

TURÁN-TYPE INEQUALITIES FOR SOME q -SPECIAL FUNCTIONS

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Abstract: In this paper, we give new Turán-type inequalities for some q -special functions, using a q -analogue of a generalization of the Schwarz inequality.



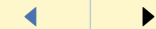
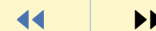
Turán-Type Inequalities

Kamel Brahim

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1. Introduction

In [9], P. Turán proved that the Legendre polynomials $P_n(x)$ satisfy the inequality

$$(1.1) \quad P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0, \quad x \in [-1, 1], \quad n = 0, 1, 2, \dots$$

and equality occurs only if $x = \pm 1$.

This inequality been the subject of much attention and several authors have provided new proofs, generalizations, extensions and refinements of (1.1).

In [7], A. Laforgia and P. Natalini established some new Turán-type inequalities for polygamma and Riemann zeta functions:

Theorem 1.1. For $n = 1, 2, \dots$ we denote by $\psi_n(x) = \psi^{(n)}(x)$ the polygamma functions defined as the n – th derivative of the psi function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0$$

with the usual notation for the gamma function. Then

$$\psi_m(x)\psi_n(x) \geq \psi_{\frac{m+n}{2}}^2(x),$$

where $\frac{m+n}{2}$ is an integer

Theorem 1.2. We denote by $\zeta(s)$ the Riemann zeta function. Then

$$(1.2) \quad (s+1) \frac{\zeta(s)}{\zeta(s+1)} \geq s \frac{\zeta(s+1)}{\zeta(s+2)}, \quad \forall s > 1.$$

The main aim of this paper is to give some new Turán-type inequalities for the q -polygamma and q -zeta [2] functions by using a q -analogue of the generalization of the Schwarz inequality.



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To make the paper more self contained we begin by giving some usual notions and notations used in q -theory. Throughout this paper we will fix $q \in]0, 1[$ and adapt the notations of the Gasper-Rahman book [4].

Let a be a complex number, the q -shifted factorial are defined by:

$$(1.3) \quad (a; q)_0 = 1; \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad n = 1, 2, \dots$$

$$(1.4) \quad (a; q)_\infty = \lim_{n \rightarrow +\infty} (a; q)_n = \prod_{k=0}^{\infty} (1 - aq^k).$$

For x complex we denote

$$(1.5) \quad [x]_q = \frac{1 - q^x}{1 - q}.$$

The q -Jackson integrals from 0 to a and from 0 to ∞ are defined by [4, 5]:

$$(1.6) \quad \int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n) q^n$$

and

$$(1.7) \quad \int_0^{\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n$$

provided the sums converge absolutely.

Jackson [5] defined the q -analogue of the Gamma function as:

$$(1.8) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x} \quad x \neq 0, -1, -2, \dots$$



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It satisfies the functional equation:

$$(1.9) \quad \Gamma_q(x + 1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1$$

and tends to $\Gamma(x)$ when q tends to 1.

Moreover, it has the q -integral representation (see [1, 3])

$$\Gamma_q(s) = K_q(s) \int_0^\infty x^{s-1} e_q^{-x} d_q x,$$

where

$$e_q^x = \frac{1}{((1-q)x; q)_\infty},$$

and

$$K_q(t) = \frac{(1-q)^{-s}}{1 + (1-q)^{-1}} \cdot \frac{(-(1-q), q)_\infty (-(1-q)^{-1}, q)_\infty}{(-(1-q)q^s, q)_\infty (-(1-q)^{-1}q^{1-s}, q)_\infty}.$$

Lemma 1.3. *Let $a \in \mathbb{R}_+ \cup \{\infty\}$ and let f and g be two nonnegative functions. Then*

$$(1.10) \quad \left(\int_0^a g(x) f^{\frac{m+n}{2}}(x) d_q x \right)^2 \leq \left(\int_0^a g(x) f^m(x) d_q x \right) \left(\int_0^a g(x) f^n(x) d_q x \right),$$

where m and n belong to a set S of real numbers, such that the integrals (1.10) exist.

Proof. Letting $a > 0$, by definition of the q -Jackson integral, we have

$$(1.11) \quad \int_0^a g(x) f^{\frac{m+n}{2}}(x) d_q x = (1-q)a \sum_{p=0}^\infty g(aq^p) f^{\frac{m+n}{2}}(aq^p) q^p$$

$$= \lim_{N \rightarrow +\infty} (1-q)a \sum_{p=0}^N g(aq^p) f^{\frac{m+n}{2}}(aq^p) q^p$$



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By the use of the Schwarz inequality for finite sums, we obtain

$$(1.12) \quad \left(\sum_{p=0}^N g(aq^p) f^{\frac{m+n}{2}}(aq^p) q^p \right)^2 \\ \leq \left(\sum_{p=0}^N g(aq^p) f^m(aq^p) q^p \right) \left(\sum_{p=0}^N g(aq^p) f^n(aq^p) q^p \right).$$

The result follows from the relation (1.11) and (1.12).

To obtain the inequality for $a = \infty$, it suffices to write the inequality (1.10) for $a = q^{-N}$, then tend N to ∞ . □

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2. The q -Polygamma Functions

The q -analogue of the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is defined as the logarithmic derivative of the q -gamma function, $\psi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}$.

From (1.8), we get for $x > 0$

$$\begin{aligned}\psi_q(x) &= -\text{Log}(1-q) + \text{Log } q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} \\ &= -\text{Log}(1-q) + \text{Log } q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n}.\end{aligned}$$

The last equality implies that

$$(2.1) \quad \psi_q(x) = -\text{Log}(1-q) + \frac{\text{Log } q}{1-q} \int_0^q \frac{t^{x-1}}{1-t} d_q t.$$

Theorem 2.1. For $n = 1, 2, \dots$, put $\psi_{q,n} = \psi_q^{(n)}$ the n -th derivative of the function ψ_q . Then

$$(2.2) \quad \psi_{q,n}(x)\psi_{q,m}(x) \geq \psi_{q, \frac{m+n}{2}}^2(x),$$

where $\frac{m+n}{2}$ is an integer.

Proof. Let m and n be two integers of the same parity.

From the relation (2.1) we deduce that

$$\psi_{q,n}(x) = \frac{\text{Log } q}{1-q} \int_0^q \frac{(\text{Log } t)^n t^{x-1}}{1-t} d_q t.$$



Applying Lemma 1.3 with $g(t) = \frac{t^{x-1}}{1-t}$, $f(t) = (-\text{Log } t)$ and $a = q$, we obtain

$$(2.3) \quad \int_0^q \frac{t^{x-1}}{1-t} (-\text{Log } t)^n d_q t \int_0^q \frac{t^{x-1}}{1-t} (-\text{Log } t)^m d_q t \geq \left[\int_0^q \frac{t^{x-1}}{1-t} (-\text{Log } t)^{\frac{m+n}{2}} d_q t \right]^2,$$

which gives, since $m + n$ is even,

$$(2.4) \quad \psi_{q,n}(x) \psi_{q,m}(x) \geq \psi_{q, \frac{m+n}{2}}^2(x).$$

□

Taking $m = n + 2$, one obtains:

Corollary 2.2. *For all $x > 0$ we have*

$$(2.5) \quad \frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)} \geq \frac{\psi_{q,n+1}(x)}{\psi_{q,n+2}(x)}, \quad n = 1, 2, \dots$$



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3. The q -Zeta function

For $x > 0$, we put

$$\alpha(x) = \frac{\text{Log}(x)}{\text{Log}(q)} - E\left(\frac{\text{Log}(x)}{\text{Log}(q)}\right)$$

and

$$\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}},$$

where $E\left(\frac{\text{Log}(x)}{\text{Log}(q)}\right)$ is the integer part of $\frac{\text{Log}(x)}{\text{Log}(q)}$.

In [2], the authors defined the q -Zeta function as follows

$$(3.1) \quad \zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}.$$

They proved that it is a q -analogue of the classical Riemann Zeta function and we have for all $s \in \mathbb{C}$ such that $\Re(s) > 1$,

$$\zeta_q(s) = \frac{1}{\tilde{\Gamma}_q(s)} \int_0^{\infty} t^{s-1} Z_q(t) d_q t,$$

where for all $t > 0$,

$$Z_q(t) = \sum_{n=1}^{\infty} e_q^{-\{n\}_q t} \quad \text{and} \quad \tilde{\Gamma}_q(t) = \frac{\Gamma_q(t)}{K_q(t)}.$$

Theorem 3.1. For all $s > 1$ we have

$$(3.2) \quad [s+1]_q \frac{\zeta_q(s)}{\zeta_q(s+1)} \geq q[s]_q \frac{\zeta_q(s+1)}{\zeta_q(s+2)}.$$



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Proof. For $s > 1$ the function q -zeta satisfies the following relation

$$(3.3) \quad \zeta_q(s) = \frac{1}{\tilde{\Gamma}_q(s)} \int_0^\infty t^{s-1} Z_q(t) d_q t.$$

Applying Lemma 1.3 with $g(t) = Z_q(t)$, $f(t) = t$ we obtain

$$(3.4) \quad \int_0^\infty t^{s-1} Z_q(t) d_q t \int_0^\infty t^{s+1} Z_q(t) d_q t \geq \left[\int_0^\infty t^s Z_q(t) d_q t \right]^2.$$

Further, using (3.3), this inequality becomes

$$(3.5) \quad \zeta_q(s) \tilde{\Gamma}_q(s) \zeta_q(s+2) \tilde{\Gamma}_q(s+2) \geq [\zeta_q(s+1)]^2 \left[\tilde{\Gamma}_q(s+1) \right]^2.$$

So, by using the relation $\tilde{\Gamma}_q(s+1) = q^{-s} [s]_q \tilde{\Gamma}_q(s)$, we obtain

$$(3.6) \quad [s+1]_q \zeta_q(s) \zeta_q(s+2) \geq q [s]_q [\zeta_q(s+1)]^2$$

which completes the proof. □

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