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GÂTEAUX DERIVATIVE AND ORTHOGONALITY IN C_1-CLASSES

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ABSTRACT. The general problem in this paper is minimizing the $C_1(H)$-norm of suitable affine mappings from $B(H)$ to $C_1(H)$, using convex and differential analysis (Gâteaux derivative) as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The main results obtained characterize global minima in terms of (Banach space) orthogonality, and constitute an interesting combination of infinite-dimensional differential analysis, convex analysis, operator theory and duality.

Key words and phrases: Elementary operators, C_1-classes, orthogonality, Gateaux derivative.

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1. INTRODUCTION

Suppose $B = B(H)$ is the algebra of bounded linear operators on the complex infinite dimensional separable Hilbert space H, and let $T \in B$ be compact: then [13] we write $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for the "singular values" of T, i.e. the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$, counted according to multiplicity and arranged in decreasing order. If $1 \leq p < \infty$ we define the Schatten p-class $C_p = C_P(H)$ as the set of those compact $T \in B$ with finite p-norm

$$\|T\|_p = \left(\sum_{j=1}^{\infty} s_j(T)^p\right)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty;$$

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here tr denotes the trace functional. Thus $C_1 = C_1(H)$ is the trace class, $C_2 = C_2(H)$ is the Hilbert-Schmidt class. We write $C_\infty = C_\infty(H)$ for the compact operators, with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

the usual operator norm of T.

If V is a Banach space then a mapping $f : V \to \mathbb{C}$ is said to be "Gateâux differentiable" at a point $a \in V$ if the limit

$$\lim_{t\to 0^+} \frac{1}{t}\left(f(a+tx) - f(a)\right)$$

exists for each point $x \in V$. If this applies to the norm $f = \|\cdot\|$, then $a \in V$ is said to be a smooth point, and the functional $d_a f = \mathrm{Re}(D_a)$ is [13] sublinear, with

$$\|D_a\| = 1; \quad D_a = \|a\| = -D_a(-a).$$

For elements a, b of a Banach space V, we say that b is orthogonal to a, written $b \perp a$, provided

$$\|a\| = dist(a, \mathbb{C}b),$$

such that the line $a + \mathbb{C}b$ is tangential to the ball of center 0 and radius $\|a\|$; when $V = H$ is a Hilbert space this agrees with the usual inner product $\langle a; b\rangle = 0$. Thus when $b \perp a$ then the expression $\|a + \lambda b\|$ has a global minimum when $\lambda = 0 \in \mathbb{C}$.

In this paper we show how such a global minimum can be detected by the sign of the Gateâux derivative, and apply it to range-kernel orthogonality for certain kinds of elementary operators.

2. Global Minima

Our main result characterises certain kinds of global minima in terms of the Gateâux derivative of a norm:

Theorem 2.1. *If $\varphi : V \to V$ is linear and $a \in V$ then the mapping*

$$x \mapsto \|a + \varphi(x)\| \quad (V \to \mathbb{R}) \tag{2.1}$$

has a global minimum at $b \in V$ if and only if

$$\forall x \in V, \quad D_{a+\varphi(b)}(x) \geq 0. \tag{2.2}$$

Proof. Necessity follows from the linearity of φ. Conversely with $L = D_{a+\varphi(b)}$,

$$\|a + \varphi(b)\| = -L(-a - \varphi(b)) + L(\varphi(x) - \varphi(b)) \leq L(a + \varphi(x)) \leq \|a + \varphi(x)\|.$$

□

It is well known that this holds for all $a \in V = C_p(H)$, since [7] C_p is always uniformly convex; specifically, with $a = u|a|$ the polar decomposition

$$D_a(x) = \|a\|_p^{-(p-1)}\, tr\left(|a|^{p-1} u x^*\right), \quad x \in C_p.$$

This fails when either $p = 1$ or $p = \infty$. The norm [2] Gateâux differentiable at $0 \neq a \in C_1(H)$ if and only if either a or a^* is injective, with for $a = u|a|$

$$D_a(x) = \mathrm{Re}\, tr(u^* x) \quad \text{if } a \text{ one one},$$

$$D_a(x) = \mathrm{Re}\, tr(ux) \quad \text{if } a^* \text{ one one}.$$

We now offer a characterization of the global minimum of a map $x \mapsto \|a_\varphi(x)\|$ derived from a linear map $\varphi : C_1 \to C_1$ which is adjointable in the sense that there exists $\varphi^* : C_1 \to C_1$ for which

$$\forall x, y \in C_1, \quad tr(x\varphi(y)) = tr(\varphi^*(x)y).$$

This is certainly the case for the elementary operators $E_{a,b} = L_a \circ R_b$ induced by $a, b \in B^n$:

$$E^*_{a,b} = E_{a^*,b^*}.$$

Theorem 2.2. *A necessary and sufficient condition for $\|a + \varphi(x)\|$ to have a global minimum at a smooth point $b \in C_1(H)$, with polar decomposition $a + \varphi(b) = u|a + \varphi(b)|$, is that*

$$u^* \in \ker(\varphi^*).$$

Proof. Assume that $\|a + \varphi(x)\|$ has a global minimum on C_1 at b. Then

$$D_{a+\varphi(b)}(\varphi(x)) \geq 0 \tag{2.3}$$

for all $x \in C_1$. That is,

$$\operatorname{Re}\{tr(u^*\varphi(x))\} \geq 0, \quad \forall x \in C_1.$$

Let $f \otimes g$, be the rank one operator defined by $v \mapsto \langle v, f\rangle\, g$, where f, g are arbitrary vectors in the Hilbert space H. Take $x = f \otimes g$, since the map φ satisfies (2.2) one has

$$tr(u^*\varphi(x)) = tr(\varphi^*(u^*)x).$$

Then (2.3) is equivalent to $\operatorname{Re}\{tr(\phi^*(u^*)x)\} \geq 0$, for all $x \in C_1$, or equivalently

$$\operatorname{Re}\langle \varphi^*(u^*)g, f\rangle \geq 0, \quad \forall f, g \in H.$$

If we choose $f = g$ such that $\|f\| = 1$, we get

$$\operatorname{Re}\langle \varphi^*(u^*)f, f\rangle \geq 0. \tag{2.4}$$

Note that the set

$$\{\langle \varphi^*(u^*)f, f\rangle : \|f\| = 1\}$$

is the numerical range of $\phi^*(u^*)$ on $\mathcal{U}$ which is a convex set and its closure is a closed convex set. By (2.4) it must contain one value of positive real part, under all rotation around the origin, it must contain the origin, and we get a vector $f \in H$ such that $\langle \varphi^*(u^*)f, f\rangle < \epsilon$, where ϵ is positive. Since ϵ is arbitrary, we get $\langle \varphi^*(u^*)f, f\rangle = 0$. Thus $\varphi^*(u^*) = 0$, i.e., $u^* \in \ker \varphi^*$.

Conversely, if $u^* \in \ker \varphi^*$, it is easily seen (using the same arguments above) that

$$\operatorname{Re}\{tr(u^*\phi(x))\} \geq 0, \quad \forall x \in C_1.$$

By this we get (2.3). □

3. RANGE-KERNEL ORTHOGONALITY

Anderson [1] showed that for normal operators $a \in V = B = B(H)$ on a Hilbert space H then

$$ax = xa \Rightarrow \|x + ay - ya\| \geq \|x\| :$$

the range of the derivation $\delta_a : y \mapsto ay - ya$ is orthogonal to its kernel. This result has been generalized [4, 12, 14] to more general elementary operators

$$E_{a,b} \equiv L_a \circ R_b :\mapsto \sum_{j=1}^{n} a_j x b_j$$

both on $V = B(H)$ and on the Schatten ideals $V = C_p(H)$. The Gateâux derivative was used in [6], [5], [7], [8] and [10].

We state our first corollary of Theorem 2.2. Let $\varphi = \delta_{a,b}$, where $\delta_{a,b} : B(H) \to B(H)$ is the generalized derivation defined by $\delta_{a,b}(x) = ax - xb$.

Corollary 3.1. *Let s be a smooth point in C_1, and let $s + \varphi(s)$ have the polar decomposition $s + \varphi(s) = u\,|s + \varphi(s)|$. Then $\|s + \varphi(x)\|_{C_1}$ has a global minimum on C_1 at s, if and only if, $u^* \in \ker \delta_{b,a}$.*

Proof. It is a direct consequence of Theorem 2.2. □

This result may be reformulated in the following form where the global minimum s does not appear. It characterizes the smooth point s in C_1 which is orthogonal to the range of the generalized derivation $\delta_{a,b}$.

Theorem 3.2. *Let s be a smooth point in C_1, and let $s+\varphi(s)$ have the polar decomposition $s+\varphi(s) = u\,|s+\varphi(s)|$. Then*

$$\|s+\varphi(x)\|_{C_1} \geq \|s+\varphi(s)\|_{C_1},$$

for all $x \in C_1$ if and only if $u^* \in \ker \delta_{b,a}$.

As a corollary of this theorem we have

Corollary 3.3. *Let $s \in C_1 \cap \ker \delta_{a,b}$, and let $s+\varphi(s)$ have the polar decomposition $s+\varphi(s) = u\,|s+\varphi(s)|$. Then the two following assertions are equivalent:*

(1) $\|s+(ax-xb)\|_{C_1} \geq \|s\|_{C_1}$, *for all* $x \in C_1$.
(2) $u^* \in \ker \delta_{b,a}$.

Remark 3.4. We point out that, thanks to our general results given previously with more general linear maps φ, Theorem 3.2 and its Corollary 3.3 are true for the nuclear operator $\Delta_{a,b}(x) = axb - x$ and other more general classes of operators than $\delta_{a,b}$ like the elementary operators $E_{a,b}$.

Now by using Theorem 3.2, Corollary 3.3, Remark 3.4 we obtain some interesting results see also ([14]).

Let $s = u\,|s|$ be the polar decomposition of s, where s is a smooth point in C_1 and let $\tilde{E}_{a,b} = E_{a,b} - I$.

Theorem 3.5. *Let $c = (c_1, c_2, \dots, c_n)$ be an $n-$tuple of operators in $B(H)$ such that $\sum_{i=1}^{n} c_i c_i^* \leq 1$, $\sum_{i=1}^{n} c_i^* c_i \leq 1$ and $\ker \tilde{E}_c \subseteq \ker \tilde{E}_{c^*}$. Then $s \in \ker \tilde{E}_c \cap C_1$, if and only if,*

$$\left\| s + \tilde{E}_c(x) \right\|_1 \geq \|s\|_1 \tag{3.1}$$

for all $x \in C_1$.

Proof. Let s be in $\ker \tilde{E}_c|_{C_1}$. Then it follows from Corollary 3.3 applied for the elementary operator $\tilde{E}_c$ that

$$\left\| s + \tilde{E}_c(x) \right\|_1 \geq \|s\|_1$$

for all $x \in C_1$ if and only if $u^* \in \ker \tilde{E}_c$. The hypothesis $\ker \tilde{E}_c \subseteq \ker \tilde{E}_{c^*}$, implies that $u^* \in \ker \tilde{E}_{c^*}$. Note that $u^* \in \ker \tilde{E}_c \subseteq \ker \tilde{E}_{c^*}$ if and only if

$$tr(u^* \tilde{E}_c(x)) = 0 = tr(u^* \tilde{E}_{c^*}(x)). \tag{3.2}$$

Choosing $x \in C_1$ to be the rank one operator $f \otimes g$ it follows from (3.2) that if (3.1) holds then

$$= tr\left(\left(\sum_{i=1}^{n} c_i u^* c_i - u^*\right)(f \otimes g)\right)$$
$$= \left(\sum_{i=1}^{n} c_i u^* c_i g, f\right) - (u^* g, f) = 0$$

and

$$\left(\sum_{i=1}^{n} c_i^* u^* c_i^* g, f\right) - (u^* f, g) = 0$$

for all $f, g \in H$ or

$$\tilde{E}_c(u) = 0 = \tilde{E}_{c^*}(u).$$

It is known that if $\sum_{i=1}^n c_i c_i^* \leq 1$, $\sum_{i=1}^n c_i^* c_i \leq 1$ and $\tilde{E}_c(s) = 0 = \tilde{E}_c^*(s)$, then the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|s|^2$ reduce each c_i see ([4, Theorem 8]) and ([14, Lemma 2.3]). In particular $|s|$ commutes with each c_i for all $1 \leq i \leq n$. Hence (3.1) holds if and only if,

$$\tilde{E}_c(s) = 0 = \tilde{E}_c^*(s).$$

□

Theorem 3.6. *Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$ be $n-$tuples of operators in $B(H)$ such that*

$$\sum_{i=1}^n a_i a_i^* \leq 1, \sum_{i=1}^n a_i^* a_i \leq 1, \sum_{i=1}^n b_i b_i^* \leq 1, \sum_{i=1}^n b_i^* b_i \leq 1$$

and $\ker \tilde{E}_{a,b} \subseteq \ker \tilde{E}_{a^*,b^*}$.

Then $s \in \ker \tilde{E}_{a,b} \cap C_1$, if and only if,

$$\left\| s + \tilde{E}_{a,b}(x) \right\|_1 \geq \|s\|_1$$

for all $x \in C_1$.

Proof. It suffices to take the Hilbert space $H \oplus H$, and operators

$$c_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}, \quad s = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

and apply Theorem 3.5. □

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