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REGULARITY FOR VECTOR VALUED MINIMIZERS OF SOME ANISOTROPIC INTEGRAL FUNCTIONALS

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Abstract

We deal with anisotropic integral functionals $\int_\Omega f(x, Du(x))dx$ defined on vector valued mappings $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$. We show that a suitable "monotonicity" inequality, on the density f, guarantees global pointwise bounds for minimizers u.

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Contents

1. Introduction

We consider the integral functional

$$\mathcal{F}(u) = \int_\Omega f(x, Du(x))dx \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and Ω is a bounded open set. When $N = 1$ we are dealing with scalar functions $u : \Omega \to \mathbb{R}$; on the contrary, vector valued mappings $u : \Omega \to \mathbb{R}^N$ appear when $N \geq 2$. Local and global pointwise bounds for scalar minimizers of (1.1) have been proved in [2], [7], [5], [4]. A model functional for these results is

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x) D_j u(x) D_i u(x) \right)^{\frac{p}{2}} dx, \tag{1.2}$$

where coefficients a_{ij} are measurable, bounded and elliptic. Previous results for scalar minimizers are no longer true in the vector valued case $N \geq 2$ as De Giorgi's counterexample shows, [3]. Some years later, attention has been paid to anisotropic functionals whose model is

$$\int_\Omega \left(|D_1 u(x)|^{p_1} + |D_2 u(x)|^{p_2} + \cdots + |D_n u(x)|^{p_n} \right) dx, \tag{1.3}$$

where each component $D_i u$ of the gradient $Du = (D_1 u, D_2 u, \ldots, D_n u)$ may have a (possibly) different exponent p_i: this seems useful when dealing with some reinforced materials, [9]; see also [6, Example 1.7.1, page 169]. In the framework of anisotropic functionals, global pointwise bounds have been proved for scalar minimizers in [1] and [8]. If no additional conditions are assumed, these bounds are false in the vectorial case, as the above mentioned counterexample shows, [3]. The aim of this paper is to present a "monotonicity" assumption ensuring boundedness of vector valued minimizers. In order to do that, we recall that $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ thus $Du(x)$ is a matrix with N rows and n columns; the density $f(x, A)$ in (1.1) is assumed to be measurable with respect to x, continuous with respect to A and $f : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$. Every matrix $A = \{A_i^\alpha\} \in \mathbb{R}^{N\times n}$ will have N rows $A^1, \dots, A^N$ and n columns $A_1, \dots, A_n$. In this paper we will show that the following "monotonicity" inequality guarantees global pointwise bounds for vector valued minimizers of (1.1):

$$f(x, \tilde{A}) + \mu \sum_{i=1}^{n} \left| \tilde{A}_i - A_i \right|^{p_i} \leq f(x, A) + M(x) \tag{1.4}$$

for every pair of matrices $\tilde{A}, A \in \mathbb{R}^{N\times n}$ such that there exists a row β with $\tilde{A}^\beta = 0$ and for every remaining row $\alpha \neq \beta$ we have $\tilde{A}^\alpha = A^\alpha$. In (1.4) $\mu, p_1, \dots, p_n$ are positive constants with $p_i > 1$ and $M : \Omega \to [0, +\infty)$ with $M \in L^r(\Omega)$, $r \geq 1$. If we keep in mind that $A = Du(x)$, then the left hand side of (1.4) shows $\sum_{i=1}^n |\tilde{A}_i - D_i u(x)|^{p_i}$, thus each component $D_i u$ of the gradient Du may have a possibly different exponent p_i, so we are in the anisotropic framework: $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $D_i u \in L^{p_i}(\Omega, \mathbb{R}^N)$. In this case the harmonic mean $\overline{p} = \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \right)^{-1}$ comes into play. In Section 2 we will prove the following

Theorem 1.1. *We consider the functional (1.1) under the "monotonicity" in-*

equality (1.4) with

$$\frac{\overline{p}^*}{\overline{p}}\left(1-\frac{1}{r}\right) > 1 \tag{1.5}$$

where $\overline{p}^*$ *is the Sobolev exponent of* $\overline{p} < n$*. We consider* $u = (u^1, \dots, u^N) \in W^{1,1}(\Omega, \mathbb{R}^N)$*, with* $D_i u \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$*, such that*

$$\mathcal{F}(u) < +\infty$$

and

$$\mathcal{F}(u) \le \mathcal{F}(v) \tag{1.6}$$

for every $v \in u + W_0^{1,1}(\Omega, \mathbb{R}^N)$ *with* $D_i v \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$*. Then, for every component* u^β*, we have*

$$\inf_{\partial\Omega} u^\beta - c_* \le u^\beta(x) \le \sup_{\partial\Omega} u^\beta + c_* \tag{1.7}$$

for almost every $x \in \Omega$*, where*

$$c_* = c\left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{1}{\overline{p}}} |\Omega|^{\left[\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}-1\right]\frac{1}{\overline{p}^*}} 2^{\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}\left[\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}-1\right]^{-1}},$$

$c = c(n, p_1, \dots, p_n) > 0$ *and* $|\Omega|$ *is the Lebesgue measure of* Ω*.*

A model density f for the “monotonicity” inequality (1.4) is given in the following.

Lemma 1.2. *For every* $i = 1, \dots, n$, *let us consider* $p_i \in [2, +\infty)$ *and* $a_i \in (0, +\infty)$; *we take* $m : \Omega \to [0, +\infty)$ *and* $h : \mathbb{R} \to \mathbb{R}$ *with* $-\infty < \inf_{\mathbb{R}} h$. *Let us consider* $f : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}$ *defined as follows:*

$$f(x, A) = \sum_{i=1}^{n} a_i |A_i|^{p_i} + m(x)h(\det A). \tag{1.8}$$

Then the "monotonicity" inequality (1.4) holds true with $\mu = \min_j a_j$ *and* $M(x) = m(x)[h(0) - \inf_{\mathbb{R}} h]$. *Moreover, if* $h \geq 0$, *then* $f \geq 0$ *too.*

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2. Proofs

In order to prove Theorem 1.1, we need the following

Lemma 2.1. *Let us consider the functional (1.1) under the "monotonicity" assumption (1.4). Then, for every* $v = (v^1, \dots, v^N) \in W^{1,1}(\Omega, \mathbb{R}^N)$ *with* $D_i v \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$, *for any* $\beta \in \{1, \dots, N\}$, *for all* $t \in \mathbb{R}$, *it results that*

$$\mathcal{F}(I_{\beta,t}(v)) + \mu \sum_{i=1}^{n} \int_\Omega |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} dx \leq \mathcal{F}(v) + \int_{\{v^\beta > t\}} M(x) dx \tag{2.1}$$

where $I_{\beta,t} : \mathbb{R}^N \to \mathbb{R}^N$ *is defined as follows:*

$$\forall y = (y^1, \dots, y^N) \in \mathbb{R}^N, \qquad I_{\beta,t}(y) = (I^1_{\beta,t}(y), \dots, I^N_{\beta,t}(y))$$

with

$$I^\alpha_{\beta,t}(y) = \begin{cases} y^\alpha & \text{if } \alpha \neq \beta \\ y^\beta \wedge t = \min\{y^\beta, t\} & \text{if } \alpha = \beta. \end{cases} \tag{2.2}$$

Proof. For every $v \in W^{1,1}(\Omega, \mathbb{R}^N)$, with $D_i v \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$, it results that $I_{\beta,t}(v) \in W^{1,1}(\Omega, \mathbb{R}^N)$; moreover

$$D_i(I^\alpha_{\beta,t}(v)) = \begin{cases} D_i v^\alpha & \text{if } \alpha \neq \beta \\ 1_{\{v^\beta \leq t\}} D_i v^\beta & \text{if } \alpha = \beta, \end{cases} \tag{2.3}$$

where 1_B is the characteristic function of the set B, that is, $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ if $x \notin B$. Therefore $D_i(I_{\beta,t}(v)) \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$. On $\{x \in \Omega : v^\beta(x) > t\}$ we have $D(I^\beta_{\beta,t}(v)) = 0$ and, for $\alpha \neq \beta$, $D(I^\alpha_{\beta,t}(v)) = Dv^\alpha$; so we can apply (1.4) with $\tilde{A} = D(I_{\beta,t}(v))$ and $A = Dv$; we obtain

$$f(x, D(I_{\beta,t}(v(x)))) + \mu \sum_{i=1}^{n} |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} \leq f(x, Dv(x)) + M(x) \tag{2.4}$$

for $x \in \{v^\beta > t\}$. On $\{x \in \Omega : v^\beta(x) \leq t\}$ $D(I_{\beta,t}(v)) = Dv$, thus

$$f(x, D(I_{\beta,t}(v(x)))) + \mu \sum_{i=1}^{n} |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} = f(x, Dv(x)) \tag{2.5}$$

for $x \in \{v^\beta \leq t\}$. From (2.4) and (2.5) we have

$$f(x, D(I_{\beta,t}(v(x)))) + \mu \sum_{i=1}^{n} |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} \leq f(x, Dv(x)) + M(x) 1_{\{v^\beta > t\}}(x)$$

for $x \in \Omega$. If $x \to f(x, Dv(x)) \in L^1(\Omega)$, then $x \to f(x, D(I_{\beta,t}(v(x)))) \in L^1(\Omega)$ too and, integrating the last inequality with respect to x, we get (2.1). When $x \to f(x, Dv(x)) \notin L^1(\Omega)$, we have $\mathcal{F}(v) = +\infty$ and (2.1) holds true. This ends the proof of Lemma 2.1. □

Now we are ready to prove Theorem 1.1.

Proof. Let us fix $\beta \in \{1, \dots, N\}$. If $\sup_{\partial\Omega} u^\beta = +\infty$ then the right hand side of (1.7) is satisfied. Thus we assume $\sup_{\partial\Omega} u^\beta < t_0 \le t < +\infty$ and we note that under this assumption $I_{\beta,t}(u) \in u + W_0^{1,1}(\Omega, \mathbb{R}^N)$ and $D_i(I_{\beta,t}(u)) \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$ since

$$u^\beta \wedge t = \min\{u^\beta, t\} = u^\beta - [\max\{u^\beta - t, 0\}] = u^\beta - [(u^\beta - t) \vee 0],$$

where $(u^\beta - t) \vee 0 \in W_0^{1,1}(\Omega)$ and $D_i((u^\beta - t) \vee 0) = D_i u^\beta 1_{\{u^\beta > t\}} \in L^{p_i}(\Omega)$ $\forall i \in \{1, \dots, n\}$. From (1.6) and (2.1) it results that

$$\begin{aligned} \mathcal{F}(u) &\le \mathcal{F}(I_{\beta,t}(u)) \\ &\le \mathcal{F}(u) - \mu \sum_{i=1}^n \int_\Omega |D_i(I_{\beta,t}(u(x))) - D_i u(x)|^{p_i} dx + \int_{\{u^\beta > t\}} M(x) dx, \end{aligned}$$

that is

$$\mu \sum_{i=1}^n \int_\Omega |D_i(I_{\beta,t}(u(x))) - D_i u(x)|^{p_i} dx \le \int_{\{u^\beta > t\}} M(x) dx. \tag{2.6}$$

If we define $\phi = (u^\beta - t) \vee 0$, then we can write (2.6) as follows:

$$\mu \sum_{i=1}^n \int_\Omega |D_i \phi(x)|^{p_i} dx \le \int_{\{u^\beta > t\}} M(x) dx. \tag{2.7}$$

If $r < +\infty$, we apply Hölder's inequality to $\int_{\{u^\beta > t\}} M(x) dx$ and we obtain

$$\int_{\{u^\beta > t\}} M(x) dx \le \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)}.$$

If $r = +\infty$, then

$$\int_{\{u^\beta > t\}} M(x)dx \le \|M\|_{L^\infty(\Omega)} |\{u^\beta > t\}| = \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{\left(1-\frac{1}{r}\right)}.$$

In both cases, from (2.7) it results that

$$\sum_{i=1}^{n} \int_\Omega |D_i\phi(x)|^{p_i} dx \le \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1-\frac{1}{r}\right)}$$

in particular, $\forall i \in \{1, \dots, n\}$

$$\int_\Omega |D_i\phi(x)|^{p_i} dx \le \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1-\frac{1}{r}\right)}$$

from which

$$\left(\int_\Omega |D_i\phi(x)|^{p_i} dx\right)^{\frac{1}{p_i}} \le \left[\frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1-\frac{1}{r}\right)}\right]^{\frac{1}{p_i}}$$

and finally

$$\left[\prod_{i=1}^{n} \left(\int_\Omega |D_i\phi(x)|^{p_i} dx\right)^{\frac{1}{p_i}}\right]^{\frac{1}{n}} \le \left[\frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1-\frac{1}{r}\right)}\right]^{\frac{1}{\overline{p}}}. \tag{2.8}$$

We apply the anisotropic imbedding theorem [10] and we use (2.8):

$$0 \leq \left(\int_{\{u^\beta > t\}} \left[u^\beta(x) - t \right]^{\overline{p}^*} dx \right)^{\frac{1}{\overline{p}^*}} = \|\phi\|_{L^{\overline{p}^*}(\Omega)} \leq c \left[\prod_{i=1}^{n} \left(\int_\Omega |D_i \phi(x)|^{p_i} dx \right)^{\frac{1}{p_i}} \right]^{\frac{1}{n}} \leq c \left[\frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)} \right]^{\frac{1}{\overline{p}}}, \tag{2.9}$$

where $c = c(n, p_1, \dots, p_n) > 0$. If $\|M\|_{L^r(\Omega)} = 0$, then from (2.9) it results that $u^\beta \leq t$ almost everywhere in Ω and we are done. If $\|M\|_{L^r(\Omega)} > 0$, then for $T > t$ we have

$$(T - t)^{\overline{p}^*} |\{u^\beta > T\}| = \int_{\{u^\beta > T\}} (T - t)^{\overline{p}^*} dx \leq \int_{\{u^\beta > T\}} \left[u^\beta(x) - t \right]^{\overline{p}^*} dx \leq \int_{\{u^\beta > t\}} \left[u^\beta(x) - t \right]^{\overline{p}^*} dx \tag{2.10}$$

and from (2.9) and (2.10) we get

$$|\{u^\beta > T\}| \leq c^{\overline{p}^*} \left(\frac{\|M\|_{L^r(\Omega)}}{\mu} \right)^{\frac{\overline{p}^*}{\overline{p}}} \frac{1}{(T - t)^{\overline{p}^*}} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right) \frac{\overline{p}^*}{\overline{p}}} \tag{2.11}$$

for every T, t with $T > t \geq t_0$. We set $\chi(t) = |\{u^\beta > t\}|$ and we use [7, Lemma 4.1, p. 93], that we provide below for the convenience of the reader.

Lemma 2.2. *Let* $\chi : [t_0, +\infty) \to [0, +\infty)$ *be decreasing. We assume that there exist* $k, a \in (0, +\infty)$ *and* $b \in (1, +\infty)$ *such that*

$$T > t \geq t_0 \Longrightarrow \chi(T) \leq \frac{k}{(T-t)^a} (\chi(t))^b. \tag{2.12}$$

Then it results that

$$\chi(t_0 + d) = 0 \quad \textit{where} \quad d = \left[k (\chi(t_0))^{b-1} 2^{\frac{ab}{(b-1)}} \right]^{\frac{1}{a}}. \tag{2.13}$$

We use the previous Lemma 2.2 and we have

$$|\{u^\beta > t_0 + d\}| = 0 \tag{2.14}$$

that is

$$u^\beta \leq t_0 + d \tag{2.15}$$

almost everywhere in Ω, where

$$d = c \left(\frac{\|M\|_{L^r(\Omega)}}{\mu} \right)^{\frac{1}{\overline{p}}} |\{u^\beta > t_0\}|^{\left[\left(1-\frac{1}{r}\right) \frac{\overline{p}^*}{\overline{p}} - 1 \right] \frac{1}{\overline{p}^*} 2^{\left(1-\frac{1}{r}\right) \frac{\overline{p}^*}{\overline{p}}} \left[\left(1-\frac{1}{r}\right) \frac{\overline{p}^*}{\overline{p}} - 1 \right]^{-1}}.$$

In order to get the right hand side of (1.7), we control $|\{u^\beta > t_0\}|$ by means of $|\Omega|$ and we take a sequence $\{(t_0)_m\}_m$ with $(t_0)_m \to \sup_{\partial\Omega} u^\beta$. Let us show how we obtain the left hand side of (1.7): we apply the right hand side of (1.7) to $-u$. This ends the proof of Theorem 1.1. $\square$

Now we are going to prove Lemma 1.2.

Proof. We assume that $\tilde{A}, A \in \mathbb{R}^{n\times n}$ with $\tilde{A}^\beta = 0$ and $\tilde{A}^\alpha = A^\alpha$ for $\alpha \neq \beta$. Then

$$\begin{aligned}\sum_\alpha |A_i^\alpha|^2 &= |A_i^\beta|^2 + \sum_{\alpha\neq\beta} |A_i^\alpha|^2 \\ &= |A_i^\beta - \tilde{A}_i^\beta|^2 + \sum_{\alpha\neq\beta} |\tilde{A}_i^\alpha|^2 \\ &= \sum_\alpha |A_i^\alpha - \tilde{A}_i^\alpha|^2 + \sum_\alpha |\tilde{A}_i^\alpha|^2\end{aligned} \tag{2.16}$$

so

$$|A_i|^2 = |A_i - \tilde{A}_i|^2 + |\tilde{A}_i|^2. \tag{2.17}$$

Since $p_i \geq 2$, the previous equality gives

$$|A_i|^{p_i} \geq |A_i - \tilde{A}_i|^{p_i} + |\tilde{A}_i|^{p_i}. \tag{2.18}$$

Moreover

$$h(\det A) \geq \inf_{\mathbb{R}} h = h(0) - [h(0) - \inf_{\mathbb{R}} h] = h(\det \tilde{A}) - [h(0) - \inf_{\mathbb{R}} h]. \tag{2.19}$$

Now we are able to estimate $f(x, A)$ and $f(x, \tilde{A})$ by means of (2.18) and (2.19)

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as follows:

$$f(x,\tilde{A})+(\min_j a_j)\sum_{i=1}^{n}|A_i-\tilde{A}_i|^{p_i} \tag{2.20}$$
$$\leq \sum_{i=1}^{n} a_i|\tilde{A}_i|^{p_i}+m(x)h(\det \tilde{A})+\sum_{i=1}^{n} a_i|A_i-\tilde{A}_i|^{p_i}$$
$$\leq \sum_{i=1}^{n} a_i|A_i|^{p_i}+m(x)h(\det A)+m(x)[h(0)-\inf_{\mathbb{R}} h]$$
$$= f(x,A)+m(x)[h(0)-\inf_{\mathbb{R}} h]$$

thus the "monotonicity" inequality (1.4) holds true with $\mu = \min_j a_j$ and $M(x) = m(x)[h(0)-\inf_{\mathbb{R}} h]$. This ends the proof of Lemma 1.2. □

References

[1] L. BOCCARDO, P. MARCELLINI AND C. SBORDONE, L^∞-regularity for a variational problems with sharp non standard growth conditions, *Boll. Un. Mat. Ital.*, **4-A** (1990), 219–226.

[2] E. DE GIORGI, Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat.*, **3** (1957), 25–43.

[3] E. DE GIORGI, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Un. Mat. Ital.*, **4** (1968), 135–137.

[4] M. GIAQUINTA AND E. GIUSTI, On the regularity of the minima of variational integrals, *Acta. Math.*, **148** (1982), 31–46.

[5] O. LADYZHENSKAYA AND N. URAL'TSEVA, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[6] J.L. LIONS, *Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires*, Dunod, Gauthier - Villars, Paris, 1969.

[7] G. STAMPACCHIA, *Equations Elliptiques du Second Ordre a Coefficientes Discontinus*, Semin. de Math. Superieures, Univ. de Montreal, **16**, 1966.

[8] B. STROFFOLINI, Global boundedness of solutions of anisotropic variational problems, *Boll. Un. Mat. Ital.*, **5-A** (1991), 345–352.

[9] Q. TANG, Regularity of minimizers of non-isotropic integrals of the calculus of variations, *Ann. Mat. Pura Appl.*, **164** (1993), 77–87.

[10] M. TROISI, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.*, **18** (1969), 3–24.