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SOME PRECISE ESTIMATES OF THE HYPER ORDER OF SOLUTIONS OF SOME COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $\rho(f)$ and $\rho_2(f)$ denote respectively the order and the hyper order of an entire function f. In this paper, we obtain some precise estimates of the hyper order of solutions of the following higher order linear differential equations

$$f^{(k)}+\sum_{j=0}^{k-1}A_j(z)\,e^{P_j(z)}f^{(j)}=0$$

and

$$f^{(k)}+\sum_{j=0}^{k-1}\left(A_j(z)\,e^{P_j(z)}+B_j(z)\right)f^{(j)}=0$$

where $k\geq 2$, $P_j(z)$ $(j=0,\dots,k-1)$ are nonconstant polynomials such that $\deg P_j=n$ $(j=0,\dots,k-1)$ and $A_j(z)\,(\not\equiv 0)\,,B_j(z)\,(\not\equiv 0)$ $(j=0,\dots,k-1)$ are entire functions with $\rho(A_j)<n$, $\rho(B_j)<n$ $(j=0,\dots,k-1)$. Under some conditions, we prove that every solution $f(z)\not\equiv 0$ of the above equations is of infinite order and $\rho_2(f)=n$.

Key words and phrases: Linear differential equations, Entire solutions, Hyper order.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory and with basic Wiman-Valiron theory as well (see [6], [7], [9], [10]). Let f be a meromorphic function, one defines

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{it}\right)\right|dt,$$

$$N(r,f)=\int_0^r\frac{(n(t,f)-n(0,f))}{t}dt+n(0,f)\log r,$$

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and $T(r,f) = m(r,f) + N(r,f)$ $(r > 0)$ is the Nevanlinna characteristic function of f, where $\log^+ x = \max(0, \log x)$ for $x \geq 0$ and $n(t,f)$ is the number of poles of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity. In addition, we will use $\rho(f) = \overline{\lim}_{r\to+\infty} \frac{\log T(r,f)}{\log r}$ to denote the order of growth of a meromorphic function $f(z)$. See [6, 9] for notations and definitions.

To express the rate of growth of entire solutions of infinite order, we recall the following concept.

Definition 1.1 (see [3, 11]). Let f be an entire function. Then the hyper order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log \log T(r, f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log\log M(r, f)}{\log r}, \tag{1.1}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Several authors have studied the second order linear differential equation

$$f'' + A_1(z) e^{P_1(z)} f' + A_0(z) e^{P_0(z)} f = 0, \tag{1.2}$$

where $P_1(z), P_0(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ are entire functions such that $\rho(A_1) < \deg P_1(z)$, $\rho(A_0) < \deg P_0(z)$. Gundersen showed in [4, p. 419] that if $\deg P_1(z) \neq \deg P_0(z)$, then every nonconstant solution of (1.2) is of infinite order. If $\deg P_1(z) = \deg P_0(z)$, then (1.2) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [3], Kwon has investigated the order and the hyper order of solutions of equation (1.2) in the case when $\deg P_1(z) = \deg P_0(z)$ and has obtained the following result.

Theorem A ([3]). *Let $P_1(z)$ and $P_0(z)$ be nonconstant polynomials, such that*

$$P_1(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{1.3}$$

$$P_0(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0, \tag{1.4}$$

where a_i, b_i $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n \neq 0$, $b_n \neq 0$. Let $A_1(z)$ and $A_0(z)$ $(\not\equiv 0)$ be entire functions with $\rho(A_j) < n$ $(j = 0, 1)$. Then the following four statements hold:

(i) *If either $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$, then every nonconstant solution f of (1.2) has infinite order with $\rho_2(f) \geq n$.*

(ii) *Let $a_n = b_n$ and $\deg(P_1 - P_0) = m \geq 1$, and let the orders of $A_1(z)$ and $A_0(z)$ be less than m. Then every nonconstant solution f of (1.2) has infinite order with $\rho_2(f) \geq m$.*

(iii) *Let $a_n = cb_n$ with $c > 1$ and $\deg(P_1 - cP_0) = m \geq 1$. Suppose that $\rho(A_1) < m$ and $A_0(z)$ is an entire function with $0 < \rho(A_0) < 1/2$. Then every nonconstant solution f of (1.2) has infinite order with $\rho_2(f) \geq \rho(A_0)$.*

(iv) *Let $a_n = cb_n$ with $c \geq 1$ and let $P_1(z) - cP_0(z)$ be a constant. Suppose that $\rho(A_1) < \rho(A_0) < 1/2$. Then every nonconstant solution f of (1.2) has infinite order with $\rho_2(f) \geq \rho(A_0)$.*

In [1], Chen improved the results of Theorem A(i), Theorem A(iii) for the linear differential equation (1.2) as follows:

Theorem B ([1]). *Let $P_1(z) = \sum_{i=0}^{n} a_i z^i$ and $P_0(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n \neq 0$, $b_n \neq 0$, and let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ be entire functions. Suppose that either (i) or (ii) below, holds:*

(i) $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$, $\rho(A_j) < n$ $(j = 0, 1)$;

(ii) $a_n = cb_n$ $(c > 1)$ *and* $\deg(P_1 - cP_0) = m \geq 1$, $\rho(A_j) < m$ $(j = 0, 1)$.

Then every solution $f(z) \not\equiv 0$ *of* (1.2) *satisfies* $\rho_2(f) = n$.

Recently, Chen and Shon obtained the following result:

Theorem C ([2]). *Let* $h_0 \not\equiv 0, h_1, \dots, h_{k-1}$ *be entire functions with* $\rho(h_j) < 1$ $(j = 0, \dots, k-1)$. *Let* $a_0 \neq 0, a_1, \dots, a_{k-1}$ *be complex numbers such that for* $j = 1, \dots, k-1$,

(i) $a_j = 0$, *or*
(ii) $\arg a_j = \arg a_0$ *and* $a_j = c_j a_0$ $(0 < c_j < 1)$, *or*
(iii) $\arg a_j \neq \arg a_0$.

Then every solution $f(z) \not\equiv 0$ *of the linear differential equation*

$$f^{(k)} + h_{k-1}(z) e^{a_{k-1}z} f^{(k-1)} + \dots + h_1(z) e^{a_1 z} f' + h_0(z) e^{a_0 z} f = 0 \tag{1.5}$$

satisfies $\rho(f) = \infty$ *and* $\rho_2(f) = 1$.

The main purpose of this paper is to extend and improve the results of Theorem B and Theorem C to some higher order linear differential equations. In fact we will prove the following results.

Theorem 1.1. *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \dots, k-1)$ *be nonconstant polynomials where* $a_{0,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ *are complex numbers such that* $a_{n,j} a_{n,0} \neq 0$ $(j = 1, \dots, k-1)$, *and let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, \dots, k-1)$ *be entire functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, \dots, k-1)$ *and* $\rho(A_j) < n$ $(j = 0, \dots, k-1)$. *Then every solution* $f(z) \not\equiv 0$ *of the equation*

$$f^{(k)} + A_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + \dots + A_1(z) e^{P_1(z)} f' + A_0(z) e^{P_0(z)} f = 0, \tag{1.6}$$

where $k \geq 2$, *is of infinite order and* $\rho_2(f) = n$.

Theorem 1.2. *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \dots, k-1)$ *be nonconstant polynomials where* $a_{0,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ *are complex numbers such that* $a_{n,j} a_{n,0} \neq 0$ $(j = 1, \dots, k-1)$, *and let* $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, \dots, k-1)$ *be entire functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, \dots, k-1)$ *and* $\rho(A_j) < n$, $\rho(B_j) < n$ $(j = 0, \dots, k-1)$. *Then every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$\begin{aligned} f^{(k)} &+ \left(A_{k-1}(z) e^{P_{k-1}(z)} + B_{k-1}(z)\right) f^{(k-1)} + \cdots \\ &+ \left(A_1(z) e^{P_1(z)} + B_1(z)\right) f' + \left(A_0(z) e^{P_0(z)} + B_0(z)\right) f = 0, \end{aligned} \tag{1.7}$$

where $k \geq 2$, *is of infinite order and* $\rho_2(f) = n$.

Theorem 1.3. *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \dots, k-1)$ *be nonconstant polynomials where* $a_{0,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ *are complex numbers such that* $a_{n,j} a_{n,0} \neq 0$ $(j = 1, \dots, k-1)$, *and let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, \dots, k-1)$ *be entire functions. Suppose that* $a_{n,j} = c a_{n,0}$ $(c > 1)$ *and* $\deg(P_j - cP_0) = m \geq 1$ $(j = 1, \dots, k-1)$, $\rho(A_j) < m$ $(j = 0, \dots, k-1)$. *Then every solution* $f(z) \not\equiv 0$ *of the equation* (1.6) *is of infinite order and* $\rho_2(f) = n$.

2. LEMMAS REQUIRED TO PROVE THEOREM 1.1 AND THEOREM 1.2

We need the following lemmas in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 2.1 ([5]). *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then the following two statements hold:*

(i) *There exists a constant* $A > 0$ *and a set* $E_1 \subset [0, \infty)$ *having finite linear measure such that for all* z *satisfying* $|z| = r \notin E_1$, *we have*

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq A \left[T(\alpha r, f) r^{\varepsilon} \log T(\alpha r, f) \right]^j \qquad (j \in \mathbb{N}). \tag{2.1}$$

(ii) *There exists a constant* $B > 0$ *and a set* $E_2 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\theta \in [0, 2\pi) \backslash E_2$, *then there is a constant* $R_1 = R_1(\theta) > 1$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| = r \geq R_1$, *we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{j} \qquad (j \in \mathbb{N}). \tag{2.2}$$

Lemma 2.2 ([2]). *Let* $P(z) = a_n z^n + \cdots + a_0$, $(a_n = \alpha + i\beta \neq 0)$ *be a polynomial with degree* $n \geq 1$ *and* $A(z)$ $(\not\equiv 0)$ *be an entire function with* $\sigma(A) < n$. *Set* $f(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_3 \subset [0, 2\pi)$ *that has linear measure zero, such that for any* $\theta \in [0, 2\pi) \setminus (E_3 \cup E_4)$, *where* $E_4 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, there is* $R_2 > 0$ *such that for* $|z| = r > R_2$, *the following statements hold:*

(i) *if* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leq |f(z)| \leq \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\}, \tag{2.3}$$

(ii) *if* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leq |f(z)| \leq \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\}. \tag{2.4}$$

Lemma 2.3 ([2]). *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be entire functions of finite order. If* f *is a solution of the equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{2.5}$$

then

$$\rho_2(f) \leq \max\{\rho(A_0), \ldots, \rho(A_{k-1})\}.$$

Lemma 2.4. *Let* $P(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0$ *with* $b_m \neq 0$ *be a polynomial. Then for every* $\varepsilon > 0$, *there exists* $R_3 > 0$ *such that for all* $|z| = r > R_3$ *the inequalities*

$$(1 - \varepsilon)|b_m| r^m \leq |P(z)| \leq (1 + \varepsilon)|b_m| r^m \tag{2.6}$$

hold.

Proof. Clearly,

$$|P(z)| = |a_m| |z|^m \left|1 + \frac{a_{m-1}}{a_m}\frac{1}{z} + \cdots + \frac{a_0}{a_m}\frac{1}{z^m}\right|.$$

Denote

$$R_m(z) = \frac{a_{m-1}}{a_m}\frac{1}{z} + \cdots + \frac{a_0}{a_m}\frac{1}{z^m}.$$

Obviously, $|R_m(z)| < \varepsilon$, if $|z| > R_3$ for some $\varepsilon > 0$. This means that

$$\begin{aligned}(1 - \varepsilon)|a_m| r^m &\leq (1 - |R_m(z)|)|a_m| r^m \\ &\leq |1 + R_m(z)| |a_m| r^m \\ &= |P(z)| \\ &\leq (1 + |R_m(z)|)|a_m| r^m \\ &\leq (1 + \varepsilon)|a_m| r^m.\end{aligned}$$

□

3. PROOF OF THEOREM 1.1

Assume $f(z) \not\equiv 0$ is a transcendental entire solution of (1.6). By Lemma 2.1 (ii), there exists a set $E_2 \subset [0, 2\pi)$ with linear measure zero, such that if $\theta \in [0, 2\pi) \backslash E_2$, there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r, f)\right]^{k+1} \qquad (j = 1, \ldots, k), \ (B > 0). \tag{3.1}$$

Let $P_0(z) = a_{n,0}z^n + \cdots + a_{0,0}$ $(a_{n,0} = \alpha + i\beta \neq 0)$, $\delta(P_0, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Suppose first that $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, \ldots, k-1)$. By Lemma 2.2, for any given ε $(0 < \varepsilon < 1)$, there is a set E_3 that has linear measure, and a ray $\arg z = \theta \in [0, 2\pi) \setminus (E_3 \cup E_4)$, where $E_4 = \{\theta \in [0, 2\pi)$: $\delta(P_0, \theta) = 0$ or $\delta(P_j, \theta) = 0$ $(j = 1, \ldots, k-1)\}$ (E_4 is a finite set), such that $\delta(P_0, \theta) > 0$, $\delta(P_j, \theta) < 0$ $(j = 1, \ldots, k-1)$ and for sufficiently large $|z| = r$, we have

$$\left|A_0(z)e^{P_0(z)}\right| \geq \exp\left\{(1-\varepsilon)\delta(P_0, \theta)r^n\right\} \tag{3.2}$$

and

$$\left|A_j(z)e^{P_j(z)}\right| \leq \exp\left\{(1-\varepsilon)\delta(P_j, \theta)r^n\right\} < 1 \qquad (j = 1, \ldots, k-1). \tag{3.3}$$

It follows from (1.6) that

$$\begin{aligned}\left|A_0(z)e^{P_0(z)}\right| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)}\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| \\ + \cdots + \left|A_1(z)e^{P_1(z)}\right|\left|\frac{f'(z)}{f(z)}\right|.\end{aligned} \tag{3.4}$$

Now, take a ray $\theta \in [0, 2\pi) \setminus (E_2 \cup E_3 \cup E_4)$. Hence by $(3.1) - (3.3)$ and (3.4), we get for sufficiently large $|z| = r$

$$\begin{aligned}\exp\left\{(1-\varepsilon)\delta(P_0, \theta)r^n\right\} &\leq \left(1 + \sum_{j=1}^{k-1}\exp\left\{(1-\varepsilon)\delta(P_j, \theta)r^n\right\}\right)B\left[T(2r, f)\right]^{k+1} \\ &\leq kB\left[T(2r, f)\right]^{k+1}.\end{aligned} \tag{3.5}$$

By $0 < \varepsilon < 1$ and (3.5) we get that $\rho(f) = +\infty$ and

$$\rho_2(f) = \overline{\lim_{r \to +\infty}} \frac{\log \log T(r, f)}{\log r} \geq n. \tag{3.6}$$

By Lemma 2.3, we have $\rho_2(f) = n.$

Suppose now $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, \ldots, k-1)$. Then $\delta(P_j, \theta) = c_j\delta(P_0, \theta)$ $(j = 1, \ldots, k-1)$. Put $c = \max\{c_j : j = 1, \ldots, k-1\}$. Then $0 < c < 1$. Using the same reasoning as above, for any given ε $(0 < 2\varepsilon < \frac{1-c}{1+c})$ there exists a ray $\arg z = \theta \in [0, 2\pi) \setminus (E_5 \cup E_6)$, where E_5 and E_6 are defined as in Lemma 2.2, $E_5 \cup E_6$ is of linear measure zero, satisfying $\delta(P_j, \theta) = c_j\delta(P_0, \theta) > 0$ $(j = 1, \ldots, k-1)$ and for sufficiently large $|z| = r$, we have

$$\left|A_0(z)e^{P_0(z)}\right| \geq \exp\left\{(1-\varepsilon)\delta(P_0, \theta)r^n\right\} \tag{3.7}$$

and

$$\begin{aligned}\left|A_j(z)e^{P_j(z)}\right| &\leq \exp\left\{(1+\varepsilon)\delta(P_j, \theta)r^n\right\} \\ &\leq \exp\left\{(1+\varepsilon)c\delta(P_0, \theta)r^n\right\} \ (j = 1, \ldots, k-1).\end{aligned} \tag{3.8}$$

Now, take a ray $\theta \in [0, 2\pi) \setminus (E_2 \cup E_5 \cup E_6)$. By substituting (3.1), (3.7) and (3.8) into (3.4), we get for sufficiently large $|z| = r$,

$$\text{(3.9)} \quad \exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \le (1+(k-1)\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\})B[T(2r,f)]^{k+1} \\ \le kB\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\}[T(2r,f)]^{k+1}.$$

By $0 < 2\varepsilon < \frac{1-c}{1+c}$ and (3.9) we have

$$\text{(3.10)} \qquad \exp\left\{\frac{(1-c)}{2}\delta(P_0,\theta)r^n\right\} \le kB[T(2r,f)]^{k+1}.$$

Thus, (3.10) implies $\rho(f) = +\infty$ and

$$\text{(3.11)} \qquad \rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,\ f)}{\log r} \ge n.$$

By Lemma 2.3, we have $\rho_2(f) = n$.

Now we prove that equation (1.6) cannot have a nonzero polynomial solution. Suppose first that $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, \ldots, k-1)$. Assume $f(z) \not\equiv 0$ is a polynomial solution of (1.6). By Lemma 2.2, for any given ε $(0 < \varepsilon < 1)$ there exists a ray $\arg z = \theta \in [0, 2\pi) \setminus (E_3 \cup E_4)$ satisfying $\delta(P_0, \theta) > 0$, $\delta(P_j, \theta) < 0$ $(j = 1, \ldots, k-1)$ and for sufficiently large $|z| = r$, inequalities (3.2), (3.3) hold. By (1.6) we can write

$$\text{(3.12)} \qquad A_0(z)e^{P_0(z)}f = -\left(f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)}f'\right).$$

By using (3.2), (3.3), (3.12) and Lemma 2.4 we obtain for sufficiently large $|z| = r$

$$\begin{aligned} \text{(3.13)} \qquad & (1-\varepsilon)|b_m|r^m\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \\ &\le \left|A_0(z)e^{P_0(z)}f\right| \\ &= \left|f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)}f'\right| \\ &\le k(1+\varepsilon)m|b_m|r^{m-1}, \end{aligned}$$

where $f(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_1 z + b_0$ with $b_m \neq 0$. From (3.13) we get for sufficiently large $|z| = r$

$$\text{(3.14)} \qquad \exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \le k\frac{1+\varepsilon}{1-\varepsilon}m\frac{1}{r}.$$

This is absurd since $0 < \varepsilon < 1$. By using similar reasoning as above we can prove that if $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$, then equation (1.6) cannot have nonzero polynomial solution. Hence every solution $f(z) \not\equiv 0$ of (1.6) is of infinite order and $\rho_2(f) = n$.

4. PROOF OF THEOREM 1.2

Assume $f(z) \not\equiv 0$ is a solution of (1.7). By using similar reasoning as in the proof of Theorem 1.1, it follows that $f(z)$ must be a transcendental entire solution. Suppose first that $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, \ldots, k-1)$. By Lemma 2.2, for any given ε $(0 < 2\varepsilon < \min\{1, n-\alpha\})$, where $\alpha = \max\{\rho(B_j) : j = 0, \ldots, k-1\}$, there exists a ray $\arg z = \theta$ such that $\theta \in [0, 2\pi) \setminus (E_3 \cup E_4)$, where E_3 and E_4 are defined as in Lemma 2.2, $E_3 \cup E_4$ is of linear measure zero, and $\delta(P_0, \theta) > 0$, $\delta(P_j, \theta) < 0$ $(j = 1, \ldots, k-1)$ and for sufficiently large $|z| = r$, we have

$$\text{(4.1)} \qquad \left|A_0(z)e^{P_0(z)} + B_0(z)\right| \ge (1-o(1))\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\}$$

and

$$\left|A_j(z)e^{P_j(z)}+B_j(z)\right| \le \exp\{(1-\varepsilon)\delta(P_j,\theta)r^n\}+\exp\left\{r^{\rho(B_j)+\frac{\varepsilon}{2}}\right\} \le \exp\left\{r^{\rho\left(B_j\right)+\varepsilon}\right\} \le \exp\left\{r^{\alpha+\varepsilon}\right\} \quad (j=1,\dots,k-1). \tag{4.2}$$

It follows from (1.7) that

$$\left|A_0(z)e^{P_0(z)}+B_0(z)\right| \le \left|\frac{f^{(k)}(z)}{f(z)}\right|+\left|A_{k-1}(z)e^{P_{k-1}(z)}+B_{k-1}(z)\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots +\left|A_1(z)e^{P_1(z)}+B_1(z)\right|\left|\frac{f'(z)}{f(z)}\right|. \tag{4.3}$$

Now, take a ray $\theta\in[0,2\pi)\setminus(E_2\cup E_3\cup E_4)$. Hence by (3.1) and $(4.1)-(4.3)$, we get for sufficiently large $|z|=r$

$$(1-o(1))\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \le \left(1+(k-1)\exp\left\{r^{\alpha+\varepsilon}\right\}\right)B\left[T(2r,f)\right]^{k+1} \le kB\exp\left\{r^{\alpha+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}. \tag{4.4}$$

Thus, $0<2\varepsilon<\min\{1,n-\alpha\}$ implies $\rho(f)=+\infty$ and

$$\rho_2(f)=\overline{\lim_{r\to+\infty}}\frac{\log\log T(r,\ f)}{\log r}\ge n. \tag{4.5}$$

By Lemma 2.3, we have $\rho_2(f)=n$.

Suppose now $a_{n,j}=c_ja_{n,0}$ $(0<c_j<1)$ $(j=1,\dots,k-1)$. Then $\delta(P_j,\theta)=c_j\delta(P_0,\theta)$ $(j=1,\dots,k-1)$. Put $c=\max\{c_j: j=1,\dots,k-1\}$. Then $0<c<1$. Using the same reasoning as above, for any given ε $\left(0<2\varepsilon<\frac{1-c}{1+c}\right)$ there exists a ray $\arg z=\theta\in[0,2\pi)\setminus(E_5\cup E_6)$, $E_5\cup E_6$ is of linear measure zero, satisfying $\delta(P_j,\theta)=c_j\delta(P_0,\theta)>0$ $(j=1,\dots,k-1)$ and for sufficiently large $|z|=r$, we have

$$\left|A_0(z)e^{P_0(z)}+B_0(z)\right|\ge(1-o(1))\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \tag{4.6}$$

and

$$\left|A_j(z)e^{P_j(z)}+B_j(z)\right|\le(1+o(1))\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\}\quad(j=1,\dots,k-1). \tag{4.7}$$

Now, take a ray $\theta\in[0,2\pi)\setminus(E_2\cup E_5\cup E_6)$. By substituting (3.1), (4.6) and (4.7) into (4.3), we get for sufficiently large $|z|=r$,

$$(1-o(1))\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\} \le (1+(k-1)(1+o(1))\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\})B\left[T(2r,f)\right]^{k+1} \le kB(1+o(1))\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\}\left[T(2r,f)\right]^{k+1}. \tag{4.8}$$

By $0<2\varepsilon<\frac{1-c}{1+c}$ and (4.8) we have

$$\exp\left\{\frac{(1-c)}{2}\delta(P_0,\theta)r^n\right\}\le kBd\left[T(2r,f)\right]^{k+1}, \tag{4.9}$$

where $d > 0$ is some constant. Thus, (4.9) implies $\rho(f) = +\infty$ and

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log \log T(r,\ f)}{\log r} \geq n. \tag{4.10}$$

By Lemma 2.3, we have $\rho_2(f) = n$.

5. LEMMAS REQUIRED TO PROVE THEOREM 1.3

We need the following lemmas in the proof of Theorem 1.3.

Lemma 5.1 ([8, pp. 253-255]). *Let $P_0(z) = \sum_{i=0}^{n} b_i z^i$, where n is a positive integer and $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given ε $(0 < \varepsilon < \pi/4n)$, we introduce $2n$ closed angles*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon \leq \theta \leq -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \qquad (j = 0, 1, \ldots, 2n-1). \tag{5.1}$$

Then there exists a positive number $R_4 = R_4(\varepsilon)$ such that for $|z| = r > R_4$,

$$\operatorname{Re} P_0(z) > \alpha_n r^n (1-\varepsilon) \sin(n\varepsilon), \tag{5.2}$$

if $z = re^{i\theta} \in S_j$, when j is even; while

$$\operatorname{Re} P_0(z) < -\alpha_n r^n (1-\varepsilon) \sin(n\varepsilon), \tag{5.3}$$

if $z = re^{i\theta} \in S_j$, when j is odd.

Now for any given $\theta \in [0, 2\pi)$, if $\theta \neq -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ $(j = 0, 1, \ldots, 2n-1)$, then we take ε sufficiently small, and there is some S_j $(j = 0, 1, \ldots, 2n-1)$ such that $z = re^{i\theta} \in S_j$.

Lemma 5.2 ([1]). *Let $f(z)$ be an entire function of order $\rho(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_7 \subset [1, +\infty)$ that has finite linear measure and finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_7$, we have*

$$\exp\left\{-r^{\alpha+\varepsilon}\right\} \leq |f(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{5.4}$$

Lemma 5.3 ([1]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of infinite order with the hyper order $\rho_2(f) = \sigma$, $\mu(r)$ be the maximum term, i.e., $\mu(r) = \max\{|a_n| r^n;\, n = 0, 1, \ldots\}$ and let $\nu_f(r)$ be the central index of f, i.e., $\nu_f(r) = \max\{m,\ \mu(r) = |a_m| r^m\}$. Then*

$$\overline{\lim_{r\to+\infty}} \frac{\log\log \nu_f(r)}{\log r} = \sigma. \tag{5.5}$$

Lemma 5.4 (Wiman-Valiron, [7, 10]). *Let $f(z)$ be a transcendental entire function and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z|$ outside a set E_8 of r of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \qquad (j \text{ is an integer, } r \notin E_8). \tag{5.6}$$

Lemma 5.5 ([1]). *Let $f(z)$ be an entire function with $\rho(f) = +\infty$ and $\rho_2(f) = \alpha < +\infty$, let a set $E_9 \subset [1, +\infty)$ have finite logarithmic measure. Then there exists $\left\{z_p = r_p e^{i\theta_p}\right\}$ such that $|f(z_p)| = M(r_p, f)$, $\theta_p \in [0, 2\pi)$, $\lim_{p\to+\infty} \theta_p = \theta_0 \in [0, 2\pi)$, $r_p \notin E_9$, $r_p \to +\infty$, and for any given $\varepsilon > 0$, for sufficiently large r_p, we have*

$$\overline{\lim_{p\to+\infty}} \frac{\log \nu_f(r_p)}{\log r_p} = +\infty, \tag{5.7}$$

$$\exp\left\{r_p^{\alpha-\varepsilon}\right\} \leq \nu_f(r_p) \leq \exp\left\{r_p^{\alpha+\varepsilon}\right\}. \tag{5.8}$$

Lemma 5.6. *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \ldots, k-1)$ *be nonconstant polynomials where* $a_{0,j}, \ldots, a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ *are complex numbers such that* $a_{n,j}a_{n,0} \neq 0$ $(j = 1, \ldots, k-1)$, *let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, \ldots, k-1)$ *be entire functions. Suppose that* $a_{n,j} = ca_{n,0}$ $(c > 1)$ *and* $\deg(P_j - cP_0) = m \geq 1$ $(j = 1, \ldots, k-1)$, $\rho(A_j) < m$ $(j = 0, \ldots, k-1)$. *Then every solution* $f(z) \not\equiv 0$ *of the equation* (1.6) *is of infinite order and* $\rho_2(f) \geq m$.

Proof. Assume $f(z) \not\equiv 0$ is a solution of (1.6). By using similar reasoning as in the proof of Theorem 1.1, it follows that $f(z)$ must be a transcendental entire solution. From (1.6), we have

$$\left|A_0(z)e^{(1-c)P_0(z)}\right| \leq \left|e^{-cP_0(z)}\right| \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)}\right| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + \left|A_1(z)e^{P_1(z)-cP_0(z)}\right| \left|\frac{f'(z)}{f(z)}\right|. \tag{5.9}$$

By Lemma 2.1 (i), there exists a constant $A > 0$ and a set $E_1 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Ar\left[T(2r, f)\right]^{k+1} \qquad (j = 1, \ldots, k). \tag{5.10}$$

By (5.9) and (5.10), we have for all z satisfying $|z| = r \notin E_1$

$$\left|A_0(z)e^{(1-c)P_0(z)}\right| \leq \left[\left|e^{-cP_0(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)}\right| + \cdots + \left|A_1(z)e^{P_1(z)-cP_0(z)}\right|\right] Ar\left[T(2r, f)\right]^{k+1}. \tag{5.11}$$

Since $\deg(P_j - cP_0) = m < \deg P_0 = n$ $(j = 1, \ldots, k-1)$, by Lemma 5.1 (see also [3, p. 385]), there exists a positive real number b and a curve Γ tending to infinity such that for all $z \in \Gamma$ with $|z| = r$, we have

$$\operatorname{Re} P_0(z) = 0, \qquad \operatorname{Re}(P_j(z) - cP_0(z)) \leq -br^m \qquad (j = 1, \ldots, k-1). \tag{5.12}$$

Let $\max\{\rho(A_j) \ (j = 0, \ldots, k-1)\} = \beta < m$. Then by Lemma 5.2, there exists a set $E_7 \subset [1, +\infty)$ that has finite linear measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_7$, we have

$$\exp\left\{-r^{\beta+\varepsilon}\right\} \leq |A_j(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\} \qquad (j = 0, \ldots, k-1). \tag{5.13}$$

Hence by $(5.11) - (5.13)$, we get for all $z \in \Gamma$ with $|z| = r \notin [0, 1] \cup E_1 \cup E_7$

$$\exp\left\{-r^{\beta+\varepsilon}\right\} \leq \left(1 + (k-1)\exp\left\{r^{\beta+\varepsilon}\right\}\exp\left\{-br^m\right\}\right) Ar\left[T(2r, f)\right]^{k+1}. \tag{5.14}$$

Thus $\beta + \varepsilon < m$ implies $\rho(f) = +\infty$ and

$$\rho_2(f) = \varlimsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r} \geq m.$$

□

6. PROOF OF THEOREM 1.3

Assume $f(z) \not\equiv 0$ is a solution of (1.6). Then by Lemma 5.6 and Lemma 2.3, we have $\rho(f) = \infty$ and $m \leq \rho_2(f) \leq n$. We show that $\rho_2(f) = n$. We assume that $\rho_2(f) = \lambda$ $(m \leq \lambda < n)$, and we prove that $\rho_2(f) = \lambda$ fails. By the Wiman-Valiron theory, there is a set $E_8 \subset [1, +\infty)$ with logarithmic measure $lm(E_8) < +\infty$ and we can choose z satisfying $|z| = r \notin [0, 1] \cup E_8$ and $|f(z)| = M(r, f)$, such that (5.6) holds. Set $\max\{\rho(A_j)$ $(j = 0, \ldots, k-1)\} = \beta < m$. By Lemma 5.2, for any given ε $(0 < 3\varepsilon < \min(m - \beta, n - \lambda))$, there exists a set $E_7 \subset [1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_7$, the inequalities (5.13) hold and

$$\exp\left\{-r^{m+\varepsilon}\right\} \le |\exp\{P_j(z) - cP_0(z)\}| \le \exp\left\{r^{m+\varepsilon}\right\} \qquad (j = 1, \dots, k-1). \tag{6.1}$$

By Lemma 5.5, we can choose a point range $\left\{z_p = r_p e^{i\theta_p}\right\}$ such that $|f(z_p)| = M(r_p, f)$, $\theta_p \in [0, 2\pi)$, $\lim_{p\to+\infty} \theta_p = \theta_0 \in [0, 2\pi)$, $r_p \notin [0, 1] \cup E_7 \cup E_8$, $r_p \to +\infty$, and for the above $\varepsilon > 0$, for sufficiently large r_p, we have

$$\exp\left\{r_p^{\lambda-\varepsilon}\right\} \le \nu_f(r_p) \le \exp\left\{r_p^{\lambda+\varepsilon}\right\}, \tag{6.2}$$

$$\overline{\lim_{p\to+\infty}} \frac{\log \nu_f(r_p)}{\log r_p} = +\infty. \tag{6.3}$$

Let $P_0(z) = \sum_{i=0}^{n} b_i z^i$ where n is a positive integer and $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$. By Lemma 5.1, for any given ε $(0 < 3\varepsilon < \min(m - \beta, n - \lambda, \pi/4n))$, there are $2n$ closed angles

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon \le \theta \le -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \qquad (j = 0, 1, \dots, 2n-1).$$

For the above θ_0 and $0 < 3\varepsilon < \min\left(m - \beta, n - \lambda, \frac{\pi}{4n}\right)$, there are three cases:

(1) $r_p e^{i\theta_0} \in S_j$ where j is odd;
(2) $r_p e^{i\theta_0} \in S_j$ where j is even;
(3) $\theta_0 = -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j = 0, 1, \dots, 2n-1$.

Now we have three cases to prove Theorem 1.3.

Case (1): $r_p e^{i\theta_0} \in S_j$ where j is odd. Since $\lim_{p\to+\infty} \theta_p = \theta_0$, there is a $N > 0$ such that $r_p e^{i\theta_p} \in S_j$ when $p > N$. By Lemma 5.1, we have

$$\operatorname{Re}\left\{P_0\left(r_p e^{i\theta_p}\right)\right\} < -\delta r_p^n \quad (\delta > 0), \qquad \text{i.e.,} \qquad \operatorname{Re}\left\{-P_0\left(r_p e^{i\theta_p}\right)\right\} > \delta r_p^n. \tag{6.4}$$

From (6.1) and (6.4), we obtain for sufficiently large p,

$$\begin{aligned} &\operatorname{Re}\left\{P_j\left(r_p e^{i\theta_p}\right) - P_0\left(r_p e^{i\theta_p}\right)\right\} \\ &\qquad = \operatorname{Re}\{(c-1)P_0 + (P_j - cP_0)\} \\ &\qquad < r_p^{m+\varepsilon} - (c-1)\delta r_p^n \ \ (j = 1, \dots, k-1). \end{aligned} \tag{6.5}$$

By (1.6), we have

$$-e^{-P_0(z)}\frac{f^{(k)}}{f} = A_{k-1}(z)\, e^{P_{k-1}(z)-P_0(z)}\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\, e^{P_1(z)-P_0(z)}\frac{f'}{f} + A_0(z). \tag{6.6}$$

Substituting (5.6) into (6.6), we get for $z_p = r_p e^{i\theta_p}$

$$\begin{aligned} &-\nu_f^k(r_p)(1+o(1))\exp\{-P_0(z_p)\} \\ &\quad = A_{k-1}(z_p)\exp\{P_{k-1}(z_p) - P_0(z_p)\}\, z_p \nu_f^{k-1}(r_p)(1+o(1)) + \cdots \\ &\qquad + A_1(z_p)\exp\{P_1(z_p) - P_0(z_p)\}\, z_p^{k-1}\nu_f(r_p)(1+o(1)) + z_p^k A_0(z_p). \end{aligned} \tag{6.7}$$

Thus we have, from (6.2) and (6.4)

$$\left|-\nu_f^k(r_p)(1+o(1))\exp\{-P_0(z_p)\}\right| \ge \frac{1}{2}\exp\left\{\delta r_p^n\right\}\exp\left\{k r_p^{\lambda-\varepsilon}\right\} > \exp\left\{\delta r_p^n\right\}. \tag{6.8}$$

And by (5.13), (6.5) and (6.2), we have

$$\begin{aligned}(6.9)\quad &\left|A_{k-1}(z_p)\exp\{P_{k-1}(z_p)-P_0(z_p)\}z_p\nu_f^{k-1}(r_p)(1+o(1))\right.\\ &\quad\left.+\cdots+A_1(z_p)\exp\{P_1(z_p)-P_0(z_p)\}z_p^{k-1}\nu_f(r_p)(1+o(1))+z_p^kA_0(z_p)\right|\\ &\le 2(k-1)r_p^{k-1}\exp\left\{r_p^{\beta+\varepsilon}\right\}\exp\left\{r_p^{m+\varepsilon}-(c-1)\delta r_p^n\right\}\\ &\quad\times\exp\left\{(k-1)r_p^{\lambda+\varepsilon}\right\}+r_p^k\exp\left\{r_p^{\beta+\varepsilon}\right\}\\ &\le\exp\left\{(k-1)r_p^{\lambda+2\varepsilon}\right\}.\end{aligned}$$

From (6.7) we see that (6.8) contradicts (6.9).

Case (2): $r_pe^{i\theta_0}\in S_j$ where j is even. Since $\lim_{p\to+\infty}\theta_p=\theta_0$, there is a $N>0$ such that $r_pe^{i\theta_p}\in S_j$ when $p>N$. By Lemma 5.1, we have

$$(6.10)\qquad \operatorname{Re}\left\{P_0\left(r_pe^{i\theta_p}\right)\right\}>\delta r_p^n,\ \operatorname{Re}\left\{-cP_0\left(r_pe^{i\theta_p}\right)\right\}<-c\delta r_p^n,$$

$$\begin{aligned}(6.11)\qquad \operatorname{Re}\left\{(1-c)P_0\left(r_pe^{i\theta_p}\right)\right\}<(1-c)\delta r_p^n,\ &\operatorname{Re}\left\{P_j\left(r_pe^{i\theta_p}\right)-cP_0\left(r_pe^{i\theta_p}\right)\right\}\\ &<-c\delta r_p^n\ (j=2,\dots,k-1).\end{aligned}$$

By (1.6), we have

$$\begin{aligned}(6.12)\quad -A_1(z)e^{P_1(z)-cP_0(z)}\frac{f'}{f}&=e^{-cP_0(z)}\frac{f^{(k)}}{f}+A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)}\frac{f^{(k-1)}}{f}\\ &\quad+\cdots+A_2(z)e^{P_2(z)-cP_0(z)}\frac{f''}{f}+A_0(z)e^{(1-c)P_0(z)}.\end{aligned}$$

Substituting (5.6) into (6.12) we get for $z_p=r_pe^{i\theta_p}$

$$\begin{aligned}(6.13)\quad &-A_1(z_p)\exp\{P_1(z_p)-cP_0(z_p)\}z_p^{k-1}\nu_f(r_p)(1+o(1))\\ &=\nu_f^k(r_p)(1+o(1))\exp\{-cP_0(z_p)\}+\nu_f^{k-1}(r_p)(1+o(1))\\ &\quad\times z_pA_{k-1}(z_p)\exp\{P_{k-1}(z_p)-cP_0(z_p)\}+\cdots+z_p^{k-2}\nu_f^2(r_p)(1+o(1))\\ &\quad\times A_2(z_p)\exp\{P_2(z_p)-cP_0(z_p)\}+z_p^kA_0(z_p)\exp\{(1-c)P_0(z_p)\}.\end{aligned}$$

Thus we get, from (5.13), (6.1) and (6.2)

$$\begin{aligned}(6.14)\quad &\left|-A_1(z_p)\exp\{P_1(z_p)-cP_0(z_p)\}z_p^{k-1}\nu_f(r_p)(1+o(1))\right|\\ &\ge\frac{1}{2}r_p^{k-1}\exp\left\{-r_p^{\beta+\varepsilon}\right\}\exp\left\{-r_p^{m+\varepsilon}\right\}\exp\left\{r_p^{\lambda-\varepsilon}\right\}>\exp\left\{-r_p^{m+\varepsilon}\right\}.\end{aligned}$$

And from (5.13), (6.2) and (6.10), (6.11) we have for sufficiently large p

$$\begin{aligned}(6.15)\quad &\left|\nu_f^k(r_p)(1+o(1))\exp\{-cP_0(z_p)\}+\nu_f^{k-1}(r_p)(1+o(1))\right.\\ &\quad\times z_pA_{k-1}(z_p)\exp\{P_{k-1}(z_p)-cP_0(z_p)\}+\cdots+z_p^{k-2}\nu_f^2(r_p)(1+o(1))\\ &\quad\left.\times A_2(z_p)\exp\{P_2(z_p)-cP_0(z_p)\}+z_p^kA_0(z_p)\exp\{(1-c)P_0(z_p)\}\right|\\ &\le 2\exp\left\{kr_p^{\lambda+\varepsilon}\right\}\exp\left\{-c\delta r_p^n\right\}+2(k-2)r_p^{k-2}\exp\left\{(k-1)r_p^{\lambda+\varepsilon}\right\}\\ &\quad\times\exp\left\{r_p^{\beta+\varepsilon}\right\}\exp\left\{-c\delta r_p^n\right\}+r_p^k\exp\left\{r_p^{\beta+\varepsilon}\right\}\exp\left\{(1-c)\delta r_p^n\right\}\\ &<\exp\left\{\frac{(1-c)}{2}\delta r_p^n\right\}.\end{aligned}$$

Thus $(6.13)-(6.15)$ imply a contradiction.

Case (3) : $\theta_0 = -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j = 0, 1, \dots, 2n-1$. Since $\operatorname{Re}\left\{P_0\left(r_p e^{i\theta_0}\right)\right\} = 0$ when r_p is sufficiently large and a straight line $\arg z = \theta_0$ is an asymptotic line of $\left\{r_p e^{i\theta_p}\right\}$, there is a $N > 0$ such that when $p > N$, we have

$$(6.16)\quad -1 < \operatorname{Re}\left\{P_0\left(r_p e^{i\theta_p}\right)\right\} < 1, \qquad -c < \operatorname{Re}\left\{P_j\left(r_p e^{i\theta_p}\right)\right\} < c \qquad (j = 1, \dots, k-1).$$

By considering $\operatorname{Re}\left\{P_j\left(r_p e^{i\theta_p}\right) - cP_0\left(r_p e^{i\theta_p}\right)\right\}$, we again split this into three cases.
Case (i):

$$(6.17)\qquad \operatorname{Re}\left\{P_j\left(r_p e^{i\theta_p}\right) - cP_0\left(r_p e^{i\theta_p}\right)\right\} < -dr_p^m \qquad (j = 1, \dots, k-1)$$

($d > 0$ is a constant) when p is sufficiently large. We have a slightly modified form of (6.7)

$$\begin{aligned}(6.18)\quad & -\nu_f^k(r_p)(1+o(1))\exp\{-cP_0(z_p)\} \\ &= A_{k-1}(z_p)\exp\{P_{k-1}(z_p) - cP_0(z_p)\} z_p \nu_f^{k-1}(r_p)(1+o(1)) + \cdots \\ &\quad + A_1(z_p)\exp\{P_1(z_p) - cP_0(z_p)\} z_p^{k-1}\nu_f(r_p)(1+o(1)) \\ &\quad + z_p^k A_0(z_p)\exp\{(1-c)P_0(z_p)\}.\end{aligned}$$

Thus we get, from (5.13) and $(6.16) - (6.18)$

$$\begin{aligned}\frac{1}{2}\nu_f^k(r_p)\exp\{-c\} &< \left|-\nu_f^k(r_p)(1+o(1))\exp\{-cP_0(z_p)\}\right| \\ &\le \left|A_{k-1}(z_p)\exp\{P_{k-1}(z_p) - cP_0(z_p)\} z_p\nu_f^{k-1}(r_p)(1+o(1))\right| \\ &\quad + \cdots + \left|A_1(z_p)\exp\{P_1(z_p) - cP_0(z_p)\} z_p^{k-1}\nu_f(r_p)(1+o(1))\right| \\ &\quad + \left|z_p^k A_0(z_p)\exp\{(1-c)P_0(z_p)\}\right| \\ &\le 2(k-1) r_p^{k-1}\nu_f^{k-1}(r_p)\exp\left\{r_p^{\beta+\varepsilon} - dr_p^m\right\} \\ &\quad + r_p^k \exp\left\{r_p^{\beta+\varepsilon}\right\}\exp\{(c-1)\} \\ &\le \nu_f^{k-1}(r_p)\exp\left\{r_p^{\beta+2\varepsilon}\right\}\end{aligned}$$

i.e.,

$$\frac{1}{2}\nu_f(r_p) < \exp\{c\}\exp\left\{r_p^{\beta+2\varepsilon}\right\}.$$

This is in contradiction with $\nu_f(r_p) \ge \exp\left\{r_p^{\lambda-\varepsilon}\right\}$.
Case (ii):

$$\operatorname{Re}\left\{P_j\left(r_p e^{i\theta_p}\right) - cP_0\left(r_p e^{i\theta_p}\right)\right\} > dr_p^m \qquad (j = 1, \dots, k-1),$$

i.e.,

$$(6.19)\qquad \operatorname{Re}\left\{P_0\left(r_p e^{i\theta_p}\right) - \frac{1}{c}P_j\left(r_p e^{i\theta_p}\right)\right\} < -\frac{d}{c}r_p^m \qquad (j = 1, \dots, k-1)$$

($d > 0$ is a constant) when p is sufficiently large. From (6.16), we obtain for sufficiently large p,

$$\begin{aligned}(6.20)\quad \operatorname{Re}\left\{P_s\left(r_p e^{i\theta_p}\right) - \frac{1}{c}P_j\left(r_p e^{i\theta_p}\right)\right\} &= \operatorname{Re}\left\{P_s\left(r_p e^{i\theta_p}\right)\right\} - \frac{1}{c}\operatorname{Re}\left\{P_j\left(r_p e^{i\theta_p}\right)\right\} \\ &< c+1 \quad (s = 1, \dots, j-1, j+1, \dots, k-1).\end{aligned}$$

We have a slightly modified form of (6.7)

$$
\begin{aligned}
(6.21)\quad & -\nu_f^k(r_p)(1+o(1))\exp\left\{-\frac{1}{c}P_j(z_p)\right\} \\
&= A_{k-1}(z_p)\exp\left\{P_{k-1}(z_p)-\frac{1}{c}P_j(z_p)\right\}z_p\nu_f^{k-1}(r_p)(1+o(1))+\cdots \\
&\quad + A_j(z_p)\exp\left\{\left(1-\frac{1}{c}\right)P_j(z_p)\right\}z_p^{k-j}\nu_f^j(r_p)(1+o(1))+\cdots \\
&\quad + A_1(z_p)\exp\left\{P_1(z_p)-\frac{1}{c}P_j(z_p)\right\}z_p^{k-1}\nu_f(r_p)(1+o(1)) \\
&\quad + z_p^kA_0(z_p)\exp\left\{P_0(z_p)-\frac{1}{c}P_j(z_p)\right\}.
\end{aligned}
$$

Thus we get, from (5.13), (6.16) and $(6.19)-(6.21)$

$$
\begin{aligned}
&\frac{1}{2}\nu_f^k(r_p)\exp\{-1\} \\
&\quad < \left|-\nu_f^k(r_p)(1+o(1))\exp\left\{-\frac{1}{c}P_j(z_p)\right\}\right| \\
&\quad \le \left|A_{k-1}(z_p)\exp\left\{P_{k-1}(z_p)-\frac{1}{c}P_j(z_p)\right\}z_p\nu_f^{k-1}(r_p)(1+o(1))\right| \\
&\qquad + \cdots + \left|A_j(z_p)\exp\left\{\left(1-\frac{1}{c}\right)P_j(z_p)\right\}z_p^{k-j}\nu_f^j(r_p)(1+o(1))\right| \\
&\qquad + \cdots + \left|A_1(z_p)\exp\left\{P_1(z_p)-\frac{1}{c}P_j(z_p)\right\}z_p^{k-1}\nu_f(r_p)(1+o(1))\right| \\
&\qquad + \left|z_p^kA_0(z_p)\exp\left\{P_0(z_p)-\frac{1}{c}P_j(z_p)\right\}\right| \\
&\quad \le 2(k-1)\nu_f^{k-1}(r_p)r_p^{k-1}\exp\left\{r_p^{\beta+\varepsilon}\right\} \\
&\qquad \times\exp\{c+1\}+r_p^k\exp\left\{r_p^{\beta+\varepsilon}\right\}\exp\left\{-\frac{d}{c}r_p^m\right\} \\
&\quad \le \nu_f^{k-1}(r_p)\exp\left\{r_p^{\beta+2\varepsilon}\right\}
\end{aligned}
$$

i.e.,

$$\nu_f(r_p)\exp\{-1\} < 2\exp\left\{r_p^{\beta+2\varepsilon}\right\}.$$

This is in contradiction with $\nu_f(r_p)\ge\exp\left\{r_p^{\lambda-\varepsilon}\right\}$.

Case (iii): When p is sufficiently large,

$$-1<\mathrm{Re}\left\{P_j\left(r_pe^{i\theta_p}\right)-cP_0\left(r_pe^{i\theta_p}\right)\right\}<1 \qquad (j=1,\ldots,k-1).$$

By using the same reasoning as in Case (ii) we get a contradiction. The proof of Theorem 1.3 is completed.

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