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ON A CERTAIN CLASS OF $p-$VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

In this paper, we introduce the class $A_o^*(p, A, B, \alpha)$ of p-valent functions in the unit disc $U = \{z : |z| < 1\}$. We obtain coefficient estimate, distortion and closure theorems, radii of close-to convexity, starlikeness and convexity of order $\delta \ (0 \leqslant \delta < 1)$ for this class. We also obtain class preserving integral operators for this class. Furthermore, various distortion inequalities for fractional calculus of functions in this class are also given.

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Contents

1. Introduction

Let $A(n)$ be the class of functions f, analytic and $p-$valent in $U = \{z : |z| < 1\}$ given by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad a_{p+n} > 0. \tag{1.1}$$

A function f belonging to the class $A(n)$ is said to be in the class $A_m^*(p, A, B, \alpha)$ if and only if

$$(p-1) + \mathrm{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \text{for } z \in U.$$

In the other words, $f \in A_m^*(p, A, B, \alpha)$ if and only if it satisfies the condition

$$\left|\frac{(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - p}{(A-B)(p-\alpha) + pB - B\left[(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right]}\right| < 1$$

where $-1 \leq B < A \leq 1$, $-1 \leq B < 0$ and $0 \leq \alpha < p$. Let A_m denote the subclass of $A(n)$ consisting of functions analytic and $p-$valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}; \quad a_{p+n} \geq 0. \tag{1.2}$$

Let us define

$$A_o^*(p, A, B, \alpha) = A_m^*(p, A, B, \alpha) \bigcap A_m.$$

In this paper, we obtain a coefficient estimate, distortion theorems, integral operators and radii of close-to-convexity, starlikeness and convexity, closure properties and distortion inequalities for fractional calculus. This paper is motivated by an earlier work of Nunokawa [1].

2. Coefficient Estimates

Theorem 2.1. *If the function f is defined by (1.1), then $f \in A_o^*(p, A, B, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \le (A-B)(p-\alpha)p!. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|zf^{(p)}(z) - f^{(p-1)}(z)\right| - \left|(A-B)(p-\alpha)f^{(p-1)} - Bzf^{(p)} + Bf^{(p-1)}\right| \\
&\quad = \left|-\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}\right| - \Bigg|(A-B)(p-\alpha)p!z \\
&\qquad - \left[(A-B)(p-\alpha)\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1} - B\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}\right]\Bigg| \\
&\quad \le \sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} a_{p+n} - (A-B)(p-\alpha)p! \le 0
\end{aligned}$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f \in$

$A_o^*(p,A,B,\alpha)$. To prove the converse, assume that

$$\left|\frac{(p-1)+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-p}{(A-B)(p-\alpha)+pB-B\left[(p-1)+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right]}\right|$$

$$=\left|\frac{-\sum_{n=1}^{\infty}\frac{n(p+n)!}{(n+1)!}a_{p+n}z^{n+1}}{(A-B)(p-\alpha)\left(p!z-\sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}a_{p+n}z^{n+1}\right)+B\sum_{n=1}^{\infty}\frac{n(p+n)!}{(n+1)!}a_{p+n}z^{n+1}}\right| < 1.$$

Since $\operatorname{Re}(z)\leq|z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{-\sum_{n=1}^{\infty}\frac{n(p+n)!}{(n+1)!}a_{p+n}z^{n+1}}{(A-B)(p-\alpha)\left(p!z-\sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}a_{p+n}z^{n+1}\right)+B\sum_{n=1}^{\infty}\frac{n(p+n)!}{(n+1)!}a_{p+n}z^{n+1}}\right\} < 1. \tag{2.2}$$

Choosing values of z on the real axis and letting $z\to 1^-$ through real values, we obtain

$$\sum_{n=1}^{\infty}\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!}a_{p+n}\leq(A-B)(p-\alpha)p!, \tag{2.3}$$

which obviously is required assertion (2.1). Finally, sharpness follows if we take

$$f(z) = z^p - \frac{(A-B)(p-\alpha)p!(n+1)!}{(p+n)!\left[n(1-B)+(A-B)(p-\alpha)\right]} z^{p+n}. \tag{2.4}$$

□

Corollary 2.2. *If $f \in A_o^*(p, A, B, \alpha)$, then*

$$a_{p+n} \leq \frac{(A-B)(p-\alpha)p!(n+1)!}{(p+n)!\left[n(1-B)+(A-B)(p-\alpha)\right]}. \tag{2.5}$$

The equality in (2.5) is attained for the function f given by (2.4).

3. Distortion Properties

Theorem 3.1. *If $f \in A_o^*(p,A,B,\alpha)$, then for $|z| = r < 1$*

$$r^p - \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} r^{p+1} \leq |f(z)| \leq r^p + \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} r^{p+1} \tag{3.1}$$

and

$$pr^{p-1} - \frac{2(A-B)(p-\alpha)}{(1-B)+(A-B)(p-\alpha)} r^p \leq |f'(z)| \leq pr^{p-1} + \frac{2(A-B)(p-\alpha)}{(1-B)+(A-B)(p-\alpha)} r^p. \tag{3.2}$$

All the inequalities are sharp.

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, a_{p+n} > 0.$$

From Theorem 2.1, we have

$$\frac{(p+1)!\left[(1-B)+(A-B)(p-\alpha)\right]}{2} \sum_{n=1}^{\infty} a_{p+n} \leq \sum_{n=1}^{\infty} \frac{(p+n)!\left[n(1-B)+(A-B)(p-\alpha)\right]}{(n+1)!} a_{p+n} \leq (A-B)(p-\alpha)p!$$

which

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} (p+n)a_{p+n} \leq \frac{2(A-B)(p-\alpha)}{(1-B)+(A-B)(p-\alpha)}. \tag{3.4}$$

Consequently, for $|z| = r < 1$, we obtain

$$\begin{aligned} |f(z)| &\leq r^p + r^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\leq r^p + \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} r^{p+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq r^p - r^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\geq r^p - \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} r^{p+1} \end{aligned}$$

which prove that the assertion (3.1) of Theorem 3.1 holds.

The inequalities in (3.2) can be proved in a similar manner and we omit the details. □

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z) = z^p - \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} z^{p+1}. \tag{3.5}$$

Letting $r \to 1^-$ in the left hand side of (3.1), we have the following:

Corollary 3.2. *If $f \in A_o^*(p, A, B, \alpha)$, then the disc $|z| < 1$ is mapped by f onto a domain that contains the disc*

$$|w| < \frac{(p+1)(1-B)+(A-B)(p-\alpha)(p-1)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]}.$$

The result is sharp with the extremal function f being given by (3.5).

Putting $\alpha = 0$ in Theorem 3.1 and Corollary 3.2, we get

Corollary 3.3. *If $f \in A_o^*(p, A, B, 0)$, then for $|z| = r$*

$$r^p - \frac{2p(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]} r^{p+1} \leq |f(z)| \leq r^p + \frac{2p(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]} r^{p+1}$$

and

$$pr^{p-1} - \frac{2p(A-B)}{(1-B)+p(A-B)} r^p \leq |f'(z)| \leq pr^{p-1} + \frac{2p(A-B)}{(1-B)+p(A-B)} r^p.$$

The result is sharp with the extremal function

$$f(z) = z^p - \frac{2p(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]} z^{p+1}; \quad z = \mp r. \tag{3.6}$$

Corollary 3.4. *If $f \in A_o^*(p, A, B, 0)$, then the disc $|z| < 1$ is mapped by f onto a domain that contains the disc*

$$|w| < \frac{(p+1)(1-B)+p(p-1)(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]}.$$

The result is sharp with the extremal function f being given by (3.6).

4. Radii of Close-To-Convexity, Starlikeness and Convexity

Theorem 4.1. *Let $f \in A_o^*(p, A, B, \alpha)$. Then f is $p-$valent close-to-convex of order δ $(0 \leq \delta < p)$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left\{ \left[\frac{(p+n)![n(1-B)+(A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)p!} \left(\frac{p-\delta}{p+n} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.1}$$

Theorem 4.2. *If $f \in A_o^*(p, A, B, \alpha)$, then f is $p-$valent starlike of order δ $(0 \leq \delta < p)$ in $|z| < R_2$, where*

$$R_2 = \inf_n \left\{ \left[\frac{(p+n)![n(1-B)+(A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)!p!} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.2}$$

Theorem 4.3. *If $f \in A_o^*(p, A, B, \alpha)$, then f is a $p-$valent convex function of order δ $(0 \leq \delta < p)$ in $|z| < R_3$, where*

$$R_3 = \inf_n \left\{ \left[\frac{[n(1-B)+(A-B)(p-\alpha)](p+n-1)!}{(A-B)(p-\alpha)(n+1)!(p-1)!} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.3}$$

In order to establish the required results in Theorems 4.1, 4.2 and 4.3, it is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta \quad \text{for} \quad |z| < R_1,$$

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \quad \text{for} \quad |z| < R_2 \quad \text{and}$$

$$\left| \left[1 + \frac{zf''(z)}{f'(z)} \right] - p \right| \leq p - \delta \quad \text{for} \quad |z| < R_3,$$

respectively.

Remark 1. *The results in Theorems 4.1, 4.2 and 4.3 are sharp with the extremal function f given by (2.4). Furthermore, taking $\delta = 0$ in Theorems 4.1, 4.2 and 4.3, we obtain radius of close-to-convexity, starlikeness and convexity, respectively.*

5. Integral Operators

Theorem 5.1. *Let c be a real number such that $c > -p$. If $f \in A_o^*(p, A, B, \alpha)$, then the function F defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \tag{5.1}$$

also belongs to $A_o^*(p, A, B, \alpha)$.

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}.$$

Then from the representation of F, it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

where $b_{p+n} = \left(\frac{c+p}{c+p+n}\right) a_{p+n}$. Therefore using Theorem 2.1 for the coefficients of F, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} b_{p+n} \\
&\qquad = \sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} \left(\frac{c+p}{c+p+n}\right) a_{p+n} \\
&\qquad \leq (A-B)(p-\alpha)p!
\end{aligned}$$

since $\frac{c+p}{c+p+n} < 1$ and $f \in A_o^*(p, A, B, \alpha)$. Hence $F \in A_o^*(p, A, B, \alpha)$. $\square$

Theorem 5.2. *Let c be a real number such that $c > -p$. If $F \in A_o^*(p, A, B, \alpha)$, then the function f defined by (5.1) is $p-$valent in $|z| < R^*$, where*

$$R^* = \inf_n \left\{ \left[\left(\frac{c+p}{c+p+n} \right) \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!(A-B)(p-\alpha)p!} \left(\frac{p}{p+n} \right) \right]^{\frac{1}{n}} \right\}. \tag{5.2}$$

The result is sharp. Sharpness follows if we take

$$f(z) = z^p - \left(\frac{c+p+n}{c+p} \right) \frac{(n+1)!(A-B)(p-\alpha)p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} z^{p+n}.$$

6. Closure Properties

In this section we show that the class $A_o^*(p, A, B, \alpha)$ is closed under "arithmetic mean" and "convex linear combinations".

Theorem 6.1. *Let*

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,j} z^{p+n}, \quad j = 1, 2, \dots$$

and

$$h(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

where

$$b_{p+n} = \sum_{j=1}^{\infty} \lambda_j a_{p+n,j}, \quad \lambda_j > 0$$

and $\sum_{j=1}^{\infty} \lambda_j = 1$. *If* $f_j \in A_o^*(p, A, B, \alpha)$ *for each* $j = 1, 2, \dots$, *then* $h \in A_o^*(p, A, B, \alpha)$.

Proof. If $f_j \in A_o^*(p, A, B, \alpha)$, then we have from Theorem 2.1 that

$$\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n,j} \leq (A-B)(p-\alpha)p!, \quad j = 1, 2, \dots.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} b_{p+n}$$
$$= \sum_{n=1}^{\infty} \left[\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} \left(\sum_{j=1}^{\infty} \lambda_j a_{p+n,j} \right) \right]$$
$$\leq (A-B)(p-\alpha)p!.$$

Hence, by Theorem 2.1, $h \in A_o^*(p, A, B, \alpha)$. □

Theorem 6.2. *The class $A_o^*(p, A, B, \alpha)$ is closed under convex linear combinations.*

Theorem 6.3. *Let $f_p(z) = z^p$ and*

$$f_{p+n} = z^p - \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} z^{p+n} \quad (n \geq 1).$$

Then $f \in A_o^*(p, A, B, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z), \quad z \in U,$$

where $\lambda_n \geq 0$ and $\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_n$.

Proof. Let us assume that

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z)$$
$$= z^p - \sum_{n=1}^{\infty} \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} \lambda_n z^{p+n}.$$

Then from Theorem 2.1 we have

$$\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} \times \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} \lambda_n \leq (A-B)(p-\alpha)p!.$$

Hence $f \in A_o^*(p, A, B, \alpha)$. Conversely, let $f \in A_o^*(p, A, B, \alpha)$. It follows from Corollary 2.2 that

$$a_{p+n} \leq \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}.$$

Setting

$$\lambda_n = \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)!p!} a_{p+n}, \quad n = 1, 2, \dots$$

and $\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_n$, we have

$$\begin{aligned} f(z) &= z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \\ &= z^p - \sum_{n=1}^{\infty} \lambda_n z^p + \sum_{n=1}^{\infty} \lambda_n z^p \\ &\qquad - \sum_{n=1}^{\infty} \lambda_n \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\left[n(1-B)+(A-B)(p-\alpha)\right]} z^{p+n} \\ &= \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z). \end{aligned}$$

This completes the proof of Theorem 6.3. □

7. Definitions and Applications of Fractional Calculus

In this section, we shall prove several distortion theorems for functions to general class $A_o^*(p, A, B, \alpha)$. Each of these theorems would involve certain operators of fractional calculus we find it to be convenient to recall here the following definition which were used recently by Owa [2] (and more recently, by Owa and Srivastava [3], and Srivastava and Owa [4] ; see also Srivastava et al. [5]).

Definition 7.1. *The fractional integral of order* λ *is defined, for a function* f*, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \tag{7.1}$$

where f *is an analytic function in a simply – connected region of the* z *-plane containing the origin, and the multiplicity of* $(z-\zeta)^{\lambda-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $z - \zeta > 0$.

Definition 7.2. *The fractional derivative of order* λ *is defined, for a function* f*, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{7.2}$$

where f *is constrained, and the multiplicity of* $(z-\zeta)^{-\lambda}$ *is removed, as in Definition 7.1.*

Definition 7.3. *Under the hypotheses of Definition 7.2, the fractional derivative of order* $(n+\lambda)$ *is defined by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1), \tag{7.3}$$

where $0 \leq \lambda < 1$ *and* $n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. *From Definition 7.2, we have*

$$D_z^0 f(z) = f(z) \tag{7.4}$$

which, in view of Definition 7.3 yields,

$$D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^n(z). \tag{7.5}$$

Thus, it follows from (7.4) and (7.5) that

$$\lim_{\lambda \to 0} D_z^{-\lambda} f(z) = f(z) \quad \text{and} \quad \lim_{\lambda \to 0} D_z^{1-\lambda} f(z) = f'(z).$$

Theorem 7.1. *Let the function* f *defined by (1.2) be in the class* $A_o^*(p, A, B, \alpha)$. *Then for* $z \in U$ *and* $\lambda > 0$,

$$\left| D_z^{-\lambda} f(z) \right| \geq |z|^{p+\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} - \frac{2(A-B)(p-\alpha)\Gamma(p+1)}{(\lambda+p+1)\Gamma(\lambda+p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} |z| \right\}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \leq |z|^{p+\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} + \frac{2(A-B)(p-\alpha)\Gamma(p+1)}{(\lambda+p+1)\Gamma(\lambda+p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} |z| \right\}.$$

The result is sharp.

Proof. Let

$$F(z) = \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) = z^p - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)} a_{p+n} z^{p+n} = z^p - \sum_{n=1}^{\infty} \varphi(n) a_{p+n} z^{p+n},$$

where

$$\varphi(n) = \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)}, \quad (\lambda > 0,\ n \in \mathbb{N}).$$

Then by using $0 < \varphi(n) \leq \varphi(1) = \frac{p+1}{p+\lambda+1}$ and Theorem 2.1, we observe that

$$\frac{(p+1)!\left[(1-B)+(A-B)(p-\alpha)\right]}{2!} \sum_{n=1}^{\infty} a_{p+n}$$

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$$\leq \sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} a_{p+n}$$
$$\leq (A-B)(p-\alpha)p!,$$

which shows that $F(z) \in A_o^*(p, A, B, \alpha)$. Consequently, with the aid of Theorem 3.1, we have

$$|F(z)| \geq |z^p| - \varphi(1)\,|z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$
$$\geq |z|^p - \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B)+(A-B)(p-\alpha)]}\,|z|^{p+1}$$

and

$$|F(z)| \leq |z^p| + \varphi(1)\,|z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$
$$\leq |z|^p + \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B)+(A-B)(p-\alpha)]}\,|z|^{p+1}$$

which completes the proof of Theorem 7.1.By letting $\lambda \to 0$, Theorem 7.1 reduces at once to Theorem 3.1. □

Corollary 7.2. *Under the hypotheses of Theorem 7.1, $D_z^{-\lambda} f(z)$ is included in a disk with its center at the origin and radius $R_1^{-\lambda}$ given by*

$$R_1^{-\lambda} = \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} \right\} \left\{ 1 + \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B)+(A-B)(p-\alpha)]} \right\}.$$

Theorem 7.3. *Let the function f defined by (1.2) be in the class $A_o^*(p,A,B,\alpha)$. Then,*

$$\left|D_z^\lambda f(z)\right| \geq |z|^{p-\lambda}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} - \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)\Gamma(p+1)}{\Gamma(p-\lambda+1)\Gamma(p-\lambda+2)[(1-B)+(A-B)(p-\alpha)]}|z|\right\}$$

and

$$\left|D_z^\lambda f(z)\right| \leq |z|^{p-\lambda}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} + \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)\Gamma(p+1)}{\Gamma(p-\lambda+1)\Gamma(p-\lambda+2)[(1-B)+(A-B)(p-\alpha)]}|z|\right\}$$

for $0 \leq \lambda < 1$.

Proof. Using similar arguments as given by Theorem 7.1, we can get the result. □

Corollary 7.4. *Under the hypotheses of Theorem 7.3, $D_z^\lambda f(z)$ is included in the disk with its center at the origin and radius R_2^λ given by*

$$R_2^\lambda = \left\{\frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)}\right\}\left\{1 + \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)}{\Gamma(p-\lambda+1)[(1-B)+(A-B)(p-\alpha)]}\right\}.$$

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