Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 3, Article 66, 2004

IMPROVEMENTS OF EULER-TRAPEZOIDAL TYPE INEQUALITIES WITH HIGHER-ORDER CONVEXITY AND APPLICATIONS

DAH-YAN HWANG

DEPARTMENT OF GENERAL EDUCATION
KUANG WU INSTITUTE OF TECHNOLOGY
PEITO, TAIPEI, TAIWAN 11271, R.O.C.
dyhuang@mail.apol.com.tw

Received 28 May, 2003; accepted 12 April, 2004
Communicated by C.E.M. Pearce

ABSTRACT. We improve a bounds relating to Euler's formula for the case of a function with higher-order convexity properties. These are used to improve estimates of the error involved in the use of the trapezoidal formula for integrating such a function.

Key words and phrases: Hadamard inequality, Euler formula, Convex functions, Integral inequalities, Numerical integration.

2000 *Mathematics Subject Classification.* 26D15, 26D20.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mappings f. Over the last decade this pair of inequalities (1.1) have been improved and extended in a number of ways, including the derivation of estimates of the differences between the two sides of each inequality.

In [4], Dragomir and Agarwal have made use of the latter to derive bounds for the error term in the trapezoidal formula for the numerical integration of an integrable function f such that $|f'|^q$ is convex for some $q \geq 1$. Some improvements of their result have been derived in [6] and [7].

Recently, Lj Dedic et al. [3, Theorem 2 for $r = 1$] establish the following basic result which was obtained for the difference between the two side of the right-hand Hadamard inequality.

ISSN (electronic): 1443-5756

072-03

Suppose $f:[a,b]\to\mathbb{R}$ is a real-valued twice differentiable function. If $|f''|^q$ is convex for some $q\geq 1$, then

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\frac{(b-a)^3}{12}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{1.2}$$

If $|f''|^q$ is concave, then

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\frac{(b-a)^3}{12}\left|f''\left(\frac{a+b}{2}\right)\right|. \tag{1.3}$$

In this paper, using Euler-trapezoidal formula, we shall generalize and improve the inequalities (1.2) and (1.3). Also, we apply the result to obtain a better estimates of the error in the trapezoidal formula.

2. The Euler-Trapezoidal Formula and Some Identities

In what follows, let $B_n(t)$, $n\geq 0$ be the Bernoulli polynomials and $B_n=B_n(0)$, $n\geq 0$, the Bernoulli numbers. The first few Bernoulli polynomials are

$$\begin{aligned} B_0(t) &= 1,\quad B_1(t)=t-\frac{1}{2},\quad B_2(t)=t^2-t+\frac{1}{6},\quad B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t,\\ B_4(t) &= t^4-2t^3+t^2-\frac{1}{30}\end{aligned}$$

and the first few Bernoulli numbers are

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,\quad B_4=-\frac{1}{30},\quad B_5=0.$$

For some details on the Bernoulli polynomials and the Bernoulli numbers, see for example [1, 5].

The relevant key properties of the Bernoulli polynomials are

$$\begin{aligned} &B_n'(t)=nB_{n-1}(t),\quad (n\geq 1)\\ &B_n(1+t)-B_n(t)=nt^{n-1},\quad (n\geq 0)\end{aligned}$$

(See for example, [1, Chapter 23]). Let $P_{2n}(t)=[B_n(t)-B_n]/n!$. We note that $P_{2n}(t)$, $P_{2n}(\frac{t}{2})$ and $P_{2n}(1-\frac{t}{2})$ do not change sign on (0.1).

By [2, p. 274], we have the following Euler-trapezoidal formula:

$$\begin{aligned}\int_a^b f(x)dx=\frac{b-a}{2}[f(a)+f(b)]&-\sum_{k=1}^{r-1}\frac{(b-a)^{2k}B_{2k}}{(2k)!}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\\ &+(b-a)^{2r+1}\int_0^1 P_{2r}(t)f^{(2r)}(a+t(b-a))dt.\end{aligned} \tag{2.1}$$

and by [2, p. 275], we have

$$\begin{aligned}\int_a^{a+nh} f(x)dx=T(f;h)&-\sum_{k=1}^{r-1}\frac{B_{2k}h^{2k}}{(2k)!}[f^{(2k-1)}(a+nh)-f^{(2k-1)}(a)]\\ &+h^{2r+1}\sum_{k=0}^{n-1}\int_0^1 P_{2r}(t)f^{(2r)}(a+h(t+k))dt,\end{aligned} \tag{2.2}$$

where

$$T(f;h) = h\left[\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a+nh) + \frac{1}{2}f(a+nh)\right]$$

is estimates of integral $\int_a^{a+h} f(x)dx$.

In this article, we adopt the terminology that f is $(j+2)$-convex if $f^{(j)}$ is convex, so ordinary convexity is two-convexity. A corresponding definition applies for $(j+2)$-concavity.

For our results, we need the following identities. By $B_{2j+1}\left(\frac{1}{2}\right) = 0$ for $j \geq 0$; $B_j = B_j(1)$ for $j \geq 0$ and integrating by parts, we have that the following identities hold:

$$a_{\mathrm{I}}(r) = \int_0^1 P_{2r}\left(\frac{t}{2}\right) dt = -\frac{2B_{2r+1}}{(2r+1)!} - \frac{B_{2r}}{(2r)!}, \tag{I}$$

$$a_{\mathrm{II}}(r) = \int_0^1 tP_{2r}\left(\frac{t}{2}\right) dt = -\frac{4\left(B_{2r+2}\left(\frac{1}{2}\right) - B_{2r+2}\right)}{(2r+2)!} - \frac{B_{2r}}{2(2r)!}, \tag{II}$$

$$\begin{aligned} a_{\mathrm{III}}(r) &= \int_0^1 (1-t)P_{2r}\left(\frac{t}{2}\right) dt \\ &= \frac{4\left(B_{2r+2}\left(\frac{1}{2}\right) - B_{2r+2}\right)}{(2r+2)!} - \frac{2B_{2r+1}}{(2r+1)!} - \frac{B_{2r}}{2(2r)!}, \end{aligned} \tag{III}$$

$$a_{\mathrm{IV}}(r) = \int_0^1 P_{2r}\left(1-\frac{t}{2}\right) dt = \frac{2B_{2r+1}}{(2r+1)!} - \frac{B_{2r}}{(2r)!}, \tag{IV}$$

$$a_{\mathrm{V}}(r) = \int_0^1 tP_{2r}\left(1-\frac{t}{2}\right) dt = -\frac{4\left(B_{2r+2}\left(\frac{1}{2}\right) - B_{2r+2}\right)}{(2r+2)!} - \frac{B_{2r}}{2(2r)!}, \tag{V}$$

$$\begin{aligned} a_{\mathrm{VI}}(r) &= \int_0^1 (1-t)P_{2r}\left(1-\frac{t}{2}\right) dt \\ &= -\frac{4\left(B_{2r+2}\left(\frac{1}{2}\right) - B_{2r+2}\right)}{(2r+2)!} + \frac{2B_{2r+1}}{(2r+1)!} - \frac{B_{2r}}{2(2r)!}. \end{aligned} \tag{VI}$$

3. RESULTS

In the remainder of the paper we shall use the notation

$$\begin{aligned} I_r = (-1)^r \Bigg\{ \int_a^b f(x)dx - \frac{(b-a)}{2}[f(a)+f(b)] \\ + \sum_{k=1}^{r-1} \frac{B_{2k}(b-a)^{2k}}{(2k)!}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \Bigg\}, \end{aligned}$$

As above, the empty sum for $r = 1$ is interpreted as zero.

Theorem 3.1. *Suppose $f : [a,b] \to \mathbb{R}$ is a real-valued $(2r)$-times differentiable function.*

(a) *If $|f^{(2r)}|^q$ is convex for $q \geq 1$, then*

$$\begin{aligned} |I_r| \leq (b-a)^{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{|B_{2r}|}{(2r)!}\right)^{1-\frac{1}{q}} \Bigg\{ |a_{\mathrm{III}}(r)| \cdot |f^{(2r)}(a)|^q \\ + \left(|a_{\mathrm{II}}(r)| + |a_{\mathrm{V}}(r)|\right) \cdot \left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|^q + |a_{\mathrm{VI}}(r)| \cdot |f^{(2r)}(b)|^q \Bigg\}^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

(b) *If* $|f^{(2r)}|^q$ *is concave for* $q \geq 1$, *then*

$$|I_r| \leq \frac{(b-a)^{2r+1}}{2} \left\{ |a_{\mathrm{I}}(r)| \cdot \left| f^{(2r)} \left(\frac{|a_{\mathrm{III}}(r)| \cdot a + |a_{\mathrm{II}}(r)| \cdot (\frac{a+b}{2})}{|a_{\mathrm{I}}(r)|} \right) \right| \right.$$
$$\left. + |a_{\mathrm{IV}}(r)| \cdot \left| f^{(2r)} \left(\frac{|a_{\mathrm{VI}}(r)| \cdot b + |a_{\mathrm{V}}(r)| \cdot (\frac{a+b}{2})}{|a_{\mathrm{IV}}(r)|} \right) \right| \right\}. \tag{3.2}$$

Proof. From (2.1) and the definition of I_r, we have

$$\begin{aligned} |I_r| &\leq (b-a)^{2r+1} \int_0^1 |P_{2r}(t) f^{(2r)}(a + t(b-a))| dt \\ &= (b-a)^{2r} \int_a^b \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| \cdot |f^{(2r)}(x)| dx. \end{aligned}$$

It follows from Hölder's inequality that

$$|I_r| \leq \left(\int_a^b \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| dx \right)^{1-\frac{1}{q}} \left(\int_a^b \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| \cdot |f^{(2r)}(x)|^q dx \right)^{\frac{1}{q}}. \tag{3.3}$$

Now,

$$\int_a^b \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| dx = (b-a) \left| \int_0^1 P_{2r}(t) dt \right| = \frac{(b-a)|B_{2r}|}{(2r)!}, \tag{3.4}$$

and, by the convexity of $|f^{(2r)}|^q$, we have

$$\begin{aligned} &\int_a^b \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| \cdot |f^{(2r)}(x)|^q dx \\ &= \int_a^{\frac{a+b}{2}} \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| \cdot |f^{(2r)}(x)|^q + \int_{\frac{a+b}{2}}^b \left| P_{2r} \left(\frac{x-a}{b-a} \right) \right| \cdot |f^{(2r)}(x)|^q dx \\ &= \frac{b-a}{2} \left[\int_0^1 \left| P_{2r} \left(\frac{t}{2} \right) \right| \cdot \left| f^{(2r)} \left((1-t)a + t \left(\frac{a+b}{2} \right) \right) \right|^q dt \right. \\ &\qquad \left. + \int_0^1 \left| P_{2r} \left(1 - \frac{t}{2} \right) \right| \cdot \left| f^{(2r)} \left((1-t)b + t \left(\frac{a+b}{2} \right) \right) \right|^q dt \right] \\ &\leq \frac{b-a}{2} \left[\left| \int_0^1 (1-t) P_{2r} \left(\frac{t}{2} \right) dt \right| \cdot |f^{(2r)}(a)|^q \right. \\ &\qquad + \left| \int_0^1 t P_{2r} \left(\frac{t}{2} \right) dt \right| \cdot \left| f^{(2r)} \left(\frac{a+b}{2} \right) \right|^q \\ &\qquad + \left| \int_0^1 (1-t) P_{2r} \left(1 - \frac{t}{2} \right) dt \right| \cdot |f^{(2r)}(b)|^q \\ &\qquad \left. + \left| \int_0^1 P_{2r} \left(1 - \frac{t}{2} \right) dt \right| \cdot \left| f^{(2r)} \left(\frac{a+b}{2} \right) \right|^q \right] \end{aligned} \tag{3.5}$$

Thus, by (3.3), (3.4) and (3.5) and the identities (II), (III), (V) and (VI), we have the inequality (3.1).

On the other hand, since $|f^{(2r)}|^q$ is concave implies that $|f^{(2r)}|$ is concave and by Jensen's integral inequality, we have

$$\begin{aligned}
|I_r| &\leq (b-a)^{2r}\left[\int_a^{\frac{a+b}{2}}\left|P_{2r}\left(\frac{x-a}{b-a}\right)\right|\cdot|f^{(2r)}(x)|dx+\int_{\frac{a+b}{2}}^b\left|P_{2r}\left(\frac{x-a}{b-a}\right)\right|\cdot|f^{(2r)}(x)|dx\right]\\
&=(b-a)^{2r+1}\left[\int_0^1\left|P_{2r}\left(\frac{t}{2}\right)\right|\cdot\left|f^{(2r)}\left((1-t)a+t\left(\frac{a+b}{2}\right)\right)\right|dt\right.\\
&\qquad\left.+\int_0^1\left|P_{2r}\left(1-\frac{t}{2}\right)\right|\cdot\left|f^{(2r)}\left((1-t)b+t\left(\frac{a+b}{2}\right)\right)\right|\right]\\
&\leq\frac{(b-a)^{2r+1}}{2}\left[\left|\int_0^1P_{2r}\left(\frac{t}{2}\right)dt\right|\cdot\left|f^{(2r)}\left(\frac{\left|\int_0^1P_{2r}\left(\frac{t}{2}\right)((1-t)a+t\left(\frac{a+b}{2}\right))dt\right|}{\left|\int_0^1P_{2r}\left(\frac{t}{2}\right)dt\right|}\right)\right|\right.\\
&\qquad\left.+\left|\int_0^1P_{2r}\left(1-\frac{t}{2}\right)dt\right|\cdot\left|f^{(2r)}\left(\frac{\left|\int_0^1P_{2r}\left(1-\frac{t}{2}\right)\left((1-t)b+t\left(\frac{a+b}{2}\right)\right)dt\right|}{\left|\int_0^1P_{2r}\left(1-\frac{t}{2}\right)dt\right|}\right)\right|\right].
\end{aligned}$$

Thus, by identities I, II, III, IV, V and VI, we have the inequality (3.2). □

For $r=1$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Under the assumptions of Theorem 3.1. If f is a 4-convex function, we have*

$$\begin{aligned}
(3.6)\qquad &\left|\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]\right|\\
&\qquad\leq\frac{(b-a)^3}{12}\left[\frac{3\left|f''(a)\right|^q+10\left|f''\left(\frac{a+b}{2}\right)\right|^q+3|f''(b)|^q}{16}\right]^{\frac{1}{q}}
\end{aligned}$$

and if f is a 4-concave function, we have

$$(3.7)\qquad \left|\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\frac{(b-a)^3}{12}\left[\frac{\left|f''\left(\frac{11a+5b}{16}\right)\right|+\left|f''\left(\frac{5a+11b}{16}\right)\right|}{2}\right].$$

Remark 3.3. Using the convexity of $|f''|^q$, we have

$$\left|f''\left(\frac{a+b}{2}\right)\right|^q\leq\frac{|f''(a)|^q+|f''(b)|^q}{2},$$

Hence inequality (3.6) is an improvement of inequality (1.2).

Since $|f''|^q$ is concave implies that $|f''|$ is concave, we have

$$\frac{1}{2}\left[\left|f''\left(\frac{11a+5b}{16}\right)\right|+\left|f''\left(\frac{5a+11b}{16}\right)\right|\right]\leq\left|f''\left(\frac{a+b}{2}\right)\right|.$$

Thus inequality (3.7) is an improvement of inequality (1.3).

4. APPLICATION

To obtain estimates of the error in the trapezoidal formula, we apply the results of the previous section on each interval from the subdivision

$$[a,a+b],\ \ [a+h,a+2h],\dots,[a+(n-1)h,a+nh].$$

We define

$$J_r = \int_a^{a+nh} f(x)dx - T(f;h) + \sum_{k=1}^{r-1} \frac{B_{2k}h^{2k}}{(2k)!}[f^{(2k-1)}(a+nh) - f^{(2k-1)}(a)].$$

Theorem 4.1. *If $f : [a, a+nh] \to \mathbb{R}$ is a $(2r)$-time differentiable function $(r \geq 1)$.*

(a) *If $|f^{(2r)}|^q$ is convex for some $q \geq 1$, then*

$$|J_{2r}| \leq h^{2r+1} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{|B_{2r}|}{(2r)!}\right)^{1-\frac{1}{q}} \sum_{m=1}^{n} \Bigg\{ |a_{\mathrm{III}}(r)| \cdot |f^{(2r)}(a+(m-1)h)|^q + (|a_{\mathrm{II}}(r)| + |a_{\mathrm{V}}(r)|) \cdot \left| f^{(2r)}\left(a + \left(m - \frac{1}{2}\right)h\right)\right|^q + |a_{\mathrm{VI}}(r)| \cdot |f^{(2r)}(a+mh)|^q \Bigg\}^{\frac{1}{q}}.$$

(b) *If $|f^{(2r)}|^q$ is concave for some $q \geq 1$, then*

$$|J_{2r}| \leq \frac{h^{2r+1}}{2} \sum_{m=1}^{n} \Bigg\{ |a_{\mathrm{I}}(r)| \cdot \left| f^{(2r)}\left(\frac{|a_{\mathrm{III}}(r)|(a+(m-1\cdot h) + |a_{\mathrm{II}}(r)|(a+(m-\frac{1}{2})\cdot h)}{|a_{\mathrm{I}}(r)|}\right)\right| + |a_{\mathrm{IV}}(r)| \cdot \left| f^{(2r)}\left(\frac{|a_{\mathrm{VI}}(r)|(a+mh) + |a_{\mathrm{II}}(r)| \cdot (a+(m-\frac{1}{2})h)}{|a_{\mathrm{IV}}(r)|}\right)\right| \Bigg\}.$$

Proof. Since

$$|J_r| \leq \sum_{m=1}^{n} \Bigg\{ \int_{a+(m-1)h}^{a+mh} f(x)dx - \frac{h}{2}[f(a+(m-1)h) + f(a+mh)] + \sum_{k=1}^{r-1} \frac{B_{2k}h^{2k}}{(2k)!}[f^{(2k-1)}(a+mh) - f^{(2k-1)}(a+(m-1)h)] \Bigg\},$$

we have

$$|J_r| \leq \sum_{m=1}^{n} h^{2r+1} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{|B_{2r}|}{(2r)!}\right)^{1-\frac{1}{q}} \Bigg\{ |a_{\mathrm{III}}(r)| \cdot |f^{(2r)}(a+(m-1)h)|^q + (|a_{\mathrm{II}}(r)| + |a_{\mathrm{V}}(r)|) \cdot \left| f^{(2r)}\left(a + \left(m - \frac{1}{2}\right)h\right)\right|^q + |a_{\mathrm{II}}(r)| \cdot |f^{(2r)}(a+mh)|^q \Bigg\}^{\frac{1}{q}},$$

by Theorem 3.1(a) applied to each interval $[a+(m-1)h, a+mh]$. Obviously, the proof (a) is complete.

The proof (b) is similar. □

Remark 4.2. As the same discussion in the Section 3, Theorem 4.1 for $r = 1$ is an improvement of Theorem 4 for $r = 1$ in [3].

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