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## NOTE ON QI'S INEQUALITY AND BOUGOFFA'S INEQUALITY

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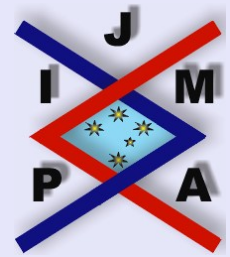


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## Abstract

In this paper, an answer to a problem proposed by L. Bougoffa is given. A consolidation of Qi's inequality and Bougoffa's inequality is obtained.

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*Key words:* Qi's inequality, Bougoffa's inequality, Integral inequality, Cauchy's Mean Value Theorem.

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# 1. Introduction

In the paper [7] F. Qi proposed the following open problem, which has attracted much attention from some mathematicians (cf. [1, 5, 6, 8]).

**Problem 1.** *Under what conditions does the inequality*

$$(1.1) \quad \int_a^b [f(x)]^t dx \geq \left( \int_a^b f(x) dx \right)^{t-1}.$$

*hold for  $t > 1$ ?*

Similar to Problem 1, in the paper [2] L. Bougoffa proposed the following:

**Problem 2.** *Under what conditions does the inequality*

$$(1.2) \quad \int_a^b [f(x)]^t dx \leq \left( \int_a^b f(x) dx \right)^{1-t}.$$

*hold for  $t < 1$ ?*

By using Hölder's inequality, L. Bougoffa obtained an answer to Problem 2 as follows

**Proposition 1.1.** *For a given positive integer  $p \geq 2$ , if  $0 < m \leq f(x) \leq M$  on  $[a, b]$  with  $M \leq m^{(p-1)^2}/(b-a)^p$ , then*

$$(1.3) \quad \int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left( \int_a^b f(x) dx \right)^{1-\frac{1}{p}}.$$



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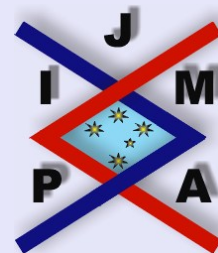
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We can see that the condition

$$(1.4) \quad 0 < m \leq f(x) \leq M \text{ on } [a, b] \text{ with } M \leq m^{(p-1)^2} / (b-a)^p$$

is not satisfied when  $\min_{[a,b]} f(x) = 0$ .

In this paper, we firstly give an answer to Problem 2, in which we allow  $\min_{[a,b]} f(x) = 0$  and  $p$  unnecessarily to be an integer. Secondly, we obtain a consolidation of Qi's inequality and Bougoffa's inequality.



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## 2. Main Results and Proofs

**Theorem 2.1.** Let  $p > 2$  be a positive number and  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) = 0$ . If  $[f^{p-2}]'(x) \geq p^p(p-2)/(p-1)^{p+1}$  for  $x \in (a, b)$ , then

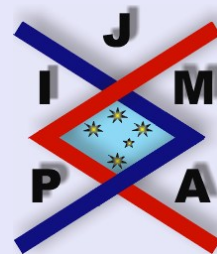
$$(2.1) \quad \int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left( \int_a^b f(x) dx \right)^{1-\frac{1}{p}}.$$

If  $0 \leq [f^{p-2}]'(x) \leq p^p(p-2)/(p-1)^{p+1}$  for  $x \in (a, b)$ , then the inequality (2.1) reverses.

*Proof.* If  $f \equiv 0$  on  $[a, b]$ , then it is trivial that the equation in (2.1) holds. Suppose now that  $f$  is not identically 0 on  $[a, b]$  and  $[f^{p-2}]'(x) \geq 0$  for  $x \in (a, b)$ , we may assume  $f(x) > 0, x \in (a, b]$ . This implies that both sides of (2.1) are not 0.

If  $[f^{p-2}]'(x) \geq p^p(p-2)/(p-1)^{p+1}$  for  $x \in (a, b)$ , then  $f(x) > 0$  for  $x \in (a, b]$ . Thus both sides of (2.1) are not 0. By using Cauchy's Mean Value Theorem twice, we have

$$(2.2) \quad \frac{\int_a^b [f(x)]^{\frac{1}{p}} dx}{\left( \int_a^b f(x) dx \right)^{1-\frac{1}{p}}} = \frac{[f(b_1)]^{\frac{1}{p}-1}}{\left(1 - \frac{1}{p}\right) \left( \int_a^{b_1} f(x) dx \right)^{-\frac{1}{p}}} \quad (a < b_1 < b)$$



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$$\begin{aligned}
&= \left( \frac{\int_a^{b_1} f(x) dx}{(1 - \frac{1}{p})^p [f(b_1)]^{p-1}} \right)^{\frac{1}{p}} \\
&= \left( \frac{1}{(1 - \frac{1}{p})^p (p-1) [f(b_2)]^{p-3} f'(b_2)} \right)^{\frac{1}{p}} \quad (a < b_2 < b_1) \\
&= \left( \frac{1}{\frac{(p-1)^{p+1}}{p^p (p-2)} [f^{p-2}]'(b_2)} \right)^{\frac{1}{p}} \\
&\leq 1.
\end{aligned}$$

So the inequality (2.1) holds.

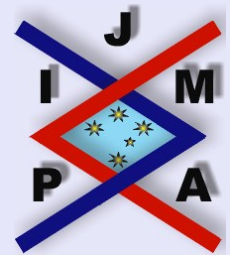
If  $0 \leq [f^{p-2}]'(x) \leq p^p(p-2)/(p-1)^{p+1}$ , then  $\frac{(p-1)^{p+1}}{p^p(p-2)} [f^{p-2}]'(b_2) \leq 1$ , which, together with (2.2), implies that the inequality (2.1) reverses.  $\square$

In the paper [3], Y. Chen and J. Kimball gave an answer to Problem 1 as follows

**Proposition 2.2.** *Let  $p > 2$  be a positive number and  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) = 0$ . If  $[f^{\frac{1}{p-2}}]'(x) \geq (p-1)^{\frac{1}{p-2}-1}$  for  $x \in (a, b)$ , then*

$$(2.3) \quad \left( \int_a^b f(x) dx \right)^{p-1} \leq \int_a^b [f(x)]^p dx.$$

If  $0 \leq [f^{\frac{1}{p-2}}]'(x) \leq (p-1)^{\frac{1}{p-2}-1}$  for  $x \in (a, b)$ , then the inequality (2.3) reverses.



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Thus, combining Theorem 2.1 and Proposition 2.2, we can obtain another result of this paper, which gives a consolidation of Qi's inequality and Bougoffa's inequality. To our best knowledge, this result is not found in the literature.

**Theorem 2.3.** Let  $p > 2$  be a positive number and  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) = 0$ .

1. If  $[f^{p-2}]'(x) \geq p^p(p-2)/(p-1)^{p+1}$  and  $[f^{\frac{1}{p-2}}]'(x) \geq (p-1)^{\frac{1}{p-2}-1}$  for  $x \in (a, b)$ , then

$$(2.4) \quad \left( \int_a^b [f(x)]^{\frac{1}{p}} dx \right)^p \leq \left( \int_a^b f(x) dx \right)^{p-1} \leq \int_a^b [f(x)]^p dx.$$

2. If  $0 \leq [f^{p-2}]'(x) \leq p^p(p-2)/(p-1)^{p+1}$  and  $0 \leq [f^{\frac{1}{p-2}}]'(x) \leq (p-1)^{\frac{1}{p-2}-1}$  for  $x \in (a, b)$ , then the inequality (2.4) reverses.

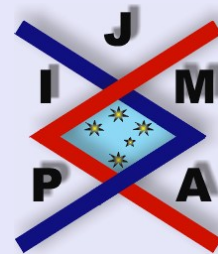
**Corollary 2.4.** Let  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) = 0$ .

1. If  $f'(x) \geq \frac{27}{16}$  for  $x \in (a, b)$ , then

$$(2.5) \quad \left( \int_a^b [f(x)]^{\frac{1}{3}} dx \right)^3 \leq \left( \int_a^b f(x) dx \right)^2 < \int_a^b [f(x)]^3 dx.$$

2. If  $0 \leq f'(x) \leq 1$  for  $x \in (a, b)$ , then

$$(2.6) \quad \left( \int_a^b [f(x)]^{\frac{1}{3}} dx \right)^3 > \left( \int_a^b f(x) dx \right)^2 \geq \int_a^b [f(x)]^3 dx.$$



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*Proof.* Set  $p = 3$  in Theorem 2.3. □

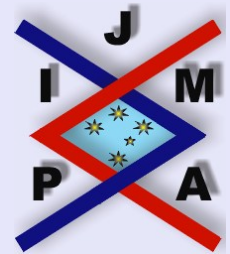
In order to illustrate a possible practical use of Corollary 2.4, we shall give two simple examples in which we can apply inequality (2.5) and (2.6).

**Example 2.1.** Let  $f(x) = e^x - e$  on  $[1, 2]$ , we see that  $f'(x) > e > \frac{27}{16}$  for  $x \in (1, 2)$ , other conditions of Corollary 2.4 are fulfilled and straightforward computation yields

$$\begin{aligned} \left( \int_1^2 (e^x - e)^{\frac{1}{3}} dx \right)^3 &\approx 1.56 < \left( \int_1^2 (e^x - e) dx \right)^2 \\ &\approx 3.81 < \int_1^2 (e^x - e)^3 dx \\ &\approx 18.74. \end{aligned}$$

**Example 2.2.** Let  $f(x) = \frac{e^x - e}{10}$  on  $[1, 2]$ , then  $\frac{e}{10} \leq f'(x) \leq \frac{e^2}{10}$ , other conditions of Corollary 2.4 are fulfilled and direct calculation produces that

$$\begin{aligned} \left[ \int_1^2 \left( \frac{e^x - e}{10} \right)^{\frac{1}{3}} dx \right]^3 &\approx 0.156 > \left( \int_1^2 \frac{e^x - e}{10} dx \right)^2 \\ &\approx 0.038 > \int_1^2 \left( \frac{e^x - e}{10} \right)^3 dx \\ &\approx 0.019. \end{aligned}$$




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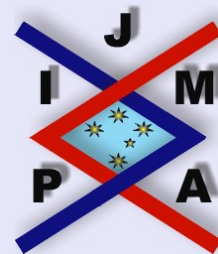
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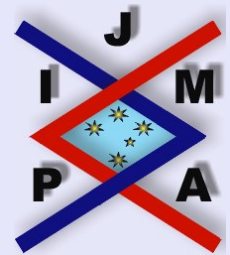
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