\xi_k)} = \theta\cdot\lim_{k\to+\infty}\{\xi_k^{-1}[\nabla f(x_0 + \xi_k d) - \nabla f(x_0)]\cdot d\} = \theta\cdot z_\theta$$

for some $z_\theta \in \partial^2 f(x_0)(d,d)$. Hence the above argument implies that given any $\theta \in \mathbb{R}^m$ we have $\theta\cdot(z_1 - z_\theta) = 0$ for some $z_\theta \in \partial^2 f(x_0)(d,d)$. Since the generalized Hessian is a compact convex set, then the strict separation theorem implies that $z_1 \in \partial^2 f(x_0)(d,d)$. □

The following example shows that the above inclusion may be strict.

Example 3.2. Consider the function

$$f(x_1, x_2) = \left([\max\{0, x_1 + x_2\}]^2, x_2^2\right).$$

Then, easy calculations show that f is differentiable with $\nabla f_1(x_1, x_2) = (0, 0)$ whenever $x_2 = -x_1$ and $\nabla f_2(x_1, x_2) = (0, 2x_2)$. Moreover ∇f is locally Lipschitz near $x_0 = (0, 0)$ and actually f is twice differentiable at any x with $x_2 \neq -x_1$

$$\nabla^2 f_1(x)(d, d) = \begin{cases} 2(d_1^2 + d_2^2) & \text{if } x_1 + x_2 > 0 \\ 0 & \text{if } x_1 + x_2 < 0 \end{cases}$$

and $\nabla^2 f_2(x)(d, d) = 2d_2^2$. Therefore, we have

$$\partial^2 f(x_0)(d, d) = \left\{\left(2\alpha\left(d_1^2 + d_2^2\right), 2d_2^2\right) \;:\; \alpha \in [0, 1]\right\}.$$

On the contrary, it is easy to check that $D^2 f(x_0)(w, d) = \{(2(d_1^2 + d_2^2), 2d_2^2)\}$.

4. CHARACTERIZATIONS OF CONVEX VECTOR FUNCTIONS

Theorem 4.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C-convex then*

$$f(x) - f(x_0) \in f'_D(x_0, x - x_0) + C$$

for all $x \in \mathbb{R}^n$.

Proof. Since f is C-convex at x_0 then it is locally Lipschitz at x_0 [12]. For all $x \in \mathbb{R}^n$ we have

$$t(f(x) - f(x_0)) \in f(tx + (1 - t)x_0) - f(x_0) + C$$

Let $l \in f'_D(x_0; x - x_0)$; then there exists $t_k \downarrow 0$ such that $\frac{f(x_0 + t_k(x - x_0)) - f(x_0)}{t_k} \to d$ and

$$f(x) - f(x_0) \in \frac{f(t_k(x - x_0) + x_0) - f(x_0)}{t_k} + C.$$

Taking the limit when $k \to +\infty$ this implies $f(x) - f(x_0) \in f'_D(x_0, x - x_0) + C$ □

Corollary 4.2. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C-convex and differentiable at x_0 then*

$$f(x) - f(x_0) \in \nabla f(x_0)(x - x_0) + C$$

for all $x \in \mathbb{R}^n$.

The following result characterizes the convexity of f in terms of $D^2 f$.

Theorem 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable C-convex function at $x_0 \in \mathbb{R}^n$. Then we have*

$$D^2 f(x_0)(x - x_0, 0) \subset C$$

for all $x \in \mathbb{R}^n$.

Proof. If $D^2 f(x_0)(x - x_0, 0)$ is empty the thesis is trivial. Otherwise, let $l \in D^2 f(x_0)(x - x_0, 0)$. Then there exists $t_k \downarrow 0$ such that

$$l = \lim_{k \to +\infty} \frac{f(x_0 + t_k(x - x_0)) - f(x_0) - t_k \nabla f(x_0)(x - x_0)}{t_k^2}$$

Since f is a differentiable C-convex function then $f(x_0 + t_k(x - x_0)) - f(x_0) - t_k \nabla f(x_0)(x - x_0) \in C$ and this implies the thesis. □

5. OPTIMALITY CONDITIONS

We are now interested in proving optimality conditions for the problem

$$\min_{x\in X} f(x)$$

where X is a given subset of $\mathbb{R}^n$. The following definition states some notions of local approximation of X at $x_0 \in \operatorname{cl} X$.

Definition 5.1.

- The *cone of feasible directions* of X at x_0 is set:
$$F(X, x_0) = \{d \in \mathbb{R}^n : \exists\, \alpha > 0 \text{ s.t. } x_0 + td \in X, \forall t \le \alpha\}$$
- The *cone of weak feasible directions* of X at x_0 is the set:
$$WF(X, x_0) = \{d \in \mathbb{R}^n : \exists\, t_k \downarrow 0 \text{ s.t. } x_0 + t_k d \in X\}$$
- The *contingent cone* of X at x_0 is the set:
$$T(X, x_0) := \{w \in \mathbb{R}^n \ : \ \exists w_k \to w,\ \exists t_k \downarrow 0 \text{ s.t. } x_0 + t_k w_k \in X\}.$$
- The *second order contingent set* of X at x_0 in the direction $d \in \mathbb{R}^n$ is the set:
$$T^2(X, x_0, d) := \{w \in \mathbb{R}^n \ : \ \exists\, t_k \downarrow 0,\ \exists\, w_k \to w \ \text{ s.t. } \ x_0 + t_k d + 2^{-1}t_k^2 w_k \in X\}.$$
- The *lower second order contingent set* of X at $x_0 \in \operatorname{cl} X$ in the direction $d \in \mathbb{R}^n$ is the set:
$$T^{ii}(X, x_0, d) := \{w \in \mathbb{R}^n \ : \ \forall\, t_k \downarrow 0,\ \exists\, w_k \to w \ \text{ s.t. } \ x_0 + t_k d + 2^{-1}t_k^2 w_k \in X\}.$$

Theorem 5.1. *Let x_0 be a local weak minimum point. Then for all $d \in F(X, x_0)$ we have*

$$f'_D(x_0; d) \cap -\operatorname{int} C = \emptyset.$$

If ∇f is locally Lipschtz at x_0 then, for all $d \in WF(X, x_0)$, we have

$$f'_D(x_0; d) \cap (-\operatorname{int} C)^c \neq \emptyset.$$

Proof. If $f'_D(x_0; d)$ is empty then the thesis is trivial. If $l \in f'_D(x_0; d) \cap -\operatorname{int} C$ then $l = \lim_{k\to+\infty} \frac{f(x_0+t_k d)-f(x_0)}{t_k}$ and $f(x_0 + t_k d) - f(x_0) \in -\operatorname{int} C$ for all k sufficiently large. Suppose now that f is locally Lipschitz. In this case $f'_D(x_0; d)$ is nonempty for all $d \in \mathbb{R}^n$. Ab absurdo, suppose $f'_D(x_0; d) \subset -\operatorname{int} C$ for some $d \in WF(X, x_0)$. Let $x_k = x_0 + t_k d$ be a sequence such that $x_k \in X$; by extracting subsequences, we have

$$l = \lim_{k\to+\infty} \frac{f(x_0 + t_k d) - f(x_0)}{t_k}$$

and $l \in f'_D(x_0; d) \subset -\operatorname{int} C$. Since $\operatorname{int} C$ is open for k "large enough" we have

$$f(x_0 + t_k d) \in f(x_0) - \operatorname{int} C.$$

□

Theorem 5.2. *If $x_0 \in X$ is a local vector weak minimum point, then for each $d \in D_{\le}(f, x_0) \cap T(X, x_0)$ the condition*

$$D^2 f(x_0)(d + w, d) \cap -\operatorname{int} C = \emptyset \tag{5.1}$$

holds for any $w \in T^{ii}(X, x_0, d)$. Furthermore, if ∇f is locally Lipschitz at x_0, then the condition

$$D^2 f(x_0)(d + w, d) \not\subseteq -\operatorname{int} C \tag{5.2}$$

holds for any $d \in D_{\le}(f, x_0) \cap T(X, x_0)$ and any $w \in T^2(X, x_0, d)$.

Proof. Ab absurdo, suppose there exist suitable d and w such that (5.1) does not hold. Then, given any $z \in D^2 f(x_0)(d+w,d) \cap -\operatorname{int} C$, there exists a sequence $t_k \downarrow 0$ such that $h_f^2(x_0, t_k, d+w, d) \to z$. By the definition of the lower second order contingent set there exists also a sequence $w_k \to w$ such that $x_k = x_0 + t_k d + 2^{-1} t_k^2 w_k \in X$. Introducing also the sequence of points $\hat{x}_k = x_0 + t_k d + 2^{-1} t_k^2 (d+w)$, we have both

$$f(x_k) - f(\hat{x}_k) = 2^{-1} t_k^2 \left[\nabla f(\hat{x}_k) \cdot (w_k - w - d) + \varepsilon_k^{(1)} \right]$$

with $\varepsilon_k^{(1)} \to 0$ and

$$f(\hat{x}_k) - f(x_0) = t_k \nabla f(x_0) \cdot d + 2^{-1} t_k^2 \left[z + \varepsilon_k^{(2)} \right]$$

with $\varepsilon_k^{(2)} \to 0$. Therefore, we have

$$f(x_k) - f(x_0) = t_k \left\{ (1 - 2^{-1} t_k) \nabla f(x_0) \cdot d + 2^{-1} t_k \left[\nabla f(\hat{x}_k) \cdot (w_k - w) + z + \varepsilon_k^{(1)} + \varepsilon_k^{(2)} \right] \right\}.$$

Since

$$\lim_{k \to \infty} \left[\nabla f(\hat{x}_k) \cdot (w_k - w) + z + \varepsilon_k^{(1)} + \varepsilon_k^{(2)} \right] = z \in -\operatorname{int} C$$

and

$$(1 - 2^{-1} t_k) \nabla f(x_0) \cdot d \in -C,$$

for k large enough we have

$$f(x_k) - f(x_0) \in -(C + \operatorname{int} C) = -\operatorname{int} C$$

in contradiction with the optimality of x_0.

Analogously, suppose there exist suitable d and w such that (5.2) does not hold. By the definition of the second order contingent cone, there exist sequences $t_k \downarrow 0$ and $w_k \to w$ such that $x_0 + t_k d + 2^{-1} t_k^2 w_k \in X$. Taking the suitable subsequence, we can suppose that $h_f^2(x_0, t_k, d+w, d) \to z$ for some $z \in C$. Then, we have $z \in D^2 f(x_0)(d+w,d) \subseteq -\operatorname{int} C$ and we achieve a contradiction just as in the previous case. □

The following example shows that the previous second order condition is not sufficient for the optimality of $\bar{x}$.

Example 5.1. Suppose $C = \mathbb{R}_+^2$ and $f : \mathbb{R}^3 \to \mathbb{R}^2$ with

$$f_1(x_1, x_2, x_3) = x_1^2 + 2x_2^3 - x_3, \quad f_2(x_1, x_2, x_3) = x_2^3 - x_3,$$

$$X = \left\{ x \in \mathbb{R}^3 : x_1^2 \leq 4x_3 \leq 2x_1^2, \;\; x_1^2 + x_2^3 \geq 0 \right\}.$$

Choosing the point $x_0 = (0, 0, 0)$, we have

$$T^2(X, x_0, d) = \begin{cases} \mathbb{R} \times \mathbb{R} \times [2^{-1} d_1^2, d_1^2] & \text{if } d_2 = 0 \\ \emptyset & \text{if } d_2 \neq 0 \end{cases}$$

for any nonzero $d \in T(X, x_0) \cap D_{\leq}(f, x_0) = \mathbb{R} \times \mathbb{R} \times \{0\}$. Therefore

$$D^2 f(x_0)(d+w, d) = (-w_3 + 2d_1^2, -w_3) \cap -\operatorname{int} \mathbb{R}_+^2 = \emptyset$$

for any $w \in T^2(X, x_0, d)$. However, x_0 is not a local weak minimum point since both f_1 and f_2 are negative along the curve described by the feasible points $x_t = (t^3, -t^2, 2^{-1} t^6)$ for $t \neq 0$.

There are at least two good explanations for such a fact. The second order contingent sets may be empty and the corresponding optimality conditions are meaningless in such a case, since they are obviously satisfied by any objective function. Furthermore, there is no convincing reason why it should be enough to test optimality only along parabolic curves, as the above example corroborates. The following result states a sufficient condition for the optimality of x_0 when f is a convex function.

Definition 5.2. A subset $X \subset \mathbb{R}^n$ is said to be star shaped at x_0 if $[x_0, x] \subset X$ for all $x \in X$.

Theorem 5.3. *Let X be a star shaped set at x_0. If f is C-convex and $f'_D(x_0; x - x_0) \subset (-\operatorname{int} C)^c$, $x \in X$, then x_0 is a weak minimum point.*

Proof. We have, $\forall x \in X$,

$$f(x) - f(x_0) \in f'_D(x_0, x - x_0) + C \subset (-\operatorname{int} C)^c + C \subset (-\operatorname{int} C)^c$$

and this implies the thesis. □

REFERENCES

[1] A. BEN-TAL AND J. ZOWE, Directional derivatives in nonsmooth optimization, *J. Optim. Theory Appl.*, **47**(4) (1985), 483–490.

[2] S. BOLINTINANU, Approximate efficiency and scalar stationarity in unbounded nonsmooth convex vector optimization problems, *J. Optim. Theory Appl.*, **106**(2) (2000), 265–296.

[3] J.M. BORWEIN AND D.M. ZHUANG, Super efficiency in convex vector optimization, *Z. Oper. Res.*, **35**(3) (1991), 175–184.

[4] S. DENG, Characterizations of the nonemptiness and compactness of solution sets in convex vector optimization, *J. Optim. Theory Appl.*, **96**(1) (1998), 123–131.

[5] S. DENG, On approximate solutions in convex vector optimization, *SIAM J. Control Optim.*, **35**(6) (1997), 2128–2136.

[6] W.E. DIEWERT, Alternative characterizations of six kinds of quasiconcavity in the nondifferentiable case with applications to nonsmooth programming, *Generalized Concavity in Optimization and Economics* (S. Schaible and W.T. Ziemba eds.), Academic Press, New York, 1981, 51–95.

[7] B.M. GLOVER, Generalized convexity in nondifferentiable programming, *Bull. Austral. Math. Soc.*, **30** (1984), 193–218.

[8] V. GOROKHOVIK, Convex and nonsmooth problems of vector optimization, (Russian) "*Navuka i Tkhnika*", Minsk, 1990.

[9] A. GUERRAGGIO AND D.T. LUC, Optimality conditions for $C^{1,1}$ vector optimization problems, *J. Optim. Theory Appl.*, **109**(3) (2001), 615–629.

[10] J.B. HIRIART-URRUTY, J.J. STRODIOT AND V.H. NGUYEN, Generalized Hessian matrix and second order optimality conditions for problems with $C^{1,1}$ data, *Appl. Math. Optim.*, **11** (1984), 43–56.

[11] X.X. HUANG AND X.Q. YANG, Characterizations of nonemptiness and compactness of the set of weakly efficient solutions for convex vector optimization and applications, *J. Math. Anal. Appl.*, **264**(2) (2001), 270–287.

[12] D.T. LUC, N.X. TAN AND P.N. TINH, Convex vector functions and their subdifferentials, *Acta Math. Vietnam.*, **23**(1) (1998), 107–127.

[13] P.N. TINH, D.T. LUC AND N.X. TAN, Subdifferential characterization of quasiconvex and convex vector functions, *Vietnam J. Math.*, **26**(1) (1998), 53–69.

[14] K. WINKLER, Characterizations of efficient points in convex vector optimization problems, *Math. Methods Oper. Res.*, **53**(2) (2001), 205–214.