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NOTE ON QI'S INEQUALITY AND BOUGOFFA'S INEQUALITY

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Abstract

In this paper, an answer to a problem proposed by L. Bougoffa is given. A consolidation of Qi's inequality and Bougoffa's inequality is obtained.

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1. Introduction

In the paper [7] F. Qi proposed the following open problem, which has attracted much attention from some mathematicians (cf. [1, 5, 6, 8]).

Problem 1. *Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1} \tag{1.1}$$

hold for $t > 1$?

Similar to Problem 1, in the paper [2] L. Bougoffa proposed the following:

Problem 2. *Under what conditions does the inequality*

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x) dx \right)^{1-t} \tag{1.2}$$

hold for $t < 1$?

By using Hölder's inequality, L. Bougoffa obtained an answer to Problem 2 as follows

Proposition 1.1. *For a given positive integer $p \geq 2$, if $0 < m \leq f(x) \leq M$ on $[a, b]$ with $M \leq m^{(p-1)^2}/(b-a)^p$, then*

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b f(x) dx \right)^{1-\frac{1}{p}}. \tag{1.3}$$

We can see that the condition

$$0 < m \le f(x) \le M \text{ on } [a,b] \text{ with } M \le m^{(p-1)^2}/(b-a)^p \tag{1.4}$$

is not satisfied when $\min_{[a,b]} f(x) = 0$.

In this paper, we firstly give an answer to Problem 2, in which we allow $\min_{[a,b]} f(x) = 0$ and p unnecessarily to be an integer. Secondly, we obtain a consolidation of Qi's inequality and Bougoffa's inequality.

2. Main Results and Proofs

Theorem 2.1. *Let $p > 2$ be a positive number and $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = 0$. If $[f^{p-2}]'(x) \geq p^p(p-2)/(p-1)^{p+1}$ for $x \in (a, b)$, then*

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}. \tag{2.1}$$

If $0 \leq [f^{p-2}]'(x) \leq p^p(p-2)/(p-1)^{p+1}$ for $x \in (a, b)$, then the inequality (2.1) reverses.

Proof. If $f \equiv 0$ on $[a, b]$, then it is trivial that the equation in (2.1) holds. Suppose now that f is not identically 0 on $[a, b]$ and $[f^{p-2}]'(x) \geq 0$ for $x \in (a, b)$, we may assume $f(x) > 0, x \in (a, b]$. This implies that both sides of (2.1) are not 0.

If $[f^{p-2}]'(x) \geq p^p(p-2)/(p-1)^{p+1}$ for $x \in (a, b)$, then $f(x) > 0$ for $x \in (a, b]$. Thus both sides of (2.1) are not 0. By using Cauchy's Mean Value Theorem twice, we have

$$\frac{\int_a^b [f(x)]^{\frac{1}{p}} dx}{\left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}} = \frac{[f(b_1)]^{\frac{1}{p}-1}}{(1-\frac{1}{p})\left(\int_a^{b_1} f(x)dx\right)^{-\frac{1}{p}}} \quad (a < b_1 < b) \tag{2.2}$$

$$= \left(\frac{\int_a^{b_1} f(x)dx}{(1-\frac{1}{p})^p [f(b_1)]^{p-1}} \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{(1-\frac{1}{p})^p (p-1)[f(b_2)]^{p-3} f'(b_2)} \right)^{\frac{1}{p}} \quad (a < b_2 < b_1)$$

$$= \left(\frac{1}{\frac{(p-1)^{p+1}}{p^p(p-2)} [f^{p-2}]'(b_2)} \right)^{\frac{1}{p}}$$

$$\leq 1.$$

So the inequality (2.1) holds.

If $0 \leq [f^{p-2}]'(x) \leq p^p(p-2)/(p-1)^{p+1}$, then $\frac{(p-1)^{p+1}}{p^p(p-2)}[f^{p-2}]'(b_2) \leq 1$, which, together with (2.2), implies that the inequality (2.1) reverses. □

In the paper [3], Y. Chen and J. Kimball gave an answer to Problem 1 as follows

Proposition 2.2. *Let $p > 2$ be a positive number and $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = 0$. If $[f^{\frac{1}{p-2}}]'(x) \geq (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a,b)$, then*

$$\left(\int_a^b f(x)dx \right)^{p-1} \leq \int_a^b [f(x)]^p dx. \tag{2.3}$$

If $0 \leq [f^{\frac{1}{p-2}}]'(x) \leq (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a,b)$, then the inequality (2.3) reverses.

Thus, combining Theorem 2.1 and Proposition 2.2, we can obtain another result of this paper, which gives a consolidation of Qi's inequality and Bougoffa's inequality. To our best knowledge, this result is not found in the literature.

Theorem 2.3. *Let $p > 2$ be a positive number and $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = 0$.*

1. *If $[f^{p-2}]'(x) \geq p^p(p-2)/(p-1)^{p+1}$ and $[f^{\frac{1}{p-2}}]'(x) \geq (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a, b)$, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{p}} dx\right)^p \leq \left(\int_a^b f(x)dx\right)^{p-1} \leq \int_a^b [f(x)]^p dx. \tag{2.4}$$

2. *If $0 \leq [f^{p-2}]'(x) \leq p^p(p-2)/(p-1)^{p+1}$ and $0 \leq [f^{\frac{1}{p-2}}]'(x) \leq (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a, b)$, then the inequality (2.4) reverses.*

Corollary 2.4. *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = 0$.*

1. *If $f'(x) \geq \frac{27}{16}$ for $x \in (a, b)$, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{3}} dx\right)^3 \leq \left(\int_a^b f(x)dx\right)^2 < \int_a^b [f(x)]^3 dx. \tag{2.5}$$

2. *If $0 \leq f'(x) \leq 1$ for $x \in (a, b)$, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{3}} dx\right)^3 > \left(\int_a^b f(x)dx\right)^2 \geq \int_a^b [f(x)]^3 dx. \tag{2.6}$$

Proof. Set $p = 3$ in Theorem 2.3. □

In order to illustrate a possible practical use of Corollary 2.4, we shall give two simple examples in which we can apply inequality (2.5) and (2.6).

Example 2.1. *Let $f(x) = e^x - e$ on $[1,2]$, we see that $f'(x) > e > \frac{27}{16}$ for $x \in (1,2)$, other conditions of Corollary 2.4 are fulfilled and straightforward computation yields*

$$\left(\int_1^2 (e^x - e)^{\frac{1}{3}} dx\right)^3 \approx 1.56 < \left(\int_1^2 (e^x - e) dx\right)^2$$
$$\approx 3.81 < \int_1^2 (e^x - e)^3 dx$$
$$\approx 18.74.$$

Example 2.2. *Let $f(x) = \frac{e^x - e}{10}$ on $[1,2]$, then $\frac{e}{10} \le f'(x) \le \frac{e^2}{10}$, other conditions of Corollary 2.4 are fulfilled and direct calculation produces that*

$$\left[\int_1^2 \left(\frac{e^x - e}{10}\right)^{\frac{1}{3}} dx\right]^3 \approx 0.156 > \left(\int_1^2 \frac{e^x - e}{10} dx\right)^2$$
$$\approx 0.038 > \int_1^2 \left(\frac{e^x - e}{10}\right)^3 dx$$
$$\approx 0.019.$$

References

[1] L. BOUGOFFA, Notes on Qi type integral inequalities, *J. Inequal. Pure Appl. Math.*, **4**(4) (2003), Art. 77. [ONLINE: http://jipam.vu.edu.au/article.php?sid=318].

[2] L. BOUGOFFA, An integral inequality similar to Qi's inequality, *J. Inequal. Pure Appl. Math.*, **6**(1) (2005), Art. 27. [ONLINE: http://jipam.vu.edu.au/article.php?sid=496].

[3] Y. CHEN AND J. KIMBALL, Note on an open problem of Feng Qi, *J. Inequal. Pure Appl. Math.*, **7**(1) (2005), Art. 4. [ONLINE: http://jipam.vu.edu.au/article.php?sid=621].

[4] J.-CH. KUANG, *Applied Inequalities*, 3rd edition, Shandong Science and Technology Press, Jinan, China, 2004. (Chinese)

[5] S. MAZOUZI AND F. QI, On an open problem regarding an integral inequality, *J. Inequal. Pure Appl. Math.*, **4**(2) (2003), Art. 31. [ONLINE: http://jipam.vu.edu.au/article.php?sid=269].

[6] T.K. POGANY, On an open problem of F. Qi, *J. Inequal. Pure Appl. Math.*, **3**(4) (2002), Art. 54. [ONLINE: http://jipam.vu.edu.au/article.php?sid=206].

[7] F. QI, Several integral inequalities, *J. Inequal. Pure Appl. Math.*, **1**(2) (2000), Art. 19. [ONLINE: http://jipam.vu.edu.au/article.php?sid=113].

[8] K.W. YU AND F. QI, A short note on an integral inequality, *RGMIA Res. Rep. Coll.*, **4**(1) (2001), Art. 4, 23– 25. [ONLINE: http://rgmia.vu.edu.au/v4n1.html].

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