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TIME-SCALE INTEGRAL INEQUALITIES

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ABSTRACT. Some recent and classical integral inequalities are extended to the general time-scale calculus, including the inequalities of Steffensen, Iyengar, Čebyšev, and Hermite-Hadamard.

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1. PRELIMINARIES ON TIME SCALES

The unification and extension of continuous calculus, discrete calculus, q-calculus, and indeed arbitrary real-number calculus to time-scale calculus was first accomplished by Hilger in his Ph.D. thesis [8]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various calculi, and in extending our understanding to a new, more general and overarching theory. The purpose of this work is to illustrate this new understanding by extending some continuous and q-calculus inequalities and some of their applications, such as those by Steffensen, Hermite-Hadamard, Iyengar, and Čebyšev, to arbitrary time scales.

The following definitions will serve as a short primer on the time-scale calculus; they can be found in Agarwal and Bohner [1], Atici and Guseinov [3], and Bohner and Peterson [4]. A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. Within that set, define the jump operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The so-called graininess functions are $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$.

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For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [3] of f at t, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U$. Common special cases again include $\mathbb{T} = \mathbb{R}$, where $f^\nabla = f'$, the usual derivative; $\mathbb{T} = \mathbb{Z}$, where the nabla derivative is the backward difference operator, $f^\nabla(t) = f(t) - f(t-1)$; q-difference equations with $0 < q < 1$ and $t > 0$,

$$f^\nabla(t) = \frac{f(t) - f(qt)}{(1-q)t}.$$

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [4] of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, $f^\Delta = f'$, the usual derivative; for $\mathbb{T} = \mathbb{Z}$ the delta derivative is the forward difference operator, $f^\Delta(t) = f(t+1) - f(t)$; in the case of q-difference equations with $q > 1$,

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}, \qquad f^\Delta(0) = \lim_{s\to 0} \frac{f(s) - f(0)}{s}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. It is known from [3] or Theorem 8.45 in [4] that if f is ld-continuous, then there is a function F such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

In the same way, from Theorem 1.74 in [4] we have that if g is right-dense continuous, there is a function G such that $G^\Delta(t) = g(t)$ and

$$\int_a^b g(t)\Delta t = G(b) - G(a).$$

The following theorem is part of Theorem 2.7 in [3] and Theorem 8.47 in [4].

Theorem 1.1 (Integration by parts). *If $a, b \in \mathbb{T}$ and f^∇, g^∇ are left-dense continuous, then*

$$\int_a^b f(t)g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(\rho(t))\nabla t$$

and

$$\int_a^b f(\rho(t))g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(t)\nabla t.$$

2. Taylor's Theorem Using Nabla Polynomials

The generalized polynomials for nabla equations [2] are the functions $\hat{h}_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows: The function $\hat{h}_0$ is

$$\hat{h}_0(t,s) \equiv 1 \quad \text{for all} \quad s, t \in \mathbb{T}, \tag{2.1}$$

and, given $\hat{h}_k$ for $k \in \mathbb{N}_0$, the function $\hat{h}_{k+1}$ is

$$\hat{h}_{k+1}(t,s) = \int_s^t \hat{h}_k(\tau, s)\nabla\tau \quad \text{for all} \quad s, t \in \mathbb{T}. \tag{2.2}$$

Note that the functions $\hat{h}_k$ are all well defined, since each is ld-continuous. If for each fixed s we let $\hat{h}_k^{\nabla}(t,s)$ denote the nabla derivative of $\hat{h}_k(t,s)$ with respect to t, then

$$\hat{h}_k^{\nabla}(t,s) = \hat{h}_{k-1}(t,s) \quad \text{for} \quad k \in \mathbb{N},\ t \in \mathbb{T}_\kappa. \tag{2.3}$$

The above definition implies

$$\hat{h}_1(t,s) = t - s \quad \text{for all} \quad s,t \in \mathbb{T}.$$

Obtaining an expression for $\hat{h}_k$ for $k > 1$ is not easy in general, but for a particular given time scale it might be easy to find these functions; see [2] for some examples.

Theorem 2.1 (Taylor's Formula [2])**.** *Let $n \in \mathbb{N}$. Suppose f is $n+1$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Let $s \in \mathbb{T}_{\kappa^n}$, $t \in \mathbb{T}$, and define the functions $\hat{h}_k$ by* (2.1) *and* (2.2)*, i.e.,*

$$\hat{h}_0(t,s) \equiv 1 \quad \textit{and} \quad \hat{h}_{k+1}(t,s) = \int_s^t \hat{h}_k(\tau,s)\nabla\tau \textit{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n} \hat{h}_k(t,s) f^{\nabla^k}(s) + \int_s^t \hat{h}_n(t,\rho(\tau)) f^{\nabla^{n+1}}(\tau)\nabla\tau.$$

We may also relate the functions $\hat{h}_k$ as introduced in (2.1) and (2.2) (which we repeat below) to the functions h_k and g_k in the delta case [1, 4], and the functions $\hat{g}_k$ in the nabla case, defined below.

Definition 2.1. For $t,s \in \mathbb{T}$ define the functions

$$h_0(t,s) = g_0(t,s) = \hat{h}_0(t,s) = \hat{g}_0(t,s) \equiv 1,$$

and given $h_n, g_n, \hat{h}_n, \hat{g}_n$ for $n \in \mathbb{N}_0$,

$$h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, g_{n+1}(t,s) = \int_s^t g_n(\sigma(\tau),s)\Delta\tau,$$
$$\hat{h}_{n+1}(t,s) = \int_s^t \hat{h}_n(\tau,s)\nabla\tau, \hat{g}_{n+1}(t,s) = \int_s^t \hat{g}_n(\rho(\tau),s)\nabla\tau.$$

The following theorem combines Theorem 9 of [2] and Theorem 1.112 of [4].

Theorem 2.2. *Let $t \in \mathbb{T}_\kappa^\kappa$ and $s \in \mathbb{T}^{\kappa^n}$. Then*

$$\hat{h}_n(t,s) = g_n(t,s) = (-1)^n h_n(s,t) = (-1)^n \hat{g}_n(s,t)$$

for all $n \geq 0$.

3. STEFFENSEN'S INEQUALITY

For a q-difference equation version of the following result and most results in this paper, including proof techniques, see [7]. In fact, the presentation of the results to follow largely mirrors the organisation of [7].

Theorem 3.1 (Steffensen's Inequality (nabla))**.** *Let $a,b \in \mathbb{T}_\kappa^\kappa$ with $a < b$ and $f,g : [a,b] \to \mathbb{R}$ be nabla-integrable functions, with f of one sign and decreasing and $0 \leq g \leq 1$ on $[a,b]$. Assume $\ell, \gamma \in [a,b]$ such that*

$$b - \ell \leq \int_a^b g(t)\nabla t \leq \gamma - a \quad \textit{if } f \geq 0, \quad t \in [a,b],$$

$$\gamma - a \leq \int_a^b g(t)\nabla t \leq b - \ell \quad \textit{if } f \leq 0, \quad t \in [a,b].$$

Then

$$\int_\ell^b f(t)\nabla t \leq \int_a^b f(t)g(t)\nabla t \leq \int_a^\gamma f(t)\nabla t. \tag{3.1}$$

Proof. The proof given in the q-difference case [7] can be extended to general time scales. As in [7], we prove only the case in (3.1) where $f \geq 0$ for the left inequality; the proofs of the other cases are similar. After subtracting within the left inequality,

$$\begin{aligned}
\int_a^b f(t)g(t)\nabla t - \int_\ell^b f(t)\nabla t \\
&= \int_a^\ell f(t)g(t)\nabla t + \int_\ell^b f(t)g(t)\nabla t - \int_\ell^b f(t)\nabla t \\
&= \int_a^\ell f(t)g(t)\nabla t - \int_\ell^b f(t)(1-g(t))\nabla t \\
&\geq \int_a^\ell f(t)g(t)\nabla t - f(\ell)\int_\ell^b (1-g(t))\nabla t \\
&= \int_a^\ell f(t)g(t)\nabla t - (b-\ell)f(\ell) + f(\ell)\int_\ell^b g(t)\nabla t \\
&\geq \int_a^\ell f(t)g(t)\nabla t - f(\ell)\int_a^b g(t)\nabla t + f(\ell)\int_\ell^b g(t)\nabla t \\
&= \int_a^\ell f(t)g(t)\nabla t - f(\ell)\left(\int_a^b g(t)\nabla t - \int_\ell^b g(t)\nabla t\right) \\
&= \int_a^\ell f(t)g(t)\nabla t - f(\ell)\int_a^\ell g(t)\nabla t \\
&= \int_a^\ell \left(f(t) - f(\ell)\right) g(t)\nabla t \geq 0
\end{aligned}$$

since f is decreasing and g is nonnegative. $\square$

Note that in the theorem above, we could easily replace the nabla integrals with delta integrals under the same hypotheses and get a completely analogous result. The following theorem more closely resembles the theorem in the continuous case; the proof is identical to that above and is omitted.

Theorem 3.2 (Steffensen's Inequality II). *Let* $a, b \in \mathbb{T}_\kappa^\kappa$ *and* $f, g : [a,b] \to \mathbb{R}$ *be nabla-integrable functions, with* f *decreasing and* $0 \leq g \leq 1$ *on* $[a,b]$. *Assume* $\lambda := \int_a^b g(t)\nabla t$ *such that* $b - \lambda, a + \lambda \in \mathbb{T}$. *Then*

$$\int_{b-\lambda}^b f(t)\nabla t \leq \int_a^b f(t)g(t)\nabla t \leq \int_a^{a+\lambda} f(t)\nabla t. \tag{3.2}$$

4. TAYLOR'S REMAINDER

Suppose f is $n+1$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Using Taylor's Theorem, Theorem 2.1, we define the remainder function by $\hat{R}_{-1,f}(\cdot,s) := f(s)$, and for $n > -1$,

$$\hat{R}_{n,f}(t,s) := f(s) - \sum_{j=0}^{n} \hat{h}_j(s,t) f^{\nabla^j}(t) = \int_t^s \hat{h}_n(s,\rho(\tau)) f^{\nabla^{n+1}}(\tau)\nabla\tau. \tag{4.1}$$

Lemma 4.1. *The following identity involving nabla Taylor's remainder holds:*

$$\int_a^b \hat{h}_{n+1}(t,\rho(s)) f^{\nabla^{n+1}}(s)\nabla s = \int_a^t \hat{R}_{n,f}(a,s)\nabla s + \int_t^b \hat{R}_{n,f}(b,s)\nabla s.$$

Proof. Proceed by mathematical induction on n. For $n = -1$,

$$\int_a^b \hat{h}_0(t,\rho(s)) f^{\nabla^0}(s)\nabla s = \int_a^b f(s)\nabla s = \int_a^t f(s)\nabla s + \int_t^b f(s)\nabla s.$$

Assume the result holds for $n = k-1$:

$$\int_a^b \hat{h}_k(t,\rho(s)) f^{\nabla^k}(s)\nabla s = \int_a^t \hat{R}_{k-1,f}(a,s)\nabla s + \int_t^b \hat{R}_{k-1,f}(b,s)\nabla s.$$

Let $n = k$. By Corollary 11 in [2], for fixed $t \in \mathbb{T}$ we have

$$\hat{h}_{k+1}^{\nabla_s}(t,s) = -\hat{h}_k(t,\rho(s)). \tag{4.2}$$

Thus using the nabla integration by parts rule, Theorem 1.1, we have

$$\begin{aligned}\int_a^b & \hat{h}_{k+1}(t,\rho(s)) f^{\nabla^{k+1}}(s)\nabla s \\ &= \int_a^b \hat{h}_k(t,\rho(s)) f^{\nabla^k}(s)\nabla s + \hat{h}_{k+1}(t,b) f^{\nabla^k}(b) - \hat{h}_{k+1}(t,a) f^{\nabla^k}(a).\end{aligned}$$

By the induction assumption and the definition of $\hat{h}_{k+1}$,

$$\begin{aligned}\int_a^b \hat{h}_{k+1}(t,\rho(s)) f^{\nabla^{k+1}}(s)\nabla s &= \int_a^t \hat{R}_{k-1,f}(a,s)\nabla s + \int_t^b \hat{R}_{k-1,f}(b,s)\nabla s \\ &\quad + \hat{h}_{k+1}(t,b) f^{\nabla^k}(b) - \hat{h}_{k+1}(t,a) f^{\nabla^k}(a) \\ &= \int_a^t \hat{R}_{k-1,f}(a,s)\nabla s + \int_t^b \hat{R}_{k-1,f}(b,s)\nabla s \\ &\quad + \int_b^t \hat{h}_k(s,b) f^{\nabla^k}(b)\nabla s - \int_a^t \hat{h}_k(s,a) f^{\nabla^k}(a)\nabla s \\ &= \int_a^t \left[\hat{R}_{k-1,f}(a,s) - \hat{h}_k(s,a) f^{\nabla^k}(a)\right]\nabla s \\ &\quad + \int_t^b \left[\hat{R}_{k-1,f}(b,s) - \hat{h}_k(s,b) f^{\nabla^k}(b)\right]\nabla s \\ &= \int_a^t \hat{R}_{k,f}(a,s)\nabla s + \int_t^b \hat{R}_{k,f}(b,s)\nabla s.\end{aligned}$$

□

Corollary 4.2. *For* $n \geq -1$,

$$\int_a^b \hat{h}_{n+1}(a,\rho(s)) f^{\nabla^{n+1}}(s)\nabla s = \int_a^b \hat{R}_{n,f}(b,s)\nabla s,$$
$$\int_a^b \hat{h}_{n+1}(b,\rho(s)) f^{\nabla^{n+1}}(s)\nabla s = \int_a^b \hat{R}_{n,f}(a,s)\nabla s.$$

Lemma 4.3. *The following identity involving delta Taylor's remainder holds:*

$$\int_a^b h_{n+1}(t,\sigma(s)) f^{\Delta^{n+1}}(s)\Delta s = \int_a^t R_{n,f}(a,s)\Delta s + \int_t^b R_{n,f}(b,s)\Delta s,$$

where

$$R_{n,f}(t,s) := f(s) - \sum_{j=0}^{n} h_j(s,t) f^{\Delta^j}(t).$$

5. APPLICATIONS OF STEFFENSEN'S INEQUALITY

In the following we generalize to arbitrary time scales some results from [7] by applying Steffensen's inequality, Theorem 3.1.

Theorem 5.1. *Let* $f : [a,b] \to \mathbb{R}$ *be an* $n+1$ *times nabla differentiable function such that* $f^{\nabla^{n+1}}$ *is increasing and* f^{∇^n} *is monontonic (either increasing or decreasing) on* $[a,b]$. *Assume* $\ell, \gamma \in [a,b]$ *such that*

$$b - \ell \leq \frac{\hat{h}_{n+2}(b,a)}{\hat{h}_{n+1}(b,\rho(a))} \leq \gamma - a \quad \text{if } f^{\nabla^n} \text{ is decreasing,}$$
$$\gamma - a \leq \frac{\hat{h}_{n+2}(b,a)}{\hat{h}_{n+1}(b,\rho(a))} \leq b - \ell \quad \text{if } f^{\nabla^n} \text{ is increasing.}$$

Then

$$f^{\nabla^n}(\gamma) - f^{\nabla^n}(a) \leq \frac{1}{\hat{h}_{n+1}(b,\rho(a))} \int_a^b \hat{R}_{n,f}(a,s)\nabla s \leq f^{\nabla^n}(b) - f^{\nabla^n}(\ell).$$

Proof. Assume f^{∇^n} is decreasing; the case where f^{∇^n} is increasing is similar and is omitted. Let $F := -f^{\nabla^{n+1}}$. Because f^{∇^n} is decreasing, $f^{\nabla^{n+1}} \leq 0$, so that $F \geq 0$ and decreasing on $[a,b]$. Define

$$g(t) := \frac{\hat{h}_{n+1}(b,\rho(t))}{\hat{h}_{n+1}(b,\rho(a))} \in [0,1], \quad t \in [a,b], \quad n \geq -1.$$

Note that F, g satisfy the assumptions of Steffensen's inequality, Theorem 3.1; using (4.2),

$$\int_a^b g(t)\nabla t = \frac{1}{\hat{h}_{n+1}(b,\rho(a))} \int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t = \frac{\hat{h}_{n+2}(b,a)}{\hat{h}_{n+1}(b,\rho(a))}.$$

Thus if

$$b - \ell \leq \frac{\hat{h}_{n+2}(b,a)}{\hat{h}_{n+1}(b,\rho(a))} \leq \gamma - a,$$

then

$$\int_\ell^b F(t)\nabla t \leq \int_a^b F(t)g(t)\nabla t \leq \int_a^\gamma F(t)\nabla t.$$

By Corollary 4.2 and the fundamental theorem of nabla calculus, this simplifies to

$$f^{\nabla^n}(t)|_{t=a}^{\gamma} \leq \frac{1}{\hat{h}_{n+1}(b,\rho(a))}\int_a^b \hat{R}_{n,f}(a,s)\nabla s \leq f^{\nabla^n}(t)|_{t=\ell}^{b}.$$

□

It is evident that an analogous result can be found for the delta integral case using the delta equivalent of Theorem 3.1.

Definition 5.1. A twice nabla-differentiable function $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$ if and only if $f^{\nabla^2} \geq 0$ on $[a,b]$.

The following corollary is the first Hermite-Hadamard inequality, derived from Theorem 5.1 with $n = 0$.

Corollary 5.2. *Let $f : [a,b] \to \mathbb{R}$ be convex and monotonic. Assume $\ell, \gamma \in [a,b]$ such that*

$$\ell \geq b - \frac{\hat{h}_2(b,a)}{b-\rho(a)}, \quad \gamma \geq \frac{\hat{h}_2(b,a)}{b-\rho(a)} + a \quad \text{if } f \text{ is decreasing,}$$

$$\ell \leq b - \frac{\hat{h}_2(b,a)}{b-\rho(a)}, \quad \gamma \leq \frac{\hat{h}_2(b,a)}{b-\rho(a)} + a \quad \text{if } f \text{ is increasing.}$$

Then

$$f(\gamma) + \frac{\rho(a)-a}{b-\rho(a)}f(a) \leq \frac{1}{b-\rho(a)}\int_a^b f(t)\nabla t \leq \frac{b-a}{b-\rho(a)}f(a) + f(b) - f(\ell).$$

Another, slightly different, form of the first Hermite-Hadamard inequality is the following; this implies that for time scales with left-scattered points there are at least two inequalities of this type.

Theorem 5.3. *Let $f : [a,b] \to \mathbb{R}$ be convex and monotonic. Assume $\ell, \gamma \in [a,b]$ such that*

$$\ell \geq a + \frac{\hat{h}_2(b,a)}{b-a}, \quad \gamma \geq b - \frac{\hat{h}_2(b,a)}{b-a} \quad \text{if } f \text{ is decreasing,}$$

$$\ell \leq a + \frac{\hat{h}_2(b,a)}{b-a}, \quad \gamma \leq b - \frac{\hat{h}_2(b,a)}{b-a} \quad \text{if } f \text{ is increasing.}$$

Then

$$f(\gamma) \leq \frac{1}{b-a}\int_a^b f(\rho(t))\nabla t \leq f(a) + f(b) - f(\ell).$$

Proof. Assume f is decreasing and convex. Then $f^{\nabla^2} \geq 0$, $f^{\nabla} \leq 0$, and f^{∇} is increasing. Then $F := -f^{\nabla}$ is decreasing and satisfies $F \geq 0$. For $G := \frac{b-t}{b-a}$, $0 \leq G \leq 1$ and F, G satisfy the hypotheses of Theorem 3.1. Now the inequality expression

$$b - \ell \leq \int_a^b G(t)\nabla t \leq \gamma - a$$

takes the form

$$b - \ell \leq \frac{1}{b-a}\int_a^b (b-t)\nabla t \leq \gamma - a.$$

Concentrating on the left inequality,

$$\ell \geq b - \frac{1}{b-a}\int_a^b (b-t)\nabla t = b - \frac{1}{b-a}\int_a^b (b-a+a-t)\nabla t,$$

which simplifies to

$$\ell \geq a + \frac{\hat{h}_2(b,a)}{b-a};$$

similarly,

$$\gamma \geq b - \frac{\hat{h}_2(b,a)}{b-a}.$$

Furthermore, note that $\int_r^s F(t)\nabla t = f(r) - f(s)$, and integration by parts yields

$$\int_a^b F(t)G(t)\nabla t = \frac{1}{b-a}\int_a^b (t-b)f^{\nabla}(t)\nabla t = f(a) - \frac{1}{b-a}\int_a^b f(\rho(t))\nabla t.$$

It follows that Steffensen's inequality takes the form

$$f(\ell) - f(b) \leq f(a) - \frac{1}{b-a}\int_a^b f(\rho(t))\nabla t \leq f(a) - f(\gamma),$$

which can be rearranged to match the theorem's stated conclusion. □

Theorem 5.4. *Let $f : [a,b] \to \mathbb{R}$ be an $n+1$ times nabla differentiable function such that*

$$m \leq f^{\nabla^{n+1}} \leq M$$

on $[a,b]$ for some real numbers $m < M$. Also, let $\ell, \gamma \in [a,b]$ such that

$$b - \ell \leq \frac{1}{M-m}\left[f^{\nabla^n}(b) - f^{\nabla^n}(a) - m(b-a)\right] \leq \gamma - a.$$

Then

$$m\hat{h}_{n+2}(b,a) + (M-m)\hat{h}_{n+2}(b,\ell) \leq \int_a^b \hat{R}_{n,f}(a,t)\nabla t$$
$$\leq M\hat{h}_{n+2}(b,a) + (m-M)\hat{h}_{n+2}(b,\gamma).$$

Proof. Let

$$k(t) := \frac{1}{M-m}\left[f(t) - m\hat{h}_{n+1}(t,a)\right], \quad F(t) := \hat{h}_{n+1}(b,\rho(t)),$$

$$G(t) := k^{\nabla^{n+1}}(t) = \frac{1}{M-m}\left[f^{\nabla^{n+1}}(t) - m\right] \in [0,1].$$

Observe that F is nonnegative and decreasing, and

$$\int_a^b G(t)\nabla t = \frac{1}{M-m}\left[f^{\nabla^n}(b) - f^{\nabla^n}(a) - m(b-a)\right].$$

Since F, G satisfy the hypotheses of Theorem 3.1, we compute the various integrals given in (3.1). First, by (4.2),

$$\int_\ell^b F(t)\nabla t = \int_\ell^b \hat{h}_{n+1}(b,\rho(t))\nabla t = -\hat{h}_{n+2}(b,t)\Big|_{t=\ell}^b = \hat{h}_{n+2}(b,\ell),$$

and

$$\int_a^\gamma F(t)\nabla t = -\hat{h}_{n+2}(b,t)\Big|_a^\gamma = \hat{h}_{n+2}(b,a) - \hat{h}_{n+2}(b,\gamma).$$

Moreover, using Corollary 4.2, we have

$$\begin{aligned}\int_a^b F(t)G(t)\nabla t &= \frac{1}{M-m}\int_a^b \hat{h}_{n+1}(b,\rho(t))\left(f^{\nabla^{n+1}}(t)-m\right)\nabla t \\ &= \frac{1}{M-m}\int_a^b \hat{R}_{n,f}(a,t)\nabla t + \frac{m}{M-m}\hat{h}_{n+2}(b,t)\Big|_a^b \\ &= \frac{1}{M-m}\int_a^b \hat{R}_{n,f}(a,t)\nabla t - \frac{m}{M-m}\hat{h}_{n+2}(b,a).\end{aligned}$$

Using Steffensen's inequality (3.1), we obtain

$$\hat{h}_{n+2}(b,\ell) \le \frac{1}{M-m}\left[\int_a^b \hat{R}_{n,f}(a,t)\nabla t - m\hat{h}_{n+2}(b,a)\right] \le \hat{h}_{n+2}(b,a) - \hat{h}_{n+2}(b,\gamma),$$

which yields the conclusion of the theorem. □

Theorem 5.5. *Let $f:[a,b]\to\mathbb{R}$ be a nabla and delta differentiable function such that*

$$m \le f^{\nabla}, f^{\Delta} \le M$$

on $[a,b]$ for some real numbers $m<M$.

(i) *If there exist $\ell,\gamma\in[a,b]$ such that*

$$b-\ell \le \frac{1}{M-m}\left[f(b)-f(a)-m(b-a)\right] \le \gamma - a,$$

then

$$\begin{aligned}m\hat{h}_2(b,a)+(M-m)\hat{h}_2(b,\ell) &\le \int_a^b f(t)\nabla t - (b-a)f(a) \\ &\le M\hat{h}_2(b,a)+(m-M)\hat{h}_2(b,\gamma).\end{aligned}$$

(ii) *If there exist $\ell,\gamma\in[a,b]$ such that*

$$\gamma - a \le \frac{1}{M-m}\left[f(b)-f(a)-m(b-a)\right] \le b-\ell,$$

then

$$\begin{aligned}mh_2(a,b)+(M-m)h_2(a,\gamma) &\le (b-a)f(b) - \int_a^b f(t)\Delta t \\ &\le Mh_2(a,b)+(m-M)h_2(a,\ell).\end{aligned}$$

Proof. The first part is just Theorem 5.4 with $n=0$. For the second part, let

$$k(t) := \frac{1}{M-m}\left[f(t)-m(t-b)\right], \quad F(t) := h_1(a,\sigma(t)),$$

$$G(t) := k^{\Delta}(t) = \frac{1}{M-m}\left[f^{\Delta}(t)-m\right] \in [0,1].$$

Clearly F is decreasing and nonpositive, and

$$\int_a^b G(t)\Delta t = \frac{1}{M-m}\left[f(b)-f(a)-m(b-a)\right] \in [\gamma-a, b-\ell].$$

Since F,G satisfy the hypotheses of Steffensen's inequality for delta integrals, we determine the corresponding integrals. First,

$$\int_\ell^b F(t)\Delta t = \int_\ell^b h_1(a,\sigma(t))\Delta t = -h_2(a,t)\Big|_{t=\ell}^b = -h_2(a,b)+h_2(a,\ell),$$

and

$$\int_a^\gamma F(t)\Delta t = -h_2(a,t)\big|_a^\gamma = -h_2(a,\gamma).$$

Moreover, using the formula for integration by parts for delta integrals,

$$\begin{aligned}\int_a^b F(t)G(t)\Delta t &= \int_a^b h_1(a,\sigma(t))k^\Delta(t)\Delta t\\ &= h_1(a,t)k(t)\big|_a^b - \int_a^b h_1^\Delta(a,t)k(t)\Delta t\\ &= \frac{1}{M-m}\left[-(b-a)f(b) + \int_a^b f(t)\Delta t + mh_2(a,b)\right].\end{aligned}$$

Using Steffensen's inequality for delta integrals, we obtain

$$\begin{aligned}-h_2(a,b)+h_2(a,\ell) &\le \frac{1}{M-m}\left[-(b-a)f(b) + \int_a^b f(t)\Delta t + mh_2(a,b)\right]\\ &\le -h_2(a,\gamma),\end{aligned}$$

which yields the conclusion of (ii). □

In [7], part (ii) of the above theorem also involved the equivalent of nabla derivatives for q-difference equations with $0<q<1$. However, the function used there, $F(t) = a - qt = a - \rho(t)$, is not of one sign on $[a,b]$, since $F(a) = a(1-q) > 0$, $F(a/q) = 0$, and $F(a/q^2) = a(1-1/q) < 0$. For this reason we introduced a delta-derivative perspective in (ii) above and in the following.

Corollary 5.6. *Let $f:[a,b]\to\mathbb{R}$ be a nabla and delta differentiable function such that*

$$m \le f^\nabla, f^\Delta \le M$$

on $[a,b]$ for some real numbers $m<M$. Assume there exist $\ell,\gamma\in[a,\rho(b)]$ such that

$$\rho(\gamma)-a \le \frac{1}{M-m}\left[f(b)-f(a)-m(b-a)\right] \le \gamma - a, \tag{5.1}$$

$$b-\ell \le \frac{1}{M-m}\left[f(b)-f(a)-m(b-a)\right] \le b-\rho(\ell). \tag{5.2}$$

Then

$$\begin{aligned}&2mh_2(a,b) + (M-m)\left[h_2(\ell,b)+h_2(a,\rho(\gamma))\right]\\ &\qquad\le \int_a^b f(t)\nabla t - \int_a^b f(t)\Delta t + (b-a)(f(b)-f(a))\\ &\qquad\le 2Mh_2(a,b) - (M-m)\left[h_2(\gamma,b)+h_2(a,\rho(\ell))\right].\end{aligned}$$

Proof. By the previous theorem, Theorem 5.5,

$$\begin{aligned}m\hat{h}_2(b,a) + (M-m)\hat{h}_2(b,\ell) &\le \int_a^b f(t)\nabla t - (b-a)f(a)\\ &\le M\hat{h}_2(b,a) + (m-M)\hat{h}_2(b,\gamma)\end{aligned} \tag{5.3}$$

using (i) and the fact that

$$b-\ell \le \frac{1}{M-m}\left[f(b)-f(a)-m(b-a)\right] \le \gamma - a;$$

in like manner

$$
\begin{aligned}
mh_2(a,b) + (M-m)h_2(a,\rho(\gamma)) &\le (b-a)f(b) - \int_a^b f(t)\Delta t \\
&\le Mh_2(a,b) + (m-M)h_2(a,\rho(\ell))
\end{aligned} \tag{5.4}
$$

using (ii) and the fact that

$$\rho(\gamma) - a \le \frac{1}{M-m}\left[f(b) - f(a) - m(b-a)\right] \le b - \rho(\ell).$$

Add (5.3) to (5.4) and use Theorem 2.2 to arrive at the conclusion. ☐

Remark 5.7. If $\mathbb{T} = \mathbb{R}$, set $\lambda := b - \ell = \gamma - a$, so that $b - \gamma = \ell - a = b - a - \lambda$. Here the nabla and delta integrals of f on $[a,b]$ are identical, and $h_2(s,t) = (t-s)^2/2$, so the conclusion of the previous corollary, Corollary 5.6, is the known [7] inequality

$$m + \frac{(M-m)\lambda^2}{(b-a)^2} \le \frac{f(b)-f(a)}{b-a} \le M - \frac{(M-m)(b-a-\lambda)^2}{(b-a)^2}.$$

If $\mathbb{T} = \mathbb{Z}$, then $h_2(s,t) = (t-s)(t-s+1)/2 = (t-s)^{\overline{2}}/2$ and

$$\int_a^b f(t)\nabla t - \int_a^b f(t)\Delta t = \sum_{t=a+1}^{b} f(t) - \sum_{t=a}^{b-1} f(t) = f(b) - f(a).$$

This time take $\lambda = b - \ell = \gamma - 1 - a$. The discrete conclusion of Corollary 5.6 is thus

$$m + \frac{(M-m)\lambda^{\overline{2}}}{(b-a)^{\overline{2}}} \le \frac{f(b)-f(a)}{b-a} \le M - \frac{(M-m)(b-a-\lambda-1)^{\overline{2}}}{(b-a)^{\overline{2}}}.$$

Corollary 5.8 (Iyengar's Inequality). *Let $f : [a,b] \to \mathbb{R}$ be a nabla and delta differentiable function such that*

$$m \le f^{\nabla}, f^{\Delta} \le M$$

on $[a,b]$ for some real numbers $m < M$. Assume there exist $\ell, \gamma \in [a,\rho(b)]$ such that (5.1), (5.2) *are satisfied. Then*

$$
\begin{aligned}
&(M-m)\left[h_2(\ell,b) + h_2(a,\rho(\ell)) - h_2(a,b)\right] \\
&\qquad \le \int_a^b f(t)\nabla t + \int_a^b f(t)\Delta t - (b-a)(f(b)+f(a)) \\
&\qquad \le (M-m)\left[h_2(a,b) - h_2(\gamma,b) - h_2(a,\rho(\gamma))\right].
\end{aligned}
$$

Proof. Subtract (5.4) from (5.3) and use Theorem 2.2 to arrive at the conclusion. ☐

Remark 5.9. Again if $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\nabla t = \int_a^b f(t)\Delta t = \int_a^b f(t)dt$ and $h_2(t,s) = (t-s)^2/2$. Moreover, $\rho(\ell) = \ell$ and $\rho(\gamma) = \gamma$; set

$$\lambda = b - \ell = \gamma - a = \frac{1}{M-m}\left[f(b) - f(a) - m(b-a)\right].$$

This transforms the conclusion of Corollary 5.8 into a continuous calculus version,

$$\left|\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{\left[f(b)-f(a)-m(b-a)\right]\left[M(b-a)+f(a)-f(b)\right]}{2(M-m)}.$$

6. APPLICATIONS OF ČEBYŠEV'S INEQUALITY

Recently, Čebyšev's inequality on time scales for delta integrals was proven [9]. We repeat the statement of it here in the case of nabla integrals for completeness.

Theorem 6.1 (Čebyšev's inequality)**.** *Let f and g be both increasing or both decreasing in $[a,b]$. Then*

$$\int_a^b f(t)g(t)\nabla t \geq \frac{1}{b-a}\int_a^b f(t)\nabla t \int_a^b g(t)\nabla t.$$

If one of the functions is increasing and the other is decreasing, then the above inequality is reversed.

The following is an application of Čebyšev's inequality, which extends a similar result in [7] to general time scales.

Theorem 6.2. *Assume that $f^{\nabla^{n+1}}$ is monotonic on $[a,b]$.*

(i) *If $f^{\nabla^{n+1}}$ is increasing, then*

$$\begin{aligned} 0 &\geq \int_a^b \hat{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a}\right]\hat{h}_{n+2}(b,a) \\ &\geq \left[f^{\nabla^{n+1}}(a) - f^{\nabla^{n+1}}(b)\right]\hat{h}_{n+2}(b,a). \end{aligned}$$

(ii) *If $f^{\nabla^{n+1}}$ is decreasing, then*

$$\begin{aligned} 0 &\leq \int_a^b \hat{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a}\right]\hat{h}_{n+2}(b,a) \\ &\leq \left[f^{\nabla^{n+1}}(a) - f^{\nabla^{n+1}}(b)\right]\hat{h}_{n+2}(b,a). \end{aligned}$$

Proof. The situation for (ii) is analogous to that of (i). Assume (i), and set $F(t) := f^{\nabla^{n+1}}(t)$, $G(t) := \hat{h}_{n+1}(b,\rho(t))$. Then F is increasing by assumption, and G is decreasing, so that by Čebyšev's nabla inequality,

$$\int_a^b F(t)G(t)\nabla t \leq \frac{1}{b-a}\int_a^b F(t)\nabla t \int_a^b G(t)\nabla t.$$

By Corollary 4.2,

$$\int_a^b F(t)G(t)\nabla t = \int_a^b f^{\nabla^{n+1}}(t)\hat{h}_{n+1}(b,\rho(t))\nabla t = \int_a^b \hat{R}_{n,f}(a,t)\nabla t.$$

We also have

$$\int_a^b F(t)\nabla t = f^{\nabla^n}(b) - f^{\nabla^n}(a), \qquad \int_a^b G(t)\nabla t = \int_a^b \hat{h}_{n+1}(b,\rho(t)) = \hat{h}_{n+2}(b,a).$$

Thus Čebyšev's inequality implies

$$\int_a^b \hat{R}_{n,f}(a,t)\nabla t \leq \frac{1}{b-a}\left[f^{\nabla^n}(b) - f^{\nabla^n}(a)\right]\hat{h}_{n+2}(b,a),$$

which subtracts to the left side of the inequality. Since $f^{\nabla^{n+1}}$ is increasing on $[a,b]$,

$$f^{\nabla^{n+1}}(a)\hat{h}_{n+2}(b,a) \leq \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a}\right]\hat{h}_{n+2}(b,a) \leq f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b,a),$$

and we have

$$\int_a^b \hat{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a}\right]\hat{h}_{n+2}(b,a) \geq \int_a^b \hat{R}_{n,f}(a,t)\nabla t - f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b,a).$$

Now Corollary 4.2 and $f^{\nabla^{n+1}}$ is increasing imply that

$$f^{\nabla^{n+1}}(b)\int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t \geq \int_a^b \hat{R}_{n,f}(a,t)\nabla t \geq f^{\nabla^{n+1}}(a)\int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t,$$

which simplifies to

$$f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b,a) \geq \int_a^b \hat{R}_{n,f}(a,t)\nabla t \geq f^{\nabla^{n+1}}(a)\hat{h}_{n+2}(b,a).$$

This, together with the earlier lines give the right side of the inequality. □

Theorem 6.3. *Assume that $f^{\Delta^{n+1}}$ is monotonic on $[a,b]$ and the function g_k is as defined in Definition 2.1.*

(i) *If $f^{\Delta^{n+1}}$ is increasing, then*

$$\begin{aligned} 0 &\leq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b-a}\right] g_{n+2}(b,a) \\ &\leq \left[f^{\Delta^{n+1}}(b) - f^{\Delta^{n+1}}(a)\right] g_{n+2}(b,a). \end{aligned}$$

(ii) *If $f^{\Delta^{n+1}}$ is decreasing, then*

$$\begin{aligned} 0 &\geq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b-a}\right] g_{n+2}(b,a) \\ &\geq \left[f^{\Delta^{n+1}}(b) - f^{\Delta^{n+1}}(a)\right] g_{n+2}(b,a). \end{aligned}$$

Proof. The situation for (ii) is analogous to that of (i). Assume (i), and set $F(t) := f^{\Delta^{n+1}}(t)$, $G(t) := (-1)^{n+1}h_{n+1}(a,\sigma(t))$. Then F and G are increasing, so that by Čebyšev's delta inequality,

$$\int_a^b F(t)G(t)\Delta t \geq \frac{1}{b-a}\int_a^b F(t)\Delta t \int_a^b G(t)\Delta t.$$

By Lemma 4.3 with $t = a$,

$$\int_a^b F(t)G(t)\Delta t = (-1)^{n+1}\int_a^b f^{\Delta^{n+1}}(t)h_{n+1}(a,\sigma(t))\Delta t = (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t.$$

We also have $\int_a^b F(t)\Delta t = f^{\Delta^n}(b) - f^{\Delta^n}(a)$, and, using Theorem 2.2,

$$\int_a^b G(t)\Delta t = (-1)^{n+1}\int_a^b h_{n+1}(a,\sigma(t))\Delta t = g_{n+2}(b,a).$$

Thus Čebyšev's inequality implies

$$(-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t \geq \frac{1}{b-a}\left[f^{\Delta^n}(b) - f^{\Delta^n}(a)\right] g_{n+2}(b,a),$$

which subtracts to the left side of the inequality. Since $f^{\Delta^{n+1}}$ is increasing on $[a,b]$,

$$f^{\Delta^{n+1}}(a)g_{n+2}(b,a) \leq \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b-a}\right] g_{n+2}(b,a) \leq f^{\Delta^{n+1}}(b)g_{n+2}(b,a),$$

and we have

$$(-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - f^{\Delta^{n+1}}(a)g_{n+2}(b,a) \\ \geq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - \left[\frac{f^{\Delta^n}(b)-f^{\Delta^n}(a)}{b-a}\right] g_{n+2}(b,a).$$

Now Theorem 2.2 and Lemma 4.3 again with $t=a$ yield

$$(-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t = \int_a^b g_{n+1}(\sigma(t),a) f^{\Delta^{n+1}}(t)\Delta t.$$

Since $f^{\Delta^{n+1}}$ is increasing,

$$f^{\Delta^{n+1}}(b)\int_a^b g_{n+1}(\sigma(t),a)\Delta t \geq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t \geq f^{\Delta^{n+1}}(a)\int_a^b g_{n+1}(\sigma(t),a)\Delta t,$$

which simplifies to

$$f^{\Delta^{n+1}}(b)g_{n+2}(b,a) \geq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t \geq f^{\Delta^{n+1}}(a)g_{n+2}(b,a).$$

This, together with the earlier lines give the right side of the inequality. □

Remark 6.4. If $\mathbb{T}=\mathbb{R}$, then combining Theorem 6.2 and Theorem 6.3 yields Theorem 3.1 in [6].

Remark 6.5. In Theorem 6.2 (i), if $n=0$, we obtain

$$\int_a^b f(t)\nabla t \leq (b-a)f(a) + \frac{\hat{h}_2(b,a)}{b-a}(f(b)-f(a)). \tag{6.1}$$

Compare that with the following result.

Theorem 6.6. *Assume that f is nabla convex on $[a,b]$; that is, $f^{\nabla^2}\geq 0$ on $[a,b]$. Then*

$$\int_a^b f(\rho(t))\nabla t \leq (b-a)f(b) - \frac{\hat{h}_2(b,a)}{b-a}(f(b)-f(a)). \tag{6.2}$$

Proof. If $F := f^{\nabla}$ and $G(t) := t-a = \hat{h}_1(t,a)$, then both F and G are increasing functions. By Čebyšev's inequality on time scales, and the definition of $\hat{h}$ in (2.2),

$$\int_a^b f^{\nabla}(t)(t-a)\nabla t \geq \frac{1}{b-a}\int_a^b f^{\nabla}(t)\nabla t \int_a^b \hat{h}_1(t,a)\nabla t.$$

Using nabla integration by parts on the left, and calculating the right yields the result. □

The following result is a Hermite-Hadamard-type inequality for time scales; compare with Corollary 5.2.

Corollary 6.7. *Let f be nabla convex on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b \frac{f(\rho(t))+f(t)}{2}\nabla t \leq \frac{f(b)+f(a)}{2}.$$

Proof. Use (6.1), (6.2) and rearrange accordingly. □

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