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# UPPER AND LOWER SOLUTIONS METHOD FOR DISCRETE INCLUSIONS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. In this note the concept of lower and upper solutions combined with the nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions for first order discrete inclusions with nonlinear boundary conditions.

*Key words and phrases:* Discrete Inclusions, Convex valued multivalued map, Fixed point, Upper and lower solutions, Nonlinear boundary conditions.

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## 1. INTRODUCTION

This note is concerned with the existence of solutions for the discrete boundary multivalued problem

(1.1) $$\Delta y(i-1) \in F(i, y(i)), \qquad i \in [1, T] = \{1, 2, \dots, T\},$$

(1.2) $$L(y(0), y(T+1)) = 0,$$

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where $F : \mathbb{N} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a compact convex valued multivalued map and $L : \mathbb{N}^2 \to \mathbb{R}$ is a nonlinear single-valued map.

Very recently Agarwal *et al* [3] applied the concept of upper and lower solutions combined with the Leray-Schauder nonlinear alternative to a class of second order discrete inclusions subjected to Dirichlet conditions. For more details on recent results and applications of difference equations we recommend for instance the monographs by Agarwal *et al* [1], [2], Pachpatte [9] and the references cited therein.

In this note we shall apply the same tool as in [3] to first order discrete inclusions with nonlinear boundary conditions which include the initial, terminal and periodic conditions. The corresponding problem for differential inclusions was studied by Benchohra and Ntouyas in [4].

## 2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout the note. $C([0,T],\mathbb{R})$ is the Banach space of all continuous functions from $[0,T]$ (discrete topology) into $\mathbb{R}$ with the norm $\|y\| = \sup_{k\in[0,T]} |y(k)|$. Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow \mathcal{P}(X)$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. $G$ is bounded on bounded sets if $G(B)$ is bounded in $X$ for each bounded set $B$ of $X$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

$G$ is called upper semicontinuous (u.s.c.) on $X$ if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighbourhood $M$ of $x_0$ such that $G(M) \subseteq N$. $G$ is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). $G$ has a fixed point if there is $x \in X$ such that $x \in G(x)$.

For more details on multivalued maps see the books of Deimling [5] and Hu and Papageorgiou [7].

Let us start by defining what we mean by a solution of problem (1.1) – (1.2).

**Definition 2.1.** A function $y \in C([0,T],\mathbb{R})$, is said to be a solution of (1.1) – (1.2) if $y$ satisfies the inclusion $\Delta y(i-1) \in F(i, y(i))$ on $\{1,\dots,T\}$ and the condition $L(y(0), y(T+1)) = 0$.

For any $y \in C([0,T],\mathbb{R})$ we define the set

$$S_{F,y} = \{v \in C([0,T],\mathbb{R}) :\ v(i) \in F(i,y(i)) \text{ for } i \in \{1,\dots,T\}\}.$$

**Definition 2.2.** A function $\alpha \in C([0,T+1],\mathbb{R})$ is said to be a lower solution of (1.1) – (1.2) if for each $i \in [0,T+1]$ there exists $v_1(i) \in F(i,\alpha(i))$ with $\Delta\alpha(i-1) \le v_1(i)$ and $L(\alpha(0),\alpha(T+1)) \le 0$.

Similarly a function $\beta \in C([0,T+1],\mathbb{R})$ is said to be an upper solution of (1.1) – (1.2) if for each $i \in [0,T+1]$ there exists $v_2(i) \in F(i,\beta(i))$ with $\Delta\beta(i-1) \ge v_2(i)$ and $L(\beta(0),\beta(T+1)) \ge 0$.

Our existence result in the next section relies on the following fixed point principle.

**Lemma 2.1** (Nonlinear Alternative [6]). *Let $X$ be a Banach space with $C \subset X$ convex. Assume $U$ is an open subset of $C$ with $0 \in U$ and $G : \overline{U} \to \mathcal{P}(C)$ is a compact multivalued map, u.s.c. with convex closed values. Then either,*

(i) *$G$ has a fixed point in $\overline{U}$; or*
(ii) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

## 3. MAIN RESULT

We are now in a position to state and prove our existence result for the problem (1.1) – (1.2). We first list the following hypotheses:

(H1) $y \longmapsto F(i, y)$ is upper semicontinuous for all $i \in [1, T]$;
(H2) for each $q > 0$, there exists $\phi_q \in C([1, T], \mathbb{R}_+)$ such that

$$\|F(i, y)\| = \sup\{|v| : v \in F(i, y)\} \leq \phi_q(i) \quad \text{for all } |y| \leq q \text{ and } i \in [1, T];$$

(H3) there exist $\alpha$ and $\beta \in C([0, T+1], \mathbb{R})$, lower and upper solutions for the problem (1.1) – (1.2) such that $\alpha \leq \beta$;
(H4) $L$ is a continuous single-valued map in $(x, y) \in [\alpha(0), \beta(0)] \times [\alpha(T+1), \beta(T+1)]$ and nonincreasing in $y \in [\alpha(T+1), \beta(T+1)]$.

**Theorem 3.1.** *Assume that hypotheses (H1) – (H4) hold. Then the problem (1.1) – (1.2) has at least one solution $y$ such that*

$$\alpha(i) \leq y(i) \leq \beta(i) \text{ for all } i \in [1, T].$$

*Proof.* Transform the problem (1.1) – (1.2) into a fixed point problem. Consider the following modified problem

$$\Delta y(i-1) + y(i) \in F_1(i, y(i)), \quad \text{on } [1, T] \tag{3.1}$$

$$y(0) = \tau(0, y(0) - L(\overline{y}(0), \overline{y}(T+1)), \tag{3.2}$$

where

$$F_1(i, y) = F(i, \tau(i, y)) + \tau(i, y),$$

$$\tau(i, y) = \max(\alpha(i), \min(y, \beta(i))$$

and

$$\overline{y}(i) = \tau(i, y).$$

A solution to (3.1) – (3.2) is a fixed point of the operator $N : C([1, T], \mathbb{R}) \longrightarrow \mathcal{P}(C([1, T], \mathbb{R}))$ defined by:

$$N(y) = \left\{ h \in C([1, T]) : h(k) = y(0) + \sum_{0<l<k} [g(l) + \overline{y}(l)] - \sum_{0<l<k} y(l), \ \ g \in \tilde{S}^1_{F,\overline{y}} \right\},$$

where

$$\tilde{S}^1_{F,\overline{y}} = \{v \in S^1_{F,\overline{y}} : v(i) \geq v_1(i) \text{ a.e. on } A_1 \text{ and } v(i) \leq v_2(i) \text{ on } A_2\},$$

$$S^1_{F,\overline{y}} = \{v \in C([1, T]) : v(i) \in F(i, (\overline{y})(i)) \text{ for } i \in [1, T]\},$$

$$A_1 = \{i \in [1, T] : y(i) < \alpha(i) \leq \beta(i)\}, \ \ A_2 = \{i \in [1, T] : \alpha(i) \leq \beta(i) < y(i)\}.$$

**Remark 3.2.** Notice that $F_1$ is an upper semicontinuous multivalued map with compact convex values, and there exists $\phi \in C([1, T], \mathbb{R}^+)$ such that

$$\|F_1(i, y)\| \leq \phi(i) + \max\left(\sup_{i\in[1,T]} |\alpha(i)|, \sup_{i\in[1,T]} |\beta(i)|\right).$$

We shall show that $N$ satisfies the assumptions of Lemma 2.1. The proof will be given in several steps.

**Step 1**: *$N(y)$ is convex for each $y \in C([1,T],\mathbb{R})$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in \tilde{S}^1_{F,\overline{y}}$ such that for each $k \in [1,T]$ we have

$$h_i(k) = y(0) + \sum_{0<l<k}[g_i(l) + \overline{y}(l)] - \sum_{0<l<k} y(l), \quad i = 1,2.$$

Let $0 \le d \le 1$. Then for each $k \in [1,T]$ we have

$$(dh_1 + (1-d)h_2)(k) = y(0) + \sum_{0<l<k}[dg_1(l) + (1-d)g_2(l) + \overline{y}(l)] - \sum_{0<l<k} y(l).$$

Since $\tilde{S}^1_{F_1,\overline{y}}$ is convex (because $F_1$ has convex values) then

$$dh_1 + (1-d)h_2 \in N(y).$$

**Step 2**: *$N$ maps bounded sets into bounded sets in $C([1,T],\mathbb{R})$.*

Indeed, it is enough to show that for each $q > 0$ there exists a positive constant $\ell^*$ such that for each $y \in B_q = \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty \le q\}$ one has $\|N(y)\|_\infty \le \ell^*$.

Let $y \in B_q$ and $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\overline{y}}$ such that for each $k \in [1,T]$ we have

$$h(k) = y(0) + \sum_{0<l<k}[g(l) + \overline{y}(l)] - \sum_{0<l<k} y(l).$$

By (H2) we have for each $i \in [1,T]$

$$\begin{aligned}|h(k)| \le |y(0)| + \sum_{l=1}^{k}|g(l)| + \sum_{l=1}^{k}|\overline{y}(l)| + \sum_{l=1}^{k}|y(l)|&\\ \le \max(|\alpha(0)|,|\beta(0)|) + k\|\phi_q\|_\infty&\\ + k\max\left(q, \sup_{i\in[1,T]}|\alpha(i)|, \sup_{i\in[1,T]}|\beta(i)|\right) + kq := \ell^*.&\end{aligned}$$

**Step 3**: *$N$ maps bounded set into equicontinuous sets of $C([1,T],\mathbb{R})$.*

Let $k_1, k_2 \in [1,T],\ k_1 < k_2$ and $B_q$ be a bounded set of $C([1,T])$ as in Step 2. Let $y \in B_q$ and $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\overline{y}}$ such that for each $k \in [1,T]$ we have

$$h(k) = y(0) + \sum_{0<l<k}[g(l) + \overline{y}(l)] - \sum_{0<l<k} y(l).$$

Then

$$|h(k_2) - h(k_1)| \le \sum_{k_1<l<k_2}[|g(l)| + |\overline{y}(l)|] + \sum_{k_1<l<k_2}|y(l)|.$$

As $k_2 \longrightarrow k_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem we can conclude that $N : C([1,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([1,T],\mathbb{R}))$ is a completely continuous multivalued map.

**Step 4:** *A priori bounds on solutions exist.*

Let $y \in C([1,T],\mathbb{R})$ and $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then

$$y(k) = \lambda\left(y(0) - \sum_{0<l<k} y(l) + \sum_{0<l<k}[g(l) + \overline{y}(l)]\right).$$

Hence

$$\begin{aligned}|y(k)| \le |y(0)| + \sum_{l=1}^{k}|g(l)| + \sum_{l=1}^{k}|\bar{y}(l)| + \sum_{l=1}^{k}|y(l)| \\ \le \max(|\alpha(0)|,|\beta(0)|) + T\|\phi\|_\infty \\ + T\max\left(\sup_{i\in[1,T]}|\alpha(i)|, \sup_{i\in[1,T]}|\beta(i)|\right) + 2\sum_{l=1}^{k}|y(l)|.\end{aligned}$$

Using the Pachpatte inequality (see [9, Theorem 2.5]) we get for each $k \in [1,T]$

$$|y(k)| \le c_*\left[1 + 2\sum_{l=1}^{T}\prod_{s=1}^{l-1} 2\right],$$

where

$$c_* = \max(|\alpha(0)|,|\beta(0)|) + T\|\phi\|_\infty + T\max\left(\sup_{i\in[1,T]}|\alpha(i)|, \sup_{i\in[1,T]}|\beta(i)|\right).$$

Thus

$$\|y\|_\infty \le c_*(1 + T2^{T+1}) := M.$$

Set

$$U = \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty < M+1\}.$$

As in Step 3 the operator $N : \overline{U} \longrightarrow \mathcal{P}(C([1,T],\mathbb{R}))$ is continuous and completely continuous.

**Step 5**: *$N$ has a closed graph.*

Let $y_n \in U \longrightarrow y_*,\ h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in \tilde{S}^1_{F,\overline{y}_n}$ such that for each $t \in J$

$$h_n(i) = y_n(0) + \sum_{0<l<i}[g_n(l) + \overline{y}_n(l)] - \sum_{0<l<i} y_n(l).$$

We must prove that there exists $g_* \in \tilde{S}^1_{F,\overline{y}_*}$ such that for each $k \in [1,T]$

$$h_*(i) = y_*(0) + \sum_{0<l<i} g_*(l) + \overline{y}_*(l)] - \sum_{0<l<i} y_*(l).$$

Since $y_n \in \overline{U},\ k \in \mathbb{N}$, then (H2) guarantees (see [2, p. 262]) that there exists a compact set $\Omega$ of $C([1,T],\mathbb{R})$ with $\{g_n\} \in \Omega$. Thus there exists a subsequence $\{y_{n_m}\}$ with $y_{n_m} \to y_*$ as $k \to \infty$ and $y_{n_m}(i) \in F(i, y_m(i))$ together with the map $y \to F(i,y)$ upper semicontinuous for each $i \in \mathbb{N}$. Since $\tau$ and $y$ are continuous, we have

$$\left\|\left(h_n - y_n(0) - \sum_{0<l<i}[\overline{y}_n(l) - y_n(l)]\right) - \left(h_* - y_*(0)\sum_{0<l<i}[\overline{y}_*(l) - y_*(l)]\right)\right\|_\infty \longrightarrow 0, \ \text{as } n \to \infty.$$

Consider the linear continuous operator (topology on $\mathbb{N}$ is the discrete topology)

$$\Gamma : C([1,T],\mathbb{R}) \longrightarrow C([1,T],\mathbb{R})$$
$$g \longmapsto (\Gamma g)(i) = \sum_{0<l<i} g(l).$$

Moreover, we have that

$$\left(h_n(i) - y_n(0) - \sum_{0<l<i} [\overline{y}_n(l) - y_n(l)]\right) = \Gamma(g_n)(i) \in F_1(i, y_n(i)).$$

Since $y_n \longrightarrow y_*$, it that

$$\left(h_*(i) - y_*(0) - \sum_{0<l<i} [\overline{y}_*(l) - y_*(l)\right) = \sum_{0<l<i} g_*(l)$$

for some $g_* \in \tilde{S}^1_{F,y_*}$.

Lemma 2.1 guarantees that $N$ has a fixed point which is a solution to problem (3.1) – (3.2).

**Step 6:** *The solution $y$ of (3.1) – (3.2) satisfies*

$$\alpha(i) \le y(i) \le \beta(i) \text{ for all } i \in J.$$

Let $y$ be a solution to (3.1) – (3.2). We prove that

$$y(i) \le \beta(i) \text{ for all } i \in [1,T].$$

Assume that $y-\beta$ attains a positive maximum on $[1,T]$ at $\overline{k}-1 \in [1,T]$ that is,

$$(y-\beta)(\overline{k}) = \max\{y(k) - \beta(k) : k \in [1,T]\} > 0.$$

By the definition of $\tau$ one has

$$\Delta y(\overline{k}) + y(\overline{k}) \in F(t, \beta(\overline{k})) + \beta(\overline{k}).$$

Thus there exists $v(i) \in F(\overline{k}, \beta(\overline{k}))$, with $v(\overline{k}) \le v_2(\overline{k})$ such that

$$\Delta y(\overline{k}-1) = v(\overline{k}) + \beta(\overline{k}-1) - y(\overline{k}),$$

$$\begin{aligned}\Delta y(\overline{k}-1) &= v(\overline{k}) - y(\overline{k}) + \beta(\overline{k})\\ &\le v_2(\overline{k}) - (y(\overline{k}) - \beta(\bar{k})) < v_2(\overline{k}).\end{aligned}$$

Using the fact that $\beta$ is an upper solution to (1.1) – (1.2) the above inequality yields

$$\begin{aligned}\beta(\overline{k}) - \beta(\overline{k}-1) &\ge v_2(\overline{k})\\ &> y(\overline{k}) - y(\overline{k}-1).\end{aligned}$$

Thus we obtain the contradiction

$$y(\overline{k}-1) - \beta(\overline{k}-1) > y(\overline{k}) - \beta(\overline{k}).$$

Thus

$$y(i) \le \beta(i) \text{ forall } i \in [1,T].$$

Analogously, we can prove that

$$y(i) \ge \alpha(i) \text{ for all } i \in [1,T].$$

This shows that the problem (3.1) – (3.2) has a solution in the interval $[\alpha, \beta]$.

Finally, we prove that every solution of (3.1) – (3.2) is also a solution to (1.1) – (1.2). We only need to show that

$$\alpha(0) \le y(0) - L(\overline{y}(0), \overline{y}(T+1)) \le \beta(0).$$

Notice first that we can prove

$$\alpha(T+1) \leq y(T+1) \leq \beta(T+1).$$

Suppose now that $y(0) - L(\overline{y}(0), \overline{y}(T+1)) < \alpha(0)$. Then $y(0) = \alpha(0)$ and

$$y(0) - L(\alpha(0), \overline{y}(T)) \leq \alpha(0).$$

Since $L$ is nonincreasing in $y$, we have

$$\alpha(0) \leq \alpha(0) - L(\alpha(0), \alpha(T+1)) \leq \alpha(0) - L(\alpha(0), \overline{y}(T+1)) < \alpha(0),$$

which is a contradiction. Analogously, we can prove that

$$y(0) - L(\overline{y}(0), \overline{y}(T+1)) \leq \beta(0).$$

Then $y$ is a solution to (1.1) – (1.2). □

**Remark 3.3.** Observe that if $L(x, y) = ax - by - c$, then Theorem 3.1 gives an existence result for the problem

$$\Delta y(i) \in F(i, y(i)), \qquad i \in [1, T] = \{1, 2, \dots, T\},$$
$$ay(0) - by(T) = c,$$

with $a,\ b \geq 0,\ a + b > 0$, which includes the periodic case $(a = b = 1,\ c = 0)$ and the initial and the terminal problem.

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