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INTEGRAL MEANS INEQUALITIES FOR FRACTIONAL DERIVATIVES OF SOME GENERAL SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Integral means inequalities are obtained for the fractional derivatives of order $p+\lambda$ $(0 \leqq p \leqq n;\ 0 \leqq \lambda < 1)$ of functions belonging to certain general subclasses of analytic functions. Relevant connections with various known integral means inequalities are also pointed out.

Key words and phrases: Integral means inequalities, Fractional derivatives, Analytic functions, Univalent functions, Extreme points, Subordination.

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1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

that are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{A}(n)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

We denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ of functions which are *univalent* in $\mathbb{U}$, and by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the subclasses of $\mathcal{T}(n)$ consisting of functions which are, respectively, *starlike of order* α $(0 \leqq \alpha < 1)$ and *convex of order* α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$. The classes $\mathcal{A}(n),\ \mathcal{T}(n),\ \mathcal{T}_\alpha(n),$ and $\mathcal{C}_\alpha(n)$ were investigated by Chatterjea [1] (and Srivastava *et al.* [9]). In particular, the following subclasses:

$$\mathcal{T} := \mathcal{T}(1), \quad \mathcal{T}^*(\alpha) := \mathcal{T}_\alpha(1), \quad \text{and} \quad \mathcal{C}(\alpha) := \mathcal{C}_\alpha(1)$$

were considered earlier by Silverman [7].

Next, following the work of Sekine and Owa [4], we denote by $\mathcal{A}(n, \vartheta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\, a_k z^k \quad (\vartheta \in \mathbb{R};\ a_k \geqq 0;\ n \in \mathbb{N}). \tag{1.1}$$

Finally, the subclasses $\mathcal{T}(n, \vartheta),\ \mathcal{T}^*_\alpha(n, \vartheta),$ and $\mathcal{C}_\alpha(n, \vartheta)$ of the class $\mathcal{A}(n, \vartheta)$ are defined in the same way as the subclasses $\mathcal{T}(n),\ \mathcal{T}_\alpha(n),$ and $\mathcal{C}_\alpha(n)$ of the class $\mathcal{A}(n)$.

We begin by recalling the following useful characterizations of the function classes $\mathcal{T}^*_\alpha(n, \vartheta)$ and $\mathcal{C}_\alpha(n, \vartheta)$ (see Sekine and Owa [4]).

Lemma 1.1. *A function* $f(z) \in \mathcal{A}(n, \vartheta)$ *of the form* (1.1) *is in the class* $\mathcal{T}^*_\alpha(n, \vartheta)$ *if and only if*

$$\sum_{k=n+1}^{\infty} (k - \alpha)\ a_k \leqq 1 - \alpha \quad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.2}$$

Lemma 1.2. *A function* $f(z) \in \mathcal{A}(n, \vartheta)$ *of the form* (1.1) *is in the class* $\mathcal{C}_\alpha(n, \vartheta)$ *if and only if*

$$\sum_{k=n+1}^{\infty} k(k - \alpha)\ a_k \leqq 1 - \alpha \quad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.3}$$

Motivated by the equalities in (1.2) and (1.3) above, Sekine *et al.* [6] defined a general subclass $\mathcal{A}(n; B_k, \vartheta)$ of the class $\mathcal{A}(n, \vartheta)$ consisting of functions $f(z)$ of the form (1.1), which satisfy the following inequality:

$$\sum_{k=n+1}^{\infty} B_k a_k \leqq 1 \quad (B_k > 0;\ n \in \mathbb{N}).$$

Thus it is easy to verify each of the following classifications and relationships:

$$\mathcal{A}(n; k, \vartheta) = \mathcal{T}^*_0(n, \vartheta) =: \mathcal{T}^*(n, \vartheta) = \mathcal{T}(n, \vartheta),$$

$$\mathcal{A}\left(n;\frac{k-\alpha}{1-\alpha},\vartheta\right)=\mathcal{T}_{\alpha}^{*}\left(n,\vartheta\right)\quad(0\leqq\alpha<1),$$

and

$$\mathcal{A}\left(n;\frac{k\left(k-\alpha\right)}{1-\alpha},\vartheta\right)=\mathcal{C}_{\alpha}\left(n,\vartheta\right)\quad\left(0\leqq\alpha<1\right).$$

As a matter of fact, Sekine *et al*. [6] also obtained each of the following basic properties of the general classes $\mathcal{A}\left(n;B_k,\vartheta\right)$.

Theorem 1.3. *$\mathcal{A}\left(n;B_k,\vartheta\right)$ is the convex subfamily of the class $\mathcal{A}\left(n,\vartheta\right)$.*

Theorem 1.4. *Let*

$$f_1\left(z\right)=z\quad\textit{and}\quad f_k\left(z\right)=z-\frac{e^{i(k-1)\vartheta}}{B_k}z^k \tag{1.4}$$
$$\left(k=n+1,n+2,n+3,\ldots;\ n\in\mathbb{N}\right).$$

Then $f\in\mathcal{A}\left(n;B_k,\vartheta\right)$ if and only if $f\left(z\right)$ can be expressed as follows:

$$f\left(z\right)=\lambda_1f_1\left(z\right)+\sum_{k=n+1}^{\infty}\lambda_kf_k(z),$$

where

$$\lambda_1+\sum_{k=n+1}^{\infty}\lambda_k=1\quad\left(\lambda_1\geqq0;\ \lambda_k\geqq0;\ n\in\mathbb{N}\right).$$

Corollary 1.5. *The extreme points of the class $\mathcal{A}\left(n;B_k,\vartheta\right)$ are the functions $f_1\left(z\right)$ and $f_k\left(z\right)$ $\left(k\geqq n+1;\ n\in\mathbb{N}\right)$ given by* (1.4).

Applying the concepts of extreme points, fractional calculus, and subordination, Sekine *et al.* [6] obtained several integral means inequalities for higher-order fractional derivatives and fractional integrals of functions belonging to the general classes $\mathcal{A}(n;B_k,\vartheta)$. Subsequently, Sekine and Owa [5] discussed the weakening of the hypotheses for B_k in those results by Sekine *et al.* [6]. In this paper, we investigate the integral means inequalities for the fractional derivatives of $f(z)$ of a general order $p+\lambda$ $(0\leqq p\leqq n;\ 0\leqq\lambda<1)$ of functions $f(z)$ belonging to the general classes $\mathcal{A}(n;B_k,\vartheta)$.

We shall make use of the following definitions of fractional derivatives (*cf.* Owa [3]; see also Srivastava and Owa [8]).

Definition 1.1. The *fractional derivative of order* λ is defined, for a function $f\left(z\right)$, by

$$D_z^{\lambda}f\left(z\right):=\frac{1}{\Gamma\left(1-\lambda\right)}\frac{d}{dz}\int_0^z\frac{f\left(\zeta\right)}{\left(z-\zeta\right)^{\lambda}}\,d\zeta\quad(0\leqq\lambda<1), \tag{1.5}$$

where the function $f\left(z\right)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $\left(z-\zeta\right)^{-\lambda}$ is removed by requiring $\log\left(z-\zeta\right)$ to be real when $z-\zeta>0$.

Definition 1.2. Under the hypotheses of Definition 1.1, the *fractional derivative of order* $n+\lambda$ is defined, for a function $f\left(z\right)$, by

$$D_z^{n+\lambda}f\left(z\right):=\frac{d^n}{dz^n}D_z^{\lambda}f\left(z\right)\quad\left(0\leqq\lambda<1;\ n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right).$$

It readily follows from (1.5) in Definition 1.1 that

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \qquad (0 \leqq \lambda < 1). \tag{1.6}$$

We shall also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2] in our investigation.

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, the function $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Theorem 1.6 (Littlewood [2]). *If the functions $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with*

$$g(z) \prec f(z),$$

then

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \qquad (\mu > 0;\ 0 < r < 1).$$

2. THE MAIN INTEGRAL MEANS INEQUALITIES

Theorem 2.1. *Suppose that $f(z) \in \mathcal{A}(n; k^{p+1}B_k, \vartheta)$ and that*

$$\frac{(h+1)^q B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \leqq B_k \quad (k \geqq n+1)$$

for some $h \geqq n$, $0 \leqq \lambda < 1$, and $0 \leqq q \leqq p \leqq n$. Also let the function $f_{h+1}(z)$ be defined by

$$f_{h+1}(z) = z - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} z^{h+1} \quad \left(f_{h+1} \in A\left(n; k^{q+1}B_k, \vartheta\right)\right). \tag{2.1}$$

Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{p+\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{p+\lambda} f_{h+1}(z)\right|^{\mu} d\theta \qquad (0 \leqq \lambda < 1;\ \mu > 0). \tag{2.2}$$

Proof. By virtue of the fractional derivative formula (1.6) and Definition 1.2, we find from (1.1) that

$$\begin{aligned} D_z^{p+\lambda} f(z) &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} a_k z^{k-1}\right) \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2-\lambda-p) \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1}\right), \end{aligned}$$

where

$$\Phi(k) := \frac{\Gamma(k-p)}{\Gamma(k+1-\lambda-p)} \qquad (0 \leqq \lambda < 1;\ k \geqq n+1;\ n \in \mathbb{N}). \tag{2.3}$$

Since $\Phi(k)$ is a *decreasing* function of k, we have

$$0 < \Phi(k) \leqq \Phi(n+1) = \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)}$$
$$(0 \leqq \lambda < 1;\ k \geqq n+1;\ n \in \mathbb{N}).$$

Similarly, from (2.1), (1.6), and Definition 1.2, we obtain, for $0 \leqq \lambda < 1$,

$$D_z^{p+\lambda} f_{h+1}(z) = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)}\ z^h\right).$$

For $z = re^{i\theta}$ and $0 < r < 1$, we must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1}\right|^{\mu} d\theta$$
$$\leqq \int_0^{2\pi} \left|1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)}\ z^h\right|^{\mu} d\theta, \quad (0 \leqq \lambda < 1;\ \mu > 0).$$

Thus, by applying Theorem 1.6, it would suffice to show that

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1} \prec 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)}\ z^h. \tag{2.4}$$

Indeed, by setting

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1} = 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)}\{w(z)\}^h,$$

we find that

$$\{w(z)\}^h = \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{e^{ih\vartheta}\Gamma(h+2)} \cdot \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1},$$

which readily yields $w(0) = 0$.

Therefore, we have

$$\begin{aligned}
&|w(z)|^h \\
&\leqq \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}\Phi(k)a_k|z|^{k-1} \\
&\leqq |z|^n \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \cdot \Phi(n+1) \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}a_k \\
&= |z|^n \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}a_k \\
&= |z|^n \frac{(h+1)^{q}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}a_k
\end{aligned}$$

$$\leqq |z|^n \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} B_k a_k$$

$$\leqq |z|^n \sum_{k=n+1}^{\infty} k^{p+1} B_k a_k \leqq |z|^n < 1 \quad (n \in \mathbb{N}), \tag{2.5}$$

by means of the hypothesis of Theorem 2.1.

In light of the last inequality in (2.5) above, we have the subordination (2.4), which evidently proves Theorem 2.1. □

3. REMARKS AND OBSERVATIONS

First of all, in its special case when $p = q = 0$, Theorem 2.1 readily yields

Corollary 3.1 (cf. Sekine and Owa [5], Theorem 6)**.** *Suppose that* $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ *and that*

$$\frac{B_{h+1}\Gamma(h+2-\lambda)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+2-\lambda)} \leqq B_k \quad (k \geqq n+1;\ n \in \mathbb{N})$$

for some $h \geqq n$ *and* $0 \leqq \lambda < 1$*. Also let the function* $f_{h+1}(z)$ *be defined by*

$$f_{h+1}(z) = z - \frac{e^{ih\vartheta}}{(h+1)B_{h+1}}\, z^{h+1} \quad \left(f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)\right). \tag{3.1}$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{\lambda} f_{h+1}(z)\right|^{\mu} d\theta \quad (0 \leqq \lambda < 1;\ \mu > 0). \tag{3.2}$$

A *further* consequence of Corollary 3.1 when $h = n$ would lead us immediately to Corollary 3.2 below.

Corollary 3.2. *Suppose that* $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ *and that*

$$B_{n+1} \leqq B_k \quad (k \geqq n+1;\ n \in \mathbb{N}). \tag{3.3}$$

Also let the function $f_{n+1}(z)$ *be defined by*

$$f_{n+1}(z) = z - \frac{e^{in\vartheta}}{(n+1)B_{n+1}}\, z^{n+1} \quad \left(f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)\right).$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{\lambda} f_{n+1}(z)\right|^{\mu} d\theta \quad (0 \leqq \lambda < 1;\ \mu > 0).$$

The hypothesis (3.3) in Corollary 3.2 is weaker than the corresponding hypothesis in an earlier result of Sekine *et al.* [6, p. 953, Theorem 6].

Next, for $p = 1$ and $q = 0$, Theorem 2.1 reduces to an integral means inequality of Sekine and Owa [5, Theorem 7] which, for $h = n$, yields another result of Sekine *et al.* [6, p. 953, Theorem 7] under weaker hypothesis as mentioned above.

Finally, by setting $p = q = 1$ in Theorem 2.1, we obtain a slightly improved version of another integral means inequalities of Sekine and Owa [5, Theorem 8] with respect to the parameter λ (see also Sekine *et al.* [6, p. 955, Theorem 8] for the case when $h = n$, just as we remarked above).

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