

CERTAIN SUBCLASSES OF p -VALENT MEROMORPHIC FUNCTIONS INVOLVING CERTAIN OPERATOR

M.K. AOUF AND A.O. MOSTAFA

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt.

EMail: mkaouf127@yahoo.com adelaeg254@yahoo.com

Received: 25 March, 2008

Accepted: 20 May, 2008

Communicated by: N.K. Govil

2000 AMS Sub. Class.: 30C45.

Key words: Analytic, p -valent, Meromorphic, Integral operator.

Abstract: In this paper, a new subclass $\sum_{p,\beta}^{\alpha}(\eta, \delta, \mu, \lambda)$ of p -valent meromorphic functions defined by certain integral operator is introduced. Some interesting properties of this class are obtained.

Acknowledgements: The authors would like to thank the referees of the paper for their helpful suggestions.



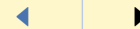
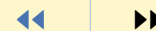
p-Valent Meromorphic Functions

M.K. Aouf and A.O. Mostafa

vol. 9, iss. 2, art. 45, 2008

[Title Page](#)

[Contents](#)



Page 1 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

Contents

1	Introduction	3
2	Main Results	6



p-Valent Meromorphic Functions

M.K. Aouf and A.O. Mostafa

vol. 9, iss. 2, art. 45, 2008

Title Page

Contents



Page 2 of 16

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 3 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

1. Introduction

Let Σ_p be the class of functions f of the form:

$$(1.1) \quad f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic and p -valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$.

Similar to [1], we define the following family of integral operators $Q_{\beta,p}^{\alpha} : \Sigma_p \rightarrow \Sigma_p$ ($\alpha \geq 0, \beta > -1; p \in \mathbb{N}$) as follows:

(i)

$$(1.2) \quad Q_{\beta,p}^{\alpha} f(z) = \left(\frac{\alpha + \beta - 1}{\beta - 1} \right) \alpha z^{-(p+\beta)} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta+p-1} f(t) dt$$

$$(1.3) \quad (\alpha > 0; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p);$$

and

(ii)

$$(1.4) \quad Q_{\beta,p}^0 f(z) = f(z) \quad (\text{for } \alpha = 0; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p).$$

From (1.2) and (1.4), we have

$$(1.5) \quad Q_{\beta,p}^{\alpha} f(z) = z^{-p} + \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k + \beta)}{\Gamma(k + \beta + \alpha)} a_{k-p} z^{k-p}$$

$$(1.6) \quad (\alpha \geq 0; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p).$$



[Title Page](#)

[Contents](#)



Page 4 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

Using the relation (1.5), it is easy to show that

$$(1.7) \quad z(Q_{\beta,p}^\alpha f(z))' = (\alpha + \beta - 1)Q_{\beta,p}^{\alpha-1} f(z) - (\alpha + \beta + p - 1)Q_{\beta,p}^\alpha f(z).$$

Definition 1.1. Let $\Sigma_{p,\beta}^\alpha(\eta, \delta, \mu, \lambda)$ be the class of functions $f \in \Sigma_p$ which satisfy:

$$(1.8) \quad \operatorname{Re} \left\{ (1 - \lambda) \left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^\mu + \lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^{\mu-1} \right\} > \eta,$$

where $g \in \Sigma_p$ satisfies the following condition:

$$(1.9) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^\alpha g(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} > \delta \quad (0 \leq \delta < 1; z \in U),$$

and η and μ are real numbers such that $0 \leq \eta < 1$, $\mu > 0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\{\lambda\} > 0$.

To establish our main results we need the following lemmas.

Lemma 1.2 ([2]). Let Ω be a set in the complex plane \mathbb{C} and let the function $\psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfy the condition $\psi(ir_2, s_1) \notin \Omega$ for all real $r_2, s_1 \leq -\frac{1+r_2^2}{2}$. If q is analytic in U with $q(0) = 1$ and $\psi(q(z), zq'(z)) \in \Omega$, $z \in U$, then $\operatorname{Re}\{q(z)\} > 0$ ($z \in U$).

Lemma 1.3 ([3]). If q is analytic in U with $q(0) = 1$, and if $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\{\lambda\} \geq 0$, then $\operatorname{Re}\{q(z) + \lambda zq'(z)\} > \alpha$ ($0 \leq \alpha < 1$) implies $\operatorname{Re}\{q(z)\} > \alpha + (1 - \alpha)(2\gamma - 1)$, where γ is given by

$$\gamma = \gamma(\operatorname{Re} \lambda) = \int_0^1 (1 + t^{\operatorname{Re}\{\lambda\}})^{-1} dt$$



Title Page

Contents



Page 5 of 16

Go Back

Full Screen

Close

which is increasing function of $\operatorname{Re}\{\lambda\}$ and $\frac{1}{2} \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

For real or complex numbers a, b and c ($c \neq 0, -1, -2, \dots$), the Gauss hypergeometric function is defined by

$$(1.10) \quad {}_2F_1(a, b; c; z) = 1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots$$

We note that the series (1.10) converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details, [4, Ch. 14]). Each of the identities (asserted by Lemma 1.4 below) is fairly well known (cf., e.g., [4, Ch. 14]).

Lemma 1.4 ([4]). For real or complex numbers a, b and c ($c \neq 0, -1, -2, \dots$),

$$(1.11) \quad \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z)$$

$(\operatorname{Re}(c) > \operatorname{Re}(b) > 0);$

$$(1.12) \quad {}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right);$$

$$(1.13) \quad {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z);$$

and

$$(1.14) \quad {}_2F_1\left(1, 1; 2; \frac{1}{2}\right) = 2 \ln 2.$$



2. Main Results

Unless otherwise mentioned, we assume throughout this paper that

$$\alpha \geq 0; \beta > -1; \alpha + \beta \neq 1; \mu > 0; 0 \leq \eta < 1; p \in \mathbb{N} \text{ and } \lambda \geq 0.$$

Theorem 2.1. Let $f \in \Sigma_{p,\beta}^\alpha(\eta, \delta, \mu, \lambda)$. Then

$$(2.1) \quad \operatorname{Re} \left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^\mu > \frac{2\mu\eta(\alpha + \beta - 1) + \lambda\delta}{2\mu(\alpha + \beta - 1) + \lambda\delta}, \quad (z \in U),$$

where the function $g \in \Sigma_p$ satisfies the condition (1.9).

Proof. Let $\gamma = \frac{2\mu\eta(\alpha+\beta-1)+\lambda\delta}{2\mu(\alpha+\beta-1)+\lambda\delta}$, and we define the function q by

$$(2.2) \quad q(z) = \frac{1}{1-\gamma} \left[\left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^\mu - \gamma \right].$$

Then q is analytic in U and $q(0) = 1$. If we set

$$(2.3) \quad h(z) = \frac{Q_{\beta,p}^\alpha g(z)}{Q_{\beta,p}^{\alpha-1} g(z)},$$

then by the hypothesis (1.9), $\operatorname{Re}\{h(z)\} > \delta$. Differentiating (2.2) with respect to z and using the identity (1.7), we have

$$(2.4) \quad (1-\lambda) \left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^\mu + \lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^{\mu-1} \\ = [(1-\gamma)q(z) + \gamma] + \frac{\lambda(1-\gamma)}{\mu(\alpha + \beta - 1)} zq'(z)h(z).$$

Title Page

Contents



Page 6 of 16

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)



Page 7 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

Let us define the function $\psi(r, s)$ by

$$(2.5) \quad \psi(r, s) = [(1 - \gamma)r + \gamma] + \frac{\lambda(1 - \gamma)}{\mu(\alpha + \beta - 1)}sh(z).$$

Using (2.5) and the fact that $f \in \Sigma_{p,\beta}^\alpha(\eta, \delta, \mu, \lambda)$, we obtain

$$\{\psi(q(z), zq'(z)); z \in U\} \subset \Omega = \{w \in C : \operatorname{Re}(w) > \eta\}.$$

Now for all real $r_2, s_1 \leq -\frac{1+r_2^2}{2}$, we have

$$\begin{aligned} \operatorname{Re}\{\psi(ir_2, s_1)\} &= \gamma + \frac{\lambda(1 - \gamma)s_1}{\mu(\alpha + \beta - 1)} \operatorname{Re} h(z) \\ &\leq \gamma - \frac{\lambda(1 - \gamma)\delta(1 + r_2^2)}{2\mu(\alpha + \beta - 1)} \\ &\leq \gamma - \frac{\lambda(1 - \gamma)\delta}{2\mu(\alpha + \beta - 1)} = \eta. \end{aligned}$$

Hence for each $z \in U$, $\psi(ir_2, s_1) \notin \Omega$. Thus by Lemma 1.2, we have $\operatorname{Re}\{q(z)\} > 0$ ($z \in U$) and hence

$$\operatorname{Re} \left(\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right)^\mu > \gamma \quad (z \in U).$$

This proves Theorem 2.1. □

Corollary 2.2. *Let the functions f and g be in Σ_p and let g satisfy the condition (1.9). If $\lambda \geq 1$ and*

$$(2.6) \quad \operatorname{Re} \left\{ (1 - \lambda) \frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} + \lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} > \eta \quad (0 \leq \eta < 1; z \in U),$$



Title Page

Contents



Page 8 of 16

Go Back

Full Screen

Close

then

$$(2.7) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} > \gamma = \frac{\eta[2(\alpha + \beta - 1) + \delta] + \delta(\lambda - 1)}{2(\alpha + \beta - 1) + \delta\lambda} \quad (z \in U).$$

Proof. We have

$$\lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} = \left\{ (1 - \lambda) \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} + \lambda \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} + (\lambda - 1) \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} \quad (z \in U).$$

Since $\lambda \geq 1$, making use of (2.6) and (2.1) (for $\mu = 1$), we deduce that

$$\operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} > \gamma = \frac{\eta[2(\alpha + \beta - 1) + \delta] + \delta(\lambda - 1)}{2(\alpha + \beta - 1) + \delta\lambda} \quad (z \in U).$$

□

Corollary 2.3. Let $\lambda \in C \setminus \{0\}$ with $\operatorname{Re}\{\lambda\} \geq 0$. If $f \in \Sigma_p$ satisfies the following condition:

$$\operatorname{Re}\{(1 - \lambda)(z^p Q_{\beta,p}^{\alpha} f(z))^{\mu} + \lambda z^p Q_{\beta,p}^{\alpha-1} f(z)(z^p Q_{\beta,p}^{\alpha} f(z))^{\mu-1}\} > \eta \quad (z \in U),$$

then

$$(2.8) \quad \operatorname{Re}\{(z^p Q_{\beta,p}^{\alpha} f(z))^{\mu}\} > \frac{2\mu\eta(\alpha + \beta - 1) + \operatorname{Re}\{\lambda\}}{2\mu(\alpha + \beta - 1) + \operatorname{Re}\{\lambda\}} \quad (z \in U).$$

Further, if $\lambda \geq 1$ and $f \in \Sigma_p$ satisfies

$$(2.9) \quad \operatorname{Re}\{(1 - \lambda)z^p Q_{\beta,p}^{\alpha} f(z) + \lambda z^p Q_{\beta,p}^{\alpha-1} f(z)\} > \eta \quad (z \in U),$$



then

$$(2.10) \quad \operatorname{Re}\{z^p Q_{\beta,p}^{\alpha-1} f(z)\} > \frac{2\eta(\alpha + \beta - 1) + \lambda - 1}{2(\alpha + \beta - 1) + \lambda} \quad (z \in U).$$

Proof. The results (2.8) and (2.10) follow by putting $g(z) = z^{-p}$ in Theorem 2.1 and Corollary 2.2, respectively. \square

Remark 1. Choosing α, λ and μ appropriately in Corollary 2.3, we have,

(i) For $\alpha = 0, \beta \neq 1$ and $\lambda = 1$ in Corollary 2.3, we have:

$$\operatorname{Re}\left\{\frac{1}{\beta-1}\left[\beta+p-1+\frac{zf'(z)}{f(z)}\right](z^p f(z))^\mu\right\} > \eta \quad (z \in U),$$

which implies that

$$\operatorname{Re}\{z^p f(z)\}^\mu > \frac{2\mu\eta(\beta-1)+1}{2\mu(\beta-1)+1} \quad (z \in U).$$

(ii) For $\alpha = 0, \beta \neq 1, \mu = 1$ and $\lambda \in C \setminus \{0\}$ with $\operatorname{Re}\{\lambda\} \geq 0$ in Corollary 2.3, we have

$$\operatorname{Re}\left\{\left(1+\frac{\lambda p}{\beta-1}\right)z^p f(z)+\frac{\lambda}{\beta-1}z^{p+1}f'(z)\right\} > \eta,$$

which implies that

$$\operatorname{Re}\{z^p f(z)\} > \frac{2\eta(\beta-1)+\operatorname{Re}\{\lambda\}}{2(\beta-1)+\operatorname{Re}\{\lambda\}} \quad (z \in U).$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 9 of 16

Go Back

Full Screen

Close



Title Page

Contents



Page 10 of 16

Go Back

Full Screen

Close

(iii) Replacing f by $-\frac{zf'}{p}$ in the result (ii), we have:

$$-\operatorname{Re} \left\{ \left[1 + \frac{\lambda}{\beta-1}(p+1) \right] \frac{z^{p+1}f'(z)}{p} + \frac{\lambda}{p(\beta-1)}z^{p+1}f''(z) \right\} > \eta \quad (0 \leq \eta < 1; z \in U),$$

which implies that

$$-\operatorname{Re} \left\{ \frac{z^{p+1}f'(z)}{p} \right\} > \frac{2\eta(\beta-1) + \operatorname{Re}\{\lambda\}}{2(\beta-1) + \operatorname{Re}\{\lambda\}} \quad (z \in U).$$

Theorem 2.4. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\{\lambda\} > 0$. If $f \in \Sigma_p$ satisfies the following condition:

$$(2.11) \quad \operatorname{Re}\{(1-\lambda)(z^p Q_{\beta,p}^\alpha f(z))^\mu + \lambda z^p Q_{\beta,p}^{\alpha-1} f(z)(z^p Q_{\beta,p}^\alpha f(z))^{\mu-1}\} > \eta \quad (z \in U),$$

then

$$(2.12) \quad \operatorname{Re}\{(z^p Q_{\beta,p}^\alpha f(z))^\mu\} > \eta + (1-\eta)(2\rho-1),$$

where

$$(2.13) \quad \rho = \frac{1}{2} {}_2F_1 \left(1, 1; \frac{\mu(\alpha+\beta-1)}{\operatorname{Re}\{\lambda\}} + 1; \frac{1}{2} \right).$$

Proof. Let

$$(2.14) \quad q(z) = (z^p Q_{\beta,p}^\alpha f(z))^\mu.$$

Then q is analytic with $q(0) = 1$. Differentiating (2.14) with respect to z and using the identity (1.7), we have

$$\begin{aligned} (1-\lambda)(z^p Q_{\beta,p}^\alpha f(z))^\mu + \lambda z^p Q_{\beta,p}^{\alpha-1} f(z)(z^p Q_{\beta,p}^\alpha f(z))^{\mu-1} \\ = q(z) + \frac{\lambda}{\mu(\alpha+\beta-1)}zq'(z), \end{aligned}$$



Title Page

Contents



Page 11 of 16

Go Back

Full Screen

Close

so that by the hypothesis (2.11), we have

$$\operatorname{Re} \left\{ q(z) + \frac{\lambda}{\mu(\alpha + \beta - 1)} z q'(z) \right\} > \eta \quad (z \in U).$$

In view of Lemma 1.3, this implies that

$$\operatorname{Re}\{q(z)\} > \eta + (1 - \eta)(2\rho - 1),$$

where

$$\rho = \rho(\operatorname{Re}\{\lambda\}) = \int_0^1 \left(1 + t \frac{\operatorname{Re}\{\lambda\}}{\mu(\alpha + \beta - 1)} \right)^{-1} dt.$$

Putting $\operatorname{Re}\{\lambda\} = \lambda_1 > 0$, we have

$$\rho = \int_0^1 \left(1 + t \frac{\lambda_1}{\mu(\alpha + \beta - 1)} \right)^{-1} dt = \frac{\mu(\alpha + \beta - 1)}{\lambda_1} \int_0^1 (1 + u)^{-1} u^{\frac{\mu(\alpha + \beta - 1)}{\lambda_1} - 1} du$$

Using (1.11), (1.12), (1.13) and (1.14), we obtain

$$\rho = \frac{1}{2} {}_2F_1 \left(1, 1; \frac{\mu(\alpha + \beta - 1)}{\lambda_1} + 1; \frac{1}{2} \right).$$

This completes the proof of Theorem 2.1. □

Corollary 2.5. Let $\lambda \in \mathbb{R}$ with $\lambda \geq 1$. If $f \in \sum_p$ satisfies

$$(2.15) \quad \operatorname{Re} \left\{ (1 - \lambda) z^p Q_{\beta,p}^\alpha f(z) + \lambda z^p Q_{\beta,p}^{\alpha-1} f(z) \right\} > \eta \quad (z \in U),$$

then

$$\operatorname{Re}\{z^p Q_{\beta,p}^{\alpha-1} f(z)\} > \eta + (1 - \eta)(2\rho_1 - 1) \left(1 - \frac{1}{\lambda} \right) \quad (z \in U),$$



[Title Page](#)

[Contents](#)



Page 12 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

where

$$\rho_1 = \frac{1}{2} {}_2F_1 \left(1, 1; \frac{(\alpha + \beta - 1)}{\lambda} + 1; \frac{1}{2} \right).$$

Proof. The result follows by using the identity

$$(2.16) \quad \lambda z^p Q_{\beta,p}^{\alpha-1} f(z) = (1 - \lambda) z^p Q_{\beta,p}^{\alpha} f(z) + \lambda z^p Q_{\beta,p}^{\alpha-1} f(z) + (\lambda - 1) z^p Q_{\beta,p}^{\alpha} f(z).$$

□

Remark 2. We note that, for $\alpha = 0, \beta = 2$ and $\lambda = \mu > 0$ in Corollary 2.3, that is, if

$$(2.17) \quad \operatorname{Re} \left\{ (1 - \lambda)(z^p f(z))^\lambda + \lambda(z^{p+1} f(z))'(z^p f(z))^{\lambda-1} \right\} > \eta \quad (z \in U),$$

then (2.8) implies that

$$(2.18) \quad \operatorname{Re} \{ (z^p f(z))^\lambda \} > \frac{2\eta + 1}{3} \quad (z \in U),$$

whereas, if $f \in \sum_p$ satisfies the condition (2.17) then by using Theorem 2.4, we have

$$\operatorname{Re} \{ (z^p f(z))^\lambda \} > 2(1 - \ln 2)\eta + (2 \ln 2 - 1) \quad (z \in U),$$

which is better than (2.18).

Theorem 2.6. Suppose that the functions f and g are in \sum_p and g satisfies the condition (1.9). If

$$(2.19) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} - \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} \right\} > -\frac{(1 - \eta)\delta}{2(\alpha + \beta - 1)} \quad (z \in U),$$



Title Page

Contents



Page 13 of 16

Go Back

Full Screen

Close

for some η ($0 \leq \eta < 1$), then

$$(2.20) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \right\} > \eta \quad (z \in U)$$

and

$$(2.21) \quad \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} > \frac{\eta[2(\alpha + \beta - 1) + \delta] - \delta}{2(\alpha + \beta - 1)} \quad (z \in U).$$

Proof. Let

$$(2.22) \quad q(z) = \frac{1}{1 - \eta} \left[\frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} - \eta \right].$$

Then q is analytic in U with $q(0) = 1$. Setting

$$(2.23) \quad \phi(z) = \frac{Q_{\beta,p}^\alpha g(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \quad (z \in U),$$

we observe that from (1.9), we have $\operatorname{Re}\{\phi(z)\} > \delta$ ($0 \leq \delta < 1$) in U . A simple computation shows that

$$\begin{aligned} \frac{(1 - \eta)zq'(z)}{\alpha + \beta - 1} \phi(z) &= \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} - \frac{Q_{\beta,p}^\alpha f(z)}{Q_{\beta,p}^\alpha g(z)} \\ &= \psi(q(z), zq'(z)), \end{aligned}$$

where

$$\psi(r, s) = \frac{(1 - \eta)s\phi(z)}{\alpha + \beta - 1}.$$



Title Page

Contents



Page 14 of 16

Go Back

Full Screen

Close

Using the hypothesis (2.19), we obtain

$$\{\psi(q(z), zq'(z)); z \in U\} \subset \Omega = \left\{ w \in \mathbb{C} : \operatorname{Re} w > -\frac{(1-\eta)\delta}{2(\alpha+\beta-1)} \right\}.$$

Now, for all real r_2 , $s_1 \leq -\frac{1+r_2^2}{2}$, we have

$$\begin{aligned} \operatorname{Re} \{\psi(ir_2, s_1)\} &= \frac{s_1(1-\eta) \operatorname{Re}\{\phi(z)\}}{\alpha+\beta-1} \\ &\leq \frac{-(1-\eta)\delta(1+r_2^2)}{2(\alpha+\beta-1)} \\ &\leq \frac{-(1-\eta)\delta}{2(\alpha+\beta-1)}. \end{aligned}$$

This shows that $\psi(ir_2, s_1) \notin \Omega$ for each $z \in U$. Hence by Lemma 1.2, we have $\operatorname{Re}\{q(z)\} > 0$ ($z \in U$). This proves (2.20). The proof of (2.21) follows by using (2.20) and (2.21) in the identity:

$$\operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} \right\} = \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)} - \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} \right\} - \operatorname{Re} \left\{ \frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)} \right\}.$$

This completes the proof of Theorem 2.6. □

Remark 3.

(i) For $\alpha = 0$ and $g(z) = z^{-p}$ in Theorem 2.6, we have

$$\operatorname{Re} \{z^{p+1} f'(z) + pz^p f(z)\} > \frac{-(1-\eta)\delta}{2} \quad (z \in U),$$

which implies that

$$\operatorname{Re} \{z^p f(z)\} > \eta \quad (z \in U)$$

and

$$\operatorname{Re} \{z^{p+1} f'(z) + (p + \beta - 1)z^p f(z)\} > \frac{\eta[2(\beta - 1) + \delta] - \delta}{2} \quad (z \in U).$$

(ii) Putting $\alpha = 0, \beta = 2$ in Theorem 2.6, we get that, if

$$\operatorname{Re} \left\{ \frac{z f'(z) + (p + 1)f(z)}{z g'(z) + (p + 1)g(z)} - \frac{f(z)}{g(z)} \right\} > \frac{-(1 - \eta)\delta}{2} \quad (z \in U),$$

then

$$\operatorname{Re} \left\{ \frac{f(z)}{g(z)} \right\} > \eta \quad (z \in U)$$

and

$$\operatorname{Re} \left\{ \frac{z f'(z) + (p + 1)f(z)}{z g'(z) + (p + 1)g(z)} \right\} > \frac{\eta(2 + \delta) - \delta}{2} \quad (z \in U).$$



p-Valent Meromorphic Functions

M.K. Aouf and A.O. Mostafa

vol. 9, iss. 2, art. 45, 2008

Title Page

Contents

◀▶

◀▶

Page 15 of 16

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5755

© 2007 Victoria University. All rights reserved.

References

- [1] E. AQLAN, J.M. JAHANGIRI AND S.R. KULKARNI, Certain integral operators applied to meromorphic p -valent functions, *J. of Nat. Geom.*, **24** (2003), 111–120.
- [2] S.S. MILLER AND P.T. MOCANU, Second order differential equations in the complex plane, *J. Math. Anal. Appl.*, **65** (1978), 289–305.
- [3] S. PONNUSAMY, Differential subordination and Bazilevic functions, *Proc. Indian Acad. Sci. (Math. Sci.)*, 105 (1995), 169–186.
- [4] E.T. WHITTAKER AND G.N. WATSON, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, 1927.



p-Valent Meromorphic Functions

M.K. Aouf and A.O. Mostafa

vol. 9, iss. 2, art. 45, 2008

Title Page

Contents



Page 16 of 16

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756