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FIXED POINTS AND THE STABILITY OF JENSEN'S FUNCTIONAL EQUATION

LIVIU CĂDARIU AND VIOREL RADU

UNIVERSITATEA "POLITEHNICA" DIN TIMIŞOARA,
DEPARTAMENTUL DE MATEMATICĂ,
PIAŢA REGINA MARIA NO.1,
1900 TIMIŞOARA, ROMÂNIA.
lcadariu@yahoo.com

UNIVERSITATEA DE VEST DIN TIMIŞOARA,
FACULTATEA DE MATEMATICĂ,
BV. VASILE PÂRVAN 4,
1900 TIMIŞOARA, ROMÂNIA.
radu@hilbert.math.uvt.ro

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ABSTRACT. We will present a fixed point method for the stability theorems of functional equations of Jensen type as given by S.-M. Jung [11] and Wang Jian [10].

Key words and phrases: Jensen functional equation, Fixed point, Stability.

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1. INTRODUCTION

The study of stability problems for functional equations is strongly related to the following question of S. M. Ulam concerning the stability of group homomorphisms:
Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

D. H. Hyers [7] gave the first affirmative answer to the question of Ulam, for Banach spaces. Subsequently, his result was extended and generalized in several ways (see e.g. [8, 18]). Th. M. Rassias in [17] and Z. Gajda in [4] considered the stability problem with unbounded Cauchy differences. The above results can be partially summarized in the following

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Theorem 1.1. *(Hyers-Rassias-Gajda)* [4, 8, 17]. *Suppose that E is a real normed space, F is a real Banach space, $f : E \to F$ is a given function, and the following condition holds*

$$(\mathbf{C_p}) \qquad \|f(x+y) - f(x) - f(y)\|_F \leq \theta(\|x\|_E^p + \|y\|_E^p), \forall x, y \in E,$$

for some $p \in [0, \infty)\backslash\{1\}$. Then there exists a unique additive function $c : E \to F$ such that

$$(\mathbf{Est_p}) \qquad ||f(x) - c(x)||_F \leq \frac{2\theta}{|2-2^p|} \|x\|_E^p, \forall x \in E.$$

This phenomenon is called *generalized Hyers-Ulam stability.* It is worth noting that almost all subsequent proofs in this very active area used the Hyers' method. Namely, the function $c : E \to F$ is explicitly constructed, starting from the given function f, by the formulae

$$(\mathbf{j_{p<1}}) \qquad c(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \qquad \text{if } p < 1;$$

$$(\mathbf{j_{p>1}}) \qquad c(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \qquad \text{if } p > 1.$$

This method is called a direct method.

There are known also other approaches, for example using the invariant mean technique introduced by Szekelyhidi (see e.g. [22, 23]), or based on the sandwich theorems (see [14]). The interested reader is referred to the expository papers [3, 18, 24] and the book [8].

One of the present authors observed recently (see [16]) that *the existence of c and the estimation* $(\mathbf{Est_p})$ can be obtained from the *fixed point alternative.*

We will show how this method can be applied to stability theorems of Jensen type, that is starting from initial conditions of the form

$$(\mathbf{J}_\varphi) \qquad \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_F \leq \varphi(x,y), \forall x, y \in E.$$

As a particular case, we obtain a new proof for the following theorem:

Theorem 1.2. *(compare with* [11, 12]*). Let $p \geq 0$ be given, with $p \neq 1$. Assume that $\delta \geq 0$ and $\theta \geq 0$ are fixed. Suppose that the mapping $f : E \to F$ satisfies the inequality*

$$(\mathbf{J_p}) \qquad \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_F \leq \delta + \theta(\|x\|^p + \|y\|^p), \forall x, y \in E,$$

Further, assume $f(0) = \delta = 0$ in the case $p > 1$.

Then there exists a unique additive mapping $j : E \to F$ such that

$$(\mathbf{Est_{p<1}}) \qquad \|f(x) - j(x)\| \leq \frac{\delta}{2^{1-p}-1} + \|f(0)\| + \frac{\theta}{2^{1-p}-1}\|x\|^p, \forall x \in E,$$

or

$$(\mathbf{Est_{p>1}}) \qquad \|f(x) - j(x)\| \leq \frac{2^{p-1}\theta}{2^{p-1}-1}\|x\|^p, \forall x \in E.$$

For the proof, see Section 3.

We think that our method of proof is working in more situations, allowing to obtain, in a simple manner, general stability theorems.

2. THE ALTERNATIVE OF FIXED POINT

For the sake of convenience and for explicit later use, we will recall two fundamental results in fixed point theory.

Theorem 2.1. *(Banach's contraction principle). Let* (X, d) *be a complete metric space, and consider a mapping* $J : X \to X$, *which is strictly contractive, that is*

$$(\mathbf{B_1}) \qquad d(Jx, Jy) \leq Ld(x, y), \forall x, y \in X,$$

for some (Lipschitz constant) $L < 1$. *Then*

(i) *The mapping* J *has one, and only one, fixed point* $x^* = J(x^*)$;

(ii) *The fixed point* x^* *is globally attractive, that is*

$$(\mathbf{B_2}) \qquad \lim_{n\to\infty} J^n x = x^*,$$

for any starting point $x \in X$;

(iii) *One has the following estimation inequalities:*

$$(\mathbf{B_3}) \qquad d(J^n x, x^*) \leq L^n d(x, x^*), \forall n \geq 0, \forall x \in X;$$

$$(\mathbf{B_4}) \qquad d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1}x), \forall n \geq 0, \forall x \in X;$$

$$(\mathbf{B_5}) \qquad d(x, x^*) \leq \frac{1}{1-L} d(x, Jx), \forall x \in X.$$

Theorem 2.2. *(The alternative of fixed point)* [13, 19]. *Suppose we are given a complete generalized metric space* (X, d) *and a strictly contractive mapping* $J : X \to X$, *with the Lipschitz constant* L. *Then, for each given element* $x \in X$, *either*

$$(\mathbf{A_1}) \qquad d(J^n x, J^{n+1}x) = +\infty, \ \forall n \geq 0,$$

or

$(\mathbf{A_2})$ *There exists a natural number* n_0 *such that:*

$(\mathbf{A_{20}})$ $d(J^n x, J^{n+1}x) < +\infty, \forall n \geq n_0$;
$(\mathbf{A_{21}})$ *The sequence* $(J^n x)$ *is convergent to a fixed point* y^* *of* J;
$(\mathbf{A_{22}})$ y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X, d(J^{n_0}x, y) < +\infty\}$;
$(\mathbf{A_{23}})$ $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy), \forall y \in Y.$

Remark 2.3.

(a) The fixed point y^*, if it exists, is not necessarily unique *in the whole space* X; it may depend on x.

(b) Actually, if $(\mathbf{A_2})$ holds, then (Y, d) is a complete *metric* space and $J(Y) \subset Y$. Therefore the properties $(A_{21}) - (A_{23})$ are easily seen to follow from Theorem 2.1.

3. A GENERALIZED THEOREM OF STABILITY FOR JENSEN'S EQUATION

Using the fixed point alternative we can prove our main result, a generalized theorem of stability for Jensen's functional equation (see also [5, 10, 11, 12]):

Theorem 3.1. *Let E be a (real or complex) linear space, F and Banach space, and $q_i = \begin{cases} 2, & i=0 \\ \frac{1}{2}, & i=1 \end{cases}$. Suppose that the mapping $f : E \to F$ satisfies the condition $f(0) = 0$ and an inequality of the form*

$$(\mathbf{J}_\varphi) \qquad \left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\|_F \le \varphi(x,y), \forall x, y \in E,$$

where $\varphi : E \times E \to [0, \infty)$ is a given function.

If there exists $L = L(i) < 1$ such that the mapping

$$x \to \psi(x) = \varphi(x, 0)$$

has the property

$$(\mathbf{H_i}) \qquad \psi(x) \le L \cdot q_i \cdot \psi\left(\frac{x}{q_i}\right), \forall x \in E,$$

and the mapping φ has the property

$$(\mathbf{H_i^*}) \qquad \lim_{n\to\infty} \frac{\varphi(2q_i^n x, 2q_i^n y)}{2q_i^n} = 0, \forall x, y \in E,$$

then there exists a unique additive mapping $j : E \to F$ such that

$$(\mathbf{Est_i}) \qquad \|f(x) - j(x)\|_F \le \frac{L^{1-i}}{1-L} \psi(x), \forall x \in E.$$

Proof. Consider the set

$$X := \{g : E \to F,\ g(0) = 0\}$$

and introduce the *generalized metric* on X :

$$d(g,h) = d_\psi(g,h) = \inf\{C \in R_+, \|g(x) - h(x)\|_F \le C\psi(x), \forall x \in E\}$$

It is easy to see that (X, d) *is complete.*

Now we will consider the (linear) mapping

$$J : X \to X, Jg(x) := \frac{1}{q_i} \cdot g(q_i x).$$

Note that $q_0 = 2$ if $(\mathbf{H}_0)$ holds, and $q_1 = 2^{-1}$ if $(\mathbf{H}_1)$ holds.

We have, for any $g, h \in X$:

$$\begin{aligned} d(g,h) < C &\Longrightarrow \|g(x) - h(x)\|_F \le C\psi(x), \forall x \in E \\ &\Longrightarrow \left\| \frac{1}{q_i} g(q_i x) - \frac{1}{q_i} h(q_i x) \right\|_F \le \frac{1}{q_i} C\psi(q_i x), \forall x \in E \\ &\Longrightarrow \left\| \frac{1}{q_i} g(q_i x) - \frac{1}{q_i} h(q_i x) \right\|_F \le LC\psi(x), \forall x \in E \\ &\Longrightarrow d(Jg, Jh) \le LC. \end{aligned}$$

Therefore we see that

$$d(Jg, Jh) \le L d(g,h), \forall g, h \in X,$$

that is J is a *strictly contractive* self-mapping of X, with the Lipschitz constant L.

If the hypothesis $(\mathbf{H}_0)$ holds, and we set $x = 2t$ and $y = 0$ in the condition $(\mathbf{J}_\varphi)$, then we see that

$$\left\| f(t) - \frac{1}{2} f(2t) \right\|_F \le \frac{1}{2}\psi(2t) \le L\psi(t), \forall t \in E,$$

that is $d(f,Jf) \le L = L^1 < \infty$. Now, if the hypothesis $(\mathbf{H}_1)$ holds, and we set $y = 0$ in the condition ($\mathbf{J}_\varphi$), then we see that

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_F \le \psi(x), \forall x \in E.$$

Therefore $d(f,Jf) \le 1 = L^0 < \infty$.

In both cases we can apply the fixed point alternative, and we obtain the existence of a mapping $j : X \to X$ such that:

- j is a fixed point of J, that is

(3.1) $$j(2x) = 2j(x), \forall x \in E.$$

The mapping j is the unique fixed point of J in the set

$$Y = \{g \in X,\ d(f,g) < \infty\}.$$

This says that j is the unique mapping with *both* the properties (3.1) and (3.2), where

(3.2) $$\exists C \in (0,\infty) \text{ such that } \|j(x) - f(x)\|_F \le C\psi(x), \forall x \in E.$$

- $d(J^n f, j) \xrightarrow[n\to\infty]{} 0$, which implies the equality

(3.3) $$\lim_{n\to\infty} \frac{f(q_i^n x)}{q_i^n} = j(x), \forall x \in X.$$

- $d(f,j) \le \dfrac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,j) \le \frac{L^{1-i}}{1-L},$$

that is ($\mathbf{Est_i}$) is seen to be true.

The additivity of j follows immediately from ($\mathbf{J}_\varphi$) and (3.3): If in ($\mathbf{J}_\varphi$) we replace x by $2q_i^n x$ and y by $2q_i^n y$, then we obtain

$$\left\| \frac{f(q_i^n(x+y))}{q_i^n} - \frac{f(2q_i^n x)}{2q_i^n} - \frac{f(2q_i^n y)}{2q_i^n} \right\|_F \le \frac{\varphi(2q_i^n x, 2q_i^n y)}{2q_i^n}, \forall x, y \in E.$$

Taking into account the hypothesis ($\mathbf{H_i^*}$) and letting $n \to \infty$, we get

$$j(x+y) = j(x) + j(y), \quad \forall x, y \in E,$$

which ends the proof. □

The proof of Theorem 1.2. If we suppose that $f(0) = 0$, then the proof follows from our Theorem 3.1 by taking

$$\varphi(x,y) := \delta + \theta(\|x\|^p + \|y\|^p), \quad \forall x, y \in E,$$

which appears in the hypothesis ($\mathbf{J_p}$). We see that

$$\frac{\varphi(2q_i^n x, 2q_i^n y)}{2q_i^n} = \frac{\delta}{2q_i^n} + (2q_i^n)^{p-1}\theta(\|x\|^p + \|y\|^p) \xrightarrow[n\to\infty]{} 0,$$

that is ($\mathbf{H_i^*}$) is true, and our method works by the following reasons:

- $\dfrac{1}{2}\psi(2x) = \dfrac{1}{2}\delta + 2^{p-1}\theta\|x\|^p \le 2^{p-1}\psi(x)$, for $p < 1$;
- $2\psi\left(\dfrac{x}{2}\right) = \dfrac{1}{2^{p-1}}\theta\|x\|^p \le \dfrac{1}{2^{p-1}}\psi(x)$, for $p > 1$,

which actually say that either $(\mathbf{H}_0)$ holds with $L = 2^{p-1}$ or $(\mathbf{H}_1)$ holds with $L = \frac{1}{2^{p-1}}$.

The general case (for $p < 1$) follows immediately by considering the mapping $\widetilde{f} = f - f(0)$:

$$\|f(x) - j(x)\| \leq \left\| \widetilde{f}(x) - j(x) \right\| + \|f(0)\| \leq \frac{\delta}{2^{1-p} - 1} + \|f(0)\| + \frac{\theta}{2^{1-p} - 1} \|x\|^p .$$

□

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