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ON A CERTAIN CLASS OF p -VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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[Abstract](#)

[Contents](#)

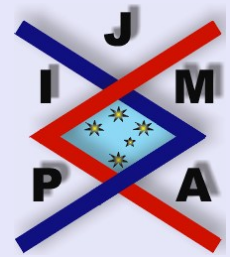


[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

In this paper, we introduce the class $A_o^*(p, A, B, \alpha)$ of p -valent functions in the unit disc $U = \{z : |z| < 1\}$. We obtain coefficient estimate, distortion and closure theorems, radii of close-to-convexity, starlikeness and convexity of order δ ($0 \leq \delta < 1$) for this class. We also obtain class preserving integral operators for this class. Furthermore, various distortion inequalities for fractional calculus of functions in this class are also given.

2000 Mathematics Subject Classification: 30C45, 30C50.

Key words: p -valent, Coefficient, Distortion, Closure, Starlike, Convex, Fractional calculus, Integral operators.

Contents

1	Introduction	3
2	Coefficient Estimates	5
3	Distortion Properties	8
4	Radii of Close-To-Convexity, Starlikeness and Convexity ...	12
5	Integral Operators	14
6	Closure Properties	16
7	Definitions and Applications of Fractional Calculus	20
	References	

On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 2 of 25

1. Introduction

Let $A(n)$ be the class of functions f , analytic and p -valent in $U = \{z : |z| < 1\}$ given by

$$(1.1) \quad f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad a_{p+n} > 0.$$

A function f belonging to the class $A(n)$ is said to be in the class $A_m^*(p, A, B, \alpha)$ if and only if

$$(p-1) + \operatorname{Re} \left\{ \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \quad \text{for } z \in U.$$

In the other words, $f \in A_m^*(p, A, B, \alpha)$ if and only if it satisfies the condition

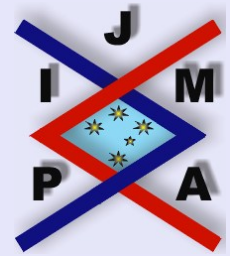
$$\left| \frac{(p-1) + \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} - p}{(A-B)(p-\alpha) + pB - B \left[(p-1) + \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} \right]} \right| < 1$$

where $-1 \leq B < A \leq 1$, $-1 \leq B < 0$ and $0 \leq \alpha < p$. Let A_m denote the subclass of $A(n)$ consisting of functions analytic and p -valent which can be expressed in the form

$$(1.2) \quad f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}; \quad a_{p+n} \geq 0.$$

Let us define

$$A_o^*(p, A, B, \alpha) = A_m^*(p, A, B, \alpha) \cap A_m.$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



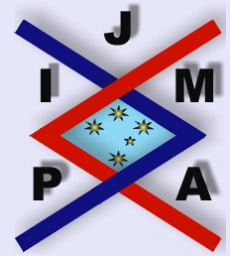
Go Back

Close

Quit

Page 3 of 25

In this paper, we obtain a coefficient estimate, distortion theorems, integral operators and radii of close-to-convexity, starlikeness and convexity, closure properties and distortion inequalities for fractional calculus. This paper is motivated by an earlier work of Nunokawa [1].



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 4 of 25

2. Coefficient Estimates

Theorem 2.1. *If the function f is defined by (1.1), then $f \in A_o^*(p, A, B, \alpha)$ if and only if*

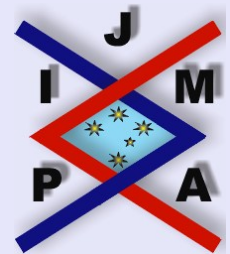
$$(2.1) \quad \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \leq (A-B)(p-\alpha)p!$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\begin{aligned} & \left| z f^{(p)}(z) - f^{(p-1)}(z) \right| - \left| (A-B)(p-\alpha) f^{(p-1)} - Bz f^{(p)} + Bf^{(p-1)} \right| \\ &= \left| - \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1} \right| - \left| (A-B)(p-\alpha)p!z \right. \\ & \quad \left. - \left[(A-B)(p-\alpha) \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1} - B \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1} \right] \right| \\ & \leq \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} - (A-B)(p-\alpha)p! \leq 0 \end{aligned}$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f \in$



On A Certain Class Of p -Valent
Functions With Negative
Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 5 of 25

$A_o^*(p, A, B, \alpha)$. To prove the converse, assume that

$$\begin{aligned} & \left| \frac{(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - p}{(A-B)(p-\alpha) + pB - B \left[(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right]} \right| \\ &= \left| \frac{-\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}}{(A-B)(p-\alpha) \left(p!z - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1} \right) + B \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}} \right| \\ & < 1. \end{aligned}$$

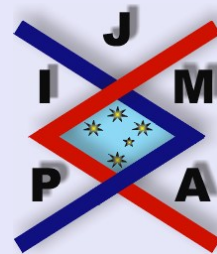
Since $\operatorname{Re}(z) \leq |z|$ for all z , we have

$$(2.2) \quad \operatorname{Re} \left\{ \frac{-\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}}{(A-B)(p-\alpha) \left(p!z - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1} \right) + B \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}} \right\} < 1.$$

Choosing values of z on the real axis and letting $z \rightarrow 1^-$ through real values, we obtain

$$(2.3) \quad \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \leq (A-B)(p-\alpha)p!,$$

which obviously is required assertion (2.1). Finally, sharpness follows if we



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 6 of 25

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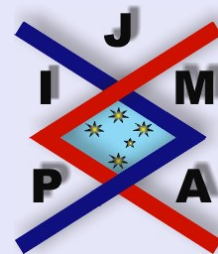
$$(2.4) \quad f(z) = z^p - \frac{(A - B)(p - \alpha)p!(n + 1)!}{(p + n)! [n(1 - B) + (A - B)(p - \alpha)]} z^{p+n}.$$

□

Corollary 2.2. *If $f \in A_o^*(p, A, B, \alpha)$, then*

$$(2.5) \quad a_{p+n} \leq \frac{(A - B)(p - \alpha)p!(n + 1)!}{(p + n)! [n(1 - B) + (A - B)(p - \alpha)]}.$$

The equality in (2.5) is attained for the function f given by (2.4).



On A Certain Class Of p -Valent
Functions With Negative
Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 7 of 25

3. Distortion Properties

Theorem 3.1. If $f \in A_o^*(p, A, B, \alpha)$, then for $|z| = r < 1$

$$(3.1) \quad r^p - \frac{2(A-B)(p-\alpha)}{(p+1)[(1-B) + (A-B)(p-\alpha)]} r^{p+1} \\ \leq |f(z)| \leq r^p + \frac{2(A-B)(p-\alpha)}{(p+1)[(1-B) + (A-B)(p-\alpha)]} r^{p+1}$$

and

$$(3.2) \quad pr^{p-1} - \frac{2(A-B)(p-\alpha)}{(1-B) + (A-B)(p-\alpha)} r^p \\ \leq |f'(z)| \leq pr^{p-1} + \frac{2(A-B)(p-\alpha)}{(1-B) + (A-B)(p-\alpha)} r^p.$$

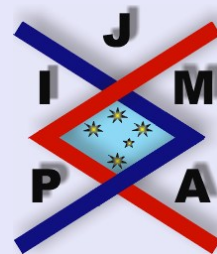
All the inequalities are sharp.

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, a_{p+n} > 0.$$

From Theorem 2.1, we have

$$\frac{(p+1)! [(1-B) + (A-B)(p-\alpha)]}{2} \sum_{n=1}^{\infty} a_{p+n} \\ \leq \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \\ \leq (A-B)(p-\alpha)p!$$



On A Certain Class Of p -Valent
Functions With Negative
Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 8 of 25

which

$$(3.3) \quad \sum_{n=1}^{\infty} a_{p+n} \leq \frac{2(A-B)(p-\alpha)}{(p+1)[(1-B) + (A-B)(p-\alpha)]}$$

and

$$(3.4) \quad \sum_{n=1}^{\infty} (p+n)a_{p+n} \leq \frac{2(A-B)(p-\alpha)}{(1-B) + (A-B)(p-\alpha)}.$$

Consequently, for $|z| = r < 1$, we obtain

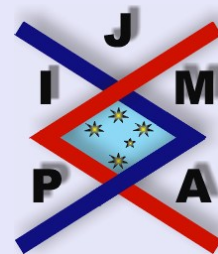
$$\begin{aligned} |f(z)| &\leq r^p + r^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\leq r^p + \frac{2(A-B)(p-\alpha)}{(p+1)[(1-B) + (A-B)(p-\alpha)]} r^{p+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq r^p - r^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\geq r^p - \frac{2(A-B)(p-\alpha)}{(p+1)[(1-B) + (A-B)(p-\alpha)]} r^{p+1} \end{aligned}$$

which prove that the assertion (3.1) of Theorem 3.1 holds.

The inequalities in (3.2) can be proved in a similar manner and we omit the details. \square



On A Certain Class Of p -Valent
Functions With Negative
Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 9 of 25

The bounds in (3.1) and (3.2) are attained for the function f given by

$$(3.5) \quad f(z) = z^p - \frac{2(A - B)(p - \alpha)}{(p + 1) [(1 - B) + (A - B)(p - \alpha)]} z^{p+1}.$$

Letting $r \rightarrow 1^-$ in the left hand side of (3.1), we have the following:

Corollary 3.2. *If $f \in A_o^*(p, A, B, \alpha)$, then the disc $|z| < 1$ is mapped by f onto a domain that contains the disc*

$$|w| < \frac{(p + 1)(1 - B) + (A - B)(p - \alpha)(p - 1)}{(p + 1) [(1 - B) + (A - B)(p - \alpha)]}.$$

The result is sharp with the extremal function f being given by (3.5).

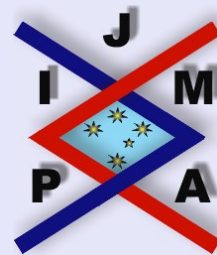
Putting $\alpha = 0$ in Theorem 3.1 and Corollary 3.2, we get

Corollary 3.3. *If $f \in A_o^*(p, A, B, 0)$, then for $|z| = r$*

$$\begin{aligned} r^p - \frac{2p(A - B)}{(p + 1) [(1 - B) + p(A - B)]} r^{p+1} \\ \leq |f(z)| \leq r^p + \frac{2p(A - B)}{(p + 1) [(1 - B) + p(A - B)]} r^{p+1} \end{aligned}$$

and

$$\begin{aligned} pr^{p-1} - \frac{2p(A - B)}{(1 - B) + p(A - B)} r^p \leq |f'(z)| \\ \leq pr^{p-1} + \frac{2p(A - B)}{(1 - B) + p(A - B)} r^p. \end{aligned}$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 10 of 25

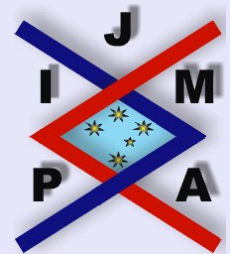
The result is sharp with the extremal function

$$(3.6) \quad f(z) = z^p - \frac{2p(A - B)}{(p + 1)[(1 - B) + p(A - B)]} z^{p+1}; \quad z = \mp r.$$

Corollary 3.4. *If $f \in A_o^*(p, A, B, 0)$, then the disc $|z| < 1$ is mapped by f onto a domain that contains the disc*

$$|w| < \frac{(p + 1)(1 - B) + p(p - 1)(A - B)}{(p + 1)[(1 - B) + p(A - B)]}.$$

The result is sharp with the extremal function f being given by (3.6).



On A Certain Class Of p -Valent
Functions With Negative
Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 11 of 25

4. Radii of Close-To-Convexity, Starlikeness and Convexity

Theorem 4.1. Let $f \in A_o^*(p, A, B, \alpha)$. Then f is p -valent close-to-convex of order δ ($0 \leq \delta < p$) in $|z| < R_1$, where

$$(4.1) \quad R_1 = \inf_n \left\{ \left[\frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)p!} \left(\frac{p-\delta}{p+n} \right) \right]^{\frac{1}{n}} \right\}.$$

Theorem 4.2. If $f \in A_o^*(p, A, B, \alpha)$, then f is p -valent starlike of order δ ($0 \leq \delta < p$) in $|z| < R_2$, where

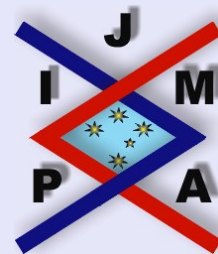
$$(4.2) \quad R_2 = \inf_n \left\{ \left[\frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)p!} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}.$$

Theorem 4.3. If $f \in A_o^*(p, A, B, \alpha)$, then f is a p -valent convex function of order δ ($0 \leq \delta < p$) in $|z| < R_3$, where

$$(4.3) \quad R_3 = \inf_n \left\{ \left[\frac{[n(1-B) + (A-B)(p-\alpha)](p+n-1)!}{(A-B)(p-\alpha)(n+1)!(p-1)!} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}.$$

In order to establish the required results in Theorems 4.1, 4.2 and 4.3, it is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta \quad \text{for } |z| < R_1,$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

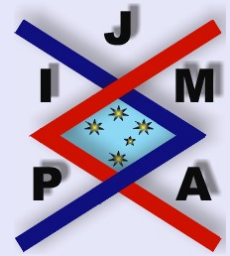
Page 12 of 25

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \quad \text{for } |z| < R_2 \quad \text{and}$$

$$\left| \left[1 + \frac{zf''(z)}{f'(z)} \right] - p \right| \leq p - \delta \quad \text{for } |z| < R_3,$$

respectively.

Remark 1. *The results in Theorems 4.1, 4.2 and 4.3 are sharp with the extremal function f given by (2.4). Furthermore, taking $\delta = 0$ in Theorems 4.1, 4.2 and 4.3, we obtain radius of close-to-convexity, starlikeness and convexity, respectively.*



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 13 of 25

5. Integral Operators

Theorem 5.1. Let c be a real number such that $c > -p$. If $f \in A_o^*(p, A, B, \alpha)$, then the function F defined by

$$(5.1) \quad F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt$$

also belongs to $A_o^*(p, A, B, \alpha)$.

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}.$$

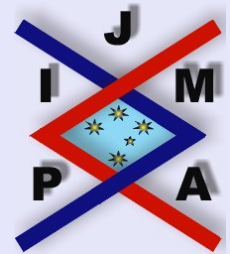
Then from the representation of F , it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

where $b_{p+n} = \left(\frac{c+p}{c+p+n} \right) a_{p+n}$. Therefore using Theorem 2.1 for the coefficients of F , we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} b_{p+n} \\ = \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} \left(\frac{c+p}{c+p+n} \right) a_{p+n} \\ \leq (A-B)(p-\alpha)p! \end{aligned}$$

since $\frac{c+p}{c+p+n} < 1$ and $f \in A_o^*(p, A, B, \alpha)$. Hence $F \in A_o^*(p, A, B, \alpha)$. \square



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 14 of 25

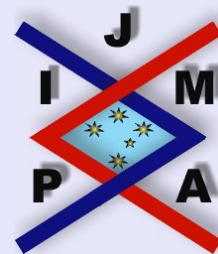
Theorem 5.2. Let c be a real number such that $c > -p$. If $F \in A_o^*(p, A, B, \alpha)$, then the function f defined by (5.1) is p -valent in $|z| < R^*$, where

(5.2) R^*

$$= \inf_n \left\{ \left[\left(\frac{c+p}{c+p+n} \right) \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!(A-B)(p-\alpha)p!} \left(\frac{p}{p+n} \right) \right]^{\frac{1}{n}} \right\}.$$

The result is sharp. Sharpness follows if we take

$$f(z) = z^p - \left(\frac{c+p+n}{c+p} \right) \frac{(n+1)!(A-B)(p-\alpha)p!}{(p+n)! [n(1-B) + (A-B)(p-\alpha)]} z^{p+n}.$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 15 of 25

6. Closure Properties

In this section we show that the class $A_o^*(p, A, B, \alpha)$ is closed under “arithmetic mean” and “convex linear combinations”.

Theorem 6.1. *Let*

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,j} z^{p+n}, \quad j = 1, 2, \dots$$

and

$$h(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

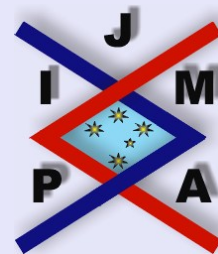
where

$$b_{p+n} = \sum_{j=1}^{\infty} \lambda_j a_{p+n,j}, \quad \lambda_j > 0$$

and $\sum_{j=1}^{\infty} \lambda_j = 1$. If $f_j \in A_o^*(p, A, B, \alpha)$ for each $j = 1, 2, \dots$, then $h \in A_o^*(p, A, B, \alpha)$.

Proof. If $f_j \in A_o^*(p, A, B, \alpha)$, then we have from Theorem 2.1 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n,j} \\ \leq (A-B)(p-\alpha)p!, \quad j = 1, 2, \dots \end{aligned}$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 16 of 25

Therefore

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} b_{p+n} \\ &= \sum_{n=1}^{\infty} \left[\frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} \left(\sum_{j=1}^{\infty} \lambda_j a_{p+n,j} \right) \right] \\ &\leq (A-B)(p-\alpha)p!. \end{aligned}$$

Hence, by Theorem 2.1, $h \in A_o^*(p, A, B, \alpha)$. □

Theorem 6.2. *The class $A_o^*(p, A, B, \alpha)$ is closed under convex linear combinations.*

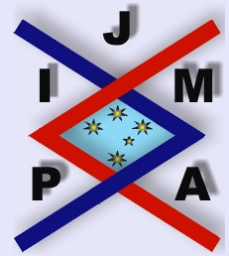
Theorem 6.3. *Let $f_p(z) = z^p$ and*

$$f_{p+n} = z^p - \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)! [n(1-B) + (A-B)(p-\alpha)]} z^{p+n} \quad (n \geq 1).$$

Then $f \in A_o^(p, A, B, \alpha)$ if and only if it can be expressed in the form*

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z), \quad z \in U,$$

where $\lambda_n \geq 0$ and $\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_n$.



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 17 of 25

Proof. Let us assume that

$$\begin{aligned} f(z) &= \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)! [n(1-B) + (A-B)(p-\alpha)]} \lambda_n z^{p+n}. \end{aligned}$$

Then from Theorem 2.1 we have

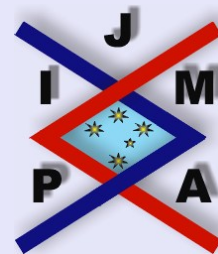
$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} \\ \times \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)! [n(1-B) + (A-B)(p-\alpha)]} \lambda_n \\ \leq (A-B)(p-\alpha)p!. \end{aligned}$$

Hence $f \in A_o^*(p, A, B, \alpha)$. Conversely, let $f \in A_o^*(p, A, B, \alpha)$. It follows from Corollary 2.2 that

$$a_{p+n} \leq \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}.$$

Setting

$$\lambda_n = \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)!p!} a_{p+n}, \quad n = 1, 2, \dots$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

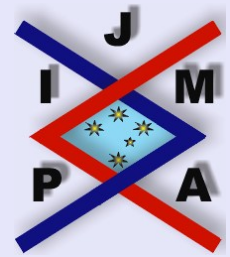
Quit

Page 18 of 25

and $\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_n$, we have

$$\begin{aligned}
 f(z) &= z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \\
 &= z^p - \sum_{n=1}^{\infty} \lambda_n z^p + \sum_{n=1}^{\infty} \lambda_n z^p \\
 &\quad - \sum_{n=1}^{\infty} \lambda_n \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)! [n(1-B) + (A-B)(p-\alpha)]} z^{p+n} \\
 &= \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z).
 \end{aligned}$$

This completes the proof of Theorem 6.3. □



**On A Certain Class Of p -Valent
Functions With Negative
Coefficients**

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 19 of 25

7. Definitions and Applications of Fractional Calculus

In this section, we shall prove several distortion theorems for functions to general class $A_o^*(p, A, B, \alpha)$. Each of these theorems would involve certain operators of fractional calculus we find it to be convenient to recall here the following definition which were used recently by Owa [2] (and more recently, by Owa and Srivastava [3], and Srivastava and Owa [4] ; see also Srivastava et al. [5]).

Definition 7.1. *The fractional integral of order λ is defined, for a function f , by*

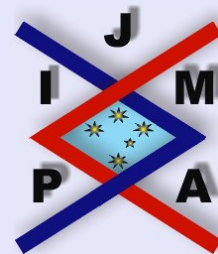
$$(7.1) \quad D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0),$$

where f is an analytic function in a simply – connected region of the z -plane containing the origin, and the multiplicity of $(z - \zeta)^{\lambda-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Definition 7.2. *The fractional derivative of order λ is defined, for a function f , by*

$$(7.2) \quad D_z^\lambda f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where f is constrained, and the multiplicity of $(z - \zeta)^{-\lambda}$ is removed, as in Definition 7.1.



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 20 of 25

Definition 7.3. Under the hypotheses of Definition 7.2, the fractional derivative of order $(n + \lambda)$ is defined by

$$(7.3) \quad D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z) \quad (0 \leq \lambda < 1),$$

where $0 \leq \lambda < 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. From Definition 7.2, we have

$$(7.4) \quad D_z^0 f(z) = f(z)$$

which, in view of Definition 7.3 yields,

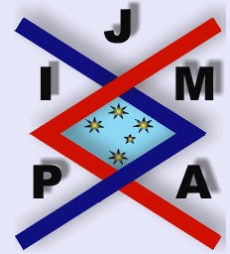
$$(7.5) \quad D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^n(z).$$

Thus, it follows from (7.4) and (7.5) that

$$\lim_{\lambda \rightarrow 0} D_z^{-\lambda} f(z) = f(z) \quad \text{and} \quad \lim_{\lambda \rightarrow 0} D_z^{1-\lambda} f(z) = f'(z).$$

Theorem 7.1. Let the function f defined by (1.2) be in the class $A_o^*(p, A, B, \alpha)$. Then for $z \in U$ and $\lambda > 0$,

$$\left| D_z^{-\lambda} f(z) \right| \geq |z|^{p+\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} - \frac{2(A-B)(p-\alpha)\Gamma(p+1)}{(\lambda+p+1)\Gamma(\lambda+p+1)[(1-B) + (A-B)(p-\alpha)]} |z| \right\}$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 21 of 25

and

$$\left| D_z^{-\lambda} f(z) \right| \leq |z|^{p+\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} + \frac{2(A-B)(p-\alpha)\Gamma(p+1)}{(\lambda+p+1)\Gamma(\lambda+p+1)[(1-B) + (A-B)(p-\alpha)]} |z| \right\}.$$

The result is sharp.

Proof. Let

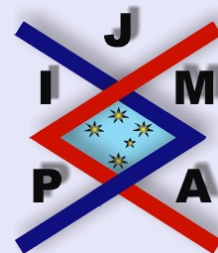
$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)} a_{p+n} z^{p+n} \\ &= z^p - \sum_{n=1}^{\infty} \varphi(n) a_{p+n} z^{p+n}, \end{aligned}$$

where

$$\varphi(n) = \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)}, \quad (\lambda > 0, n \in \mathbb{N}).$$

Then by using $0 < \varphi(n) \leq \varphi(1) = \frac{p+1}{p+\lambda+1}$ and Theorem 2.1, we observe that

$$\frac{(p+1)! [(1-B) + (A-B)(p-\alpha)]}{2!} \sum_{n=1}^{\infty} a_{p+n}$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 22 of 25

$$\begin{aligned} &\leq \sum_{n=1}^{\infty} \frac{(p+n)! [n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \\ &\leq (A-B)(p-\alpha)p!, \end{aligned}$$

which shows that $F(z) \in A_o^*(p, A, B, \alpha)$. Consequently, with the aid of Theorem 3.1, we have

$$\begin{aligned} |F(z)| &\geq |z^p| - \varphi(1) |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\geq |z|^p - \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B) + (A-B)(p-\alpha)]} |z|^{p+1} \end{aligned}$$

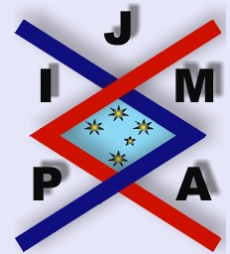
and

$$\begin{aligned} |F(z)| &\leq |z^p| + \varphi(1) |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\leq |z|^p + \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B) + (A-B)(p-\alpha)]} |z|^{p+1} \end{aligned}$$

which completes the proof of Theorem 7.1. By letting $\lambda \rightarrow 0$, Theorem 7.1 reduces at once to Theorem 3.1. \square

Corollary 7.2. *Under the hypotheses of Theorem 7.1, $D_z^{-\lambda} f(z)$ is included in a disk with its center at the origin and radius $R_1^{-\lambda}$ given by*

$$R_1^{-\lambda} = \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} \right\} \left\{ 1 + \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B) + (A-B)(p-\alpha)]} \right\}.$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 23 of 25

Theorem 7.3. Let the function f defined by (1.2) be in the class $A_o^*(p, A, B, \alpha)$. Then,

$$|D_z^\lambda f(z)| \geq |z|^{p-\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} - \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)\Gamma(p+1)}{\Gamma(p-\lambda+1)\Gamma(p-\lambda+2)[(1-B) + (A-B)(p-\alpha)]} |z| \right\}$$

and

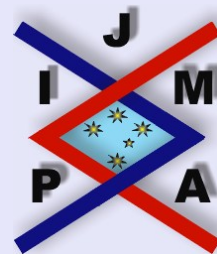
$$|D_z^\lambda f(z)| \leq |z|^{p-\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} + \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)\Gamma(p+1)}{\Gamma(p-\lambda+1)\Gamma(p-\lambda+2)[(1-B) + (A-B)(p-\alpha)]} |z| \right\}$$

for $0 \leq \lambda < 1$.

Proof. Using similar arguments as given by Theorem 7.1, we can get the result. \square

Corollary 7.4. Under the hypotheses of Theorem 7.3, $D_z^\lambda f(z)$ is included in the disk with its center at the origin and radius R_2^λ given by

$$R_2^\lambda = \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} \right\} \left\{ 1 + \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)}{\Gamma(p-\lambda+1)[(1-B) + (A-B)(p-\alpha)]} \right\}.$$



On A Certain Class Of p -Valent Functions With Negative Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

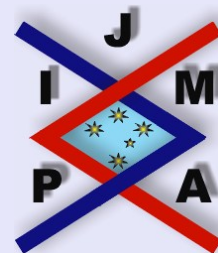
Close

Quit

Page 24 of 25

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On A Certain Class Of p -Valent
Functions With Negative
Coefficients

H.Ö. Güney and S. Sümer Eker

Title Page

Contents



Go Back

Close

Quit

Page 25 of 25