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# A REFINEMENT OF HÖLDER'S INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper, it is shown that a refinement of Hölder's inequality can be established using the positive definiteness of the Gram matrix. As applications, some improvements on Minkowski's inequality, Fan Ky's inequality and Hardy's inequality are given.

*Key words and phrases:* Inner product space, Gram matrix, Variable unit-vector, Minkowski's inequality, Fan Ky's inequality, Hardy's inequality.

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## 1. INTRODUCTION

For convenience, we need to introduce the following notations which will be frequently used throughout the paper:

$$(a^r, b^s) = \sum_{n=1}^{\infty} a_n^r b_n^s, \quad \|a\|_r = \left(\sum_{n=1}^{\infty} a_n^r\right)^{\frac{1}{r}}, \quad \|a\|_2 = \|a\|,$$

$$(f^r, g^s) = \int_0^{\infty} f^r(x) g^s(x)\, dx, \quad \|f\|_r = \left(\int_0^{\infty} f^r(x)\, dx\right)^{\frac{1}{r}}, \quad \|f\|_2 = \|f\|,$$

and

$$S_r(\alpha, y) = \left(\alpha^{r/2}, y\right) \|\alpha\|_r^{-r/2},$$

where $a = (a_1, a_2, \dots)$ are sequences of real numbers, $f : [0, \infty) \to [0, \infty)$ are measurable functions and $\alpha$ and $y$ are elements of an inner product space $E$ of real sequences.

Let $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ be sequences of real numbers in $\mathbb{R}^n$. Then Hölder's inequality can be written in the form**:**

$$(a, b) \le \|a\|_p \|b\|_q . \tag{1.1}$$

The equality in (1.1) holds if and only if $a_i^p = k b_i^q$, $i = 1, 2, \dots$, where $k$ is a constant.

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This inequality is important in function theory, functional analysis, Fourier analysis and analytic number theory, etc. However, there are drawbacks in this inequality. For example, let

$$a = (a_1, a_2, \dots, a_n, 0, \dots, 0), \quad b = (0, 0, \dots, b_{n+1}, b_{n+2}, \dots, b_{2n}), \quad a, b \in \mathbb{R}^{2n}.$$

If we let $a_i = b_j = 1$, $i = 1, 2, \dots, n$; $j = n+1, n+2, \dots, 2n$, and substitute them into (1.1), then we have $0 \le n$. In this case, Hölder's inequality is meaningless.

In the present paper we establish a new inequality that improves Hölder's inequality and remedies the defect pointed out above. At the same time, some significant refinements for a number of the classical inequalities can be established. As space is limited, only several applications of the new inequality are given.

## 2. Main Results

Let $\alpha$ and $\beta$ be elements of an inner product space $E$. Then the inner product of $\alpha$ and $\beta$ is denoted by $(\alpha, \beta)$ and the norm of $\alpha$ is given by $\|\alpha\| = \sqrt{(\alpha, \alpha)}$. In our previous papers ([1], [2]), the following result has been obtained by means of the positive definiteness of the Gram matrix.

**Lemma 2.1.** *Let $\alpha, \beta$ and $\gamma$ be three arbitrary vectors of $E$. If $\|\gamma\| = 1$, then*

$$|(\alpha, \beta)|^2 \le \|\alpha\|^2 \|\beta\|^2 - (\|\alpha\| |x| - \|\beta\| |y|)^2, \tag{2.1}$$

*where $x = (\beta, \gamma), y = (\alpha, \gamma)$. The equality in (2.1) holds if and only if $\alpha$ and $\beta$ are linearly dependent, or $\gamma$ is a linear combination of $\alpha$ and $\beta$, and $xy = 0$ but $x$ and $y$ are not simultaneously equal to zero.*

For the sake of completeness, we give here a short proof of (2.1), which can also be found in [2].

*Proof of Lemma 2.1.* Consider the Gram determinant constructed by the vectors $\alpha, \beta$ and $\gamma$:

$$G(\alpha, \beta, \gamma) = \begin{vmatrix} (\alpha, \alpha) & (\alpha, \beta) & (\alpha, \gamma) \\ (\beta, \alpha) & (\beta, \beta) & (\beta, \gamma) \\ (\gamma, \alpha) & (\gamma, \beta) & (\gamma, \gamma) \end{vmatrix}.$$

According to the positive definiteness of Gram matrix we have $G(\alpha, \beta, \gamma) \ge 0$, and $G(\alpha, \beta, \gamma) = 0$ if and only if the vectors $\alpha, \beta$ and $\gamma$ are linearly dependent.

Expanding this determinant and using the condition $\|\gamma\| = 1$ we obtain

$$\begin{aligned} G(\alpha, \beta, \gamma) &= \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \left\{ \|\alpha\|^2 x^2 - 2(\alpha, \beta)xy + \|\beta\|^2 y^2 \right\} \\ &\le \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \left\{ \|\alpha\|^2 x^2 - 2 |(\alpha, \beta)xy| + \|\beta\|^2 y^2 \right\} \\ &\le \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{ \|\alpha\| |x| - \|\beta\| |y| \}^2 \end{aligned}$$

where $x = (\beta, \gamma)$ and $y = (\alpha, \gamma)$. It follows that the equality holds if and only if the vectors $\alpha$ and $\beta$ are linearly dependent; or the vector $\gamma$ is a linear combination of the vector $\alpha$ and $\beta$, and $xy = 0$ but $x$ and $y$ are not simultaneously equal to zero. □

Applying Lemma 2.1, we can now establish the following refinement of Hölder's inequality.

**Theorem 2.2.** *Let $a_n, b_n \ge 0$, $(n = 1, 2, \dots)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|a\|_p < +\infty$ and $0 < \|b\|_q < +\infty$, then*

$$(a, b) \le \|a\|_p \|b\|_q (1 - r)^m, \tag{2.2}$$

*where*

$$r = (S_p(a,c) - S_q(b,c))^2, \qquad m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}, \qquad \|c\| = 1$$

*and*

$$\left(a^{p/2}, c\right)\left(b^{q/2}, c\right) \geq 0.$$

*The equality in (2.2) holds if and only if $a^{p/2}$and $b^{q/2}$ are linearly dependent; or if the vector $c$ is a linear combination of $a^{p/2}$and $b^{q/2}$, and $\left(a^{p/2}, c\right)\left(b^{q/2}, c\right) = 0$, but the vector $c$ is not simultaneously orthogonal to $a^{p/2}$ and $b^{q/2}$.*

*Proof.* Firstly, we consider the case $p \neq q$. Without loss of generality, we suppose that $p > q > 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $R = \frac{p}{2}$, $Q = \frac{p}{p-2}$, then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain

$$\begin{aligned} (a,b) &= \sum_{k=1}^{\infty} a_k b_k \\ &= \sum_{k=1}^{\infty} \left(a_k b_k^{q/p}\right) b_k^{1-q/p} \\ &\leq \left(\sum_{k=1}^{\infty} \left(a_k b_k^{q/p}\right)^R\right)^{\frac{1}{R}} \left(\sum_{k=1}^{\infty} \left(b_k^{1-q/p}\right)^Q\right)^{\frac{1}{Q}} \\ &= \left(a^{p/2}, b^{q/2}\right)^{2/p} \|b\|_q^{q(1-2/p)}. \end{aligned} \tag{2.3}$$

The equality in (2.3) holds if and only if $a^{p/2}$and $b^{q/2}$ are linearly dependent. In fact, the equality in (2.3) holds if and only if for any $k$, there exists $c_0$ $(c_0 \neq 0)$ such that

$$\left(a_k b_k^{q/p}\right)^R = c_0 \left(b_k^{1-q/p}\right)^Q.$$

It is easy to deduce that $a_k^{p/2} = c_0 b_k^{q/2}$.

If $\alpha, \beta$ and $\gamma$ in (2.1) are replaced by $a^{p/2}, b^{q/2}$ and $c$ respectively, then we have

$$\left(a^{p/2}, b^{q/2}\right)^2 \leq \|a\|_p^p \|b\|_q^q (1-r), \tag{2.4}$$

where $r = (S_p(a,c) - S_q(b,c))^2$. Substituting (2.4) into (2.3), we obtain after simplifications

$$(a,b) \leq \|a\|_p \|b\|_q (1-r)^{\frac{1}{p}}. \tag{2.5}$$

It is known from Lemma 2.1 that the equality in (2.5) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent; or if the vector $c$ is a linear combination of $a^{p/2}$and $b^{q/2}$, and $\left(a^{p/2}, c\right)\left(b^{q/2}, c\right) = 0$, but the vector $c$ is not simultaneously orthogonal to $a^{p/2}$ and $b^{q/2}$.

Note the symmetry of $p$ and $q$. The inequality (2.2) follows from (2.5).

Secondly, consider the case $p = 2$. By Lemma 2.1, we obtain:

$$(a,b) \leq \|a\| \|b\| (1-r)^{\frac{1}{2}}, \tag{2.6}$$

where $r = \left(\frac{(a,c)}{\|a\|} - \frac{(b,c)}{\|b\|}\right)^2$, $\|c\| = 1$ and $(a,c)(b,c) \geq 0$. The equality in (2.6) holds if and only if $a$ and $b$ are linearly dependent, or the vector $c$ is a linear combination of $a$ and $b$, and $(a,c)(b,c) = 0$, but $(a,c)$ and $(b,c)$ are not simultaneously equal to zero.

The proof of the theorem is thus completed. □

Consider the example given in the Introduction. Let $c = (c_1, c_2, \dots, c_{2n})$, $c \in \mathbb{R}^{2n}$, where $c_i = \frac{1}{\sqrt{n}}$, $i = 1, 2, \dots, n$ and $c_j = 0$, $j = n+1, n+2, \dots, 2n$. It is easy to deduce that $\|c\| = 1$ and $r = 1$. Substituting them into (2.2), it follows that the equality is valid.

The following theorem provides a similar result to Theorem 2.2.

**Theorem 2.3.** *Let $f(x), g(x) \geq 0$ $(x \in (0, +\infty))$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_q < +\infty$, then*

$$(f, g) \leq \|f\|_p \|g\|_q (1 - r)^m, \tag{2.7}$$

*where*

$$r = (S_p(f,h) - S_q(g,h))^2, \qquad m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\},$$

$$\|h\| = 1, \quad \textit{i.e.} \quad \|h\| = \left(\int_0^\infty h^2(x)dx\right)^{\frac{1}{2}} = 1$$

*and*

$$\left(f^{p/2}, h\right)\left(g^{q/2}, h\right) \geq 0.$$

*The equality in (2.3) holds if and only if $f^{p/2}$and $g^{q/2}$ are linearly dependent; or the vector $h$ is a linear combination of $f^{p/2}$and $g^{q/2}$, and $\left(f^{p/2}, h\right)\left(g^{q/2}, h\right) = 0$, but the vector $h$ is not simultaneously orthogonal to $f^{p/2}$ and $g^{q/2}$.*

Its proof is similar to that of Theorem 2.2. Hence it is omitted.

## 3. APPLICATIONS

### 3.1. A Refinement of Minkowski's Inequality.

We firstly give a refinement of Minkowski's inequality for the discrete form.

**Theorem 3.1.** *Let $a_k, b_k \geq 0$, $p > 1$. If $0 < \|a\|_p < +\infty$ and $0 < \|b\|_p < +\infty$, then*

$$\|a + b\|_p < \left(\|a\|_p + \|b\|_p\right)(1 - r)^m, \tag{3.1}$$

*where*

$$\|a + b\|_p = \left(\sum_{k=1}^{\infty} (a_k + b_k)^p\right)^{\frac{1}{p}},$$

$$r = \min\{r(a), r(b)\}, \qquad m = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\},$$

$$r(x) = \left\{\frac{\left(x^{p/2}, c\right)}{\|x\|_p^{p/2}} - \frac{\left((a+b)^{p/2}, c\right)}{\|a+b\|_p^{p/2}}\right\}^2, \qquad x = a, b;$$

$$\left((a+b)^{p/2}, c\right) = \sum_{k=1}^{\infty} (a_k + b_k)^{p/2} c_k,$$

*and $c$ is a variable unit-vector.*

*Proof.* Let $m = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$,

$$\|a + b\|_p = \left(\sum_{k=1}^{\infty} (a_k + b_k)^p\right)^{\frac{1}{p}}.$$

By Theorem 2.2, we have

$$\sum_{k=1}^{\infty} a_k (a_k + b_k)^{p-1} \le \|a\|_p \left( \sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{1-\frac{1}{p}} (1 - r(a))^m \tag{3.2}$$

and

$$\sum_{k=1}^{\infty} b_k (a_k + b_k)^{p-1} \le \|b\|_p \left( \sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{1-\frac{1}{p}} (1 - r(b))^m , \tag{3.3}$$

where

$$r(x) = \left\{ \frac{\left(x^{p/2}, c\right)}{\|x\|_p^{p/2}} - \frac{\left((a+b)^{p/2}, c\right)}{\|a+b\|_p^{p/2}} \right\}^2 , \quad x = a, b,$$

$$\|a+b\|_p^{p/2} = \left( \sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{\frac{1}{2}} ,$$

$$\left((a+b)^{p/2}, c\right) = \sum_{k=1}^{\infty} (a_k + b_k)^{p/2} c_k,$$

and $c$ is a variable unit-vector.

Adding (3.5) and (3.3) we obtain, after simplifying:

$$\|a+b\|_p \le \|a\|_p (1 - r(a))^m + \|b\|_p (1 - r(b))^m . \tag{3.4}$$

Let $r = \min\{r(a), r(b)\}$, then the inequality (3.1) follows. This completes the proof of Theorem 3.1. □

If we choose a unit-vector $c$ such that its $i$th component is 1 and the rest is zero, i.e. $c = (0, 0, \dots, 0, \underset{(i)}{1}, 0, \dots)$, then

$$r(x) = \left\{ \frac{x_i^{p/2}}{\|x\|_p^{p/2}} - \frac{(a_i + b_i)^{p/2}}{\|a+b\|_p^{p/2}} \right\}^2 \quad x = a, b.$$

Similarly, we can establish a refinement of Minkowski's integral inequality.

**Theorem 3.2.** *Let* $f(x), g(x) \ge 0$, $p > 1$. *If* $0 < \|f\|_p < +\infty$ *and* $0 < \|g\|_p < +\infty$, *then*

$$\|f+g\|_p < \left(\|f\|_p + \|g\|_p\right) (1 - r)^m, \tag{3.5}$$

*where*

$$\|f+g\|_p = \left( \int_0^{\infty} (f(x) + g(x))^p \, dx \right)^{\frac{1}{p}} ,$$

$$r = \min\{r(f), r(g)\}, \quad m = \min\left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\},$$

$$r(t) = \left\{ \frac{\left(t^{p/2}, h\right)}{\|t\|_p^{p/2}} - \frac{\left((f+g)^{p/2}, h\right)}{\|f+g\|_p^{p/2}} \right\}^2 , \quad t = f, g,$$

$$\left((f+g)^{p/2}, h\right) = \int_0^{\infty} (f(x) + g(x))^{p/2} h(x) \, dx,$$

*and $h$ is a variable unit-vector, i.e.*

$$\|h\| = \left\{ \int_0^{\infty} h^2(x)\, dx \right\}^{\frac{1}{2}} = 1.$$

Its proof is similar to that of Theorem 3.1. Hence it is omitted.

**Remark 3.3.** The variable unit-vector $h$ can be chosen in accordance with our requirements. For example, we may choose $h$ such that

$$h(x) = \sqrt{\frac{2}{\pi(1+x^2)}}.$$

### 3.2. **A Strengthening of Fan Ky's Inequality.**

**Theorem 3.4.** *Let $A, B$ and $C$ be three positive definite matrices of order $n$, $0 \le \lambda \le 1$. Then*

$$|A|^{\lambda} |B|^{1-\lambda} \le |\lambda A + (1-\lambda) B| \left( 1 - \left( \frac{|AC|^{\frac{1}{4}}}{\left|\frac{1}{2}(A+C)\right|^{\frac{1}{2}}} - \frac{|BC|^{\frac{1}{4}}}{\left|\frac{1}{2}(B+C)\right|^{\frac{1}{2}}} \right)^2 \right)^m, \tag{3.6}$$

*where $|C| = \pi^n$, $\quad m = \min\{\lambda, 1-\lambda\}$.*

*Proof.* When $\lambda = 0, 1$, the inequality (3.3) is obviously valid. Hence we need only consider the case $0 < \lambda < 1$.

If $D$ is a positive definite matrix of order $n$, then it is known from [4] that

$$J_n = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-(x,Dx)} dx = \frac{\pi^{n/2}}{|D|^{\frac{1}{2}}}, \tag{3.7}$$

where $x = (x_1, x_2, \dots, x_n)$, and $dx = dx_1 dx_2 \cdots dx_n$.

Let $F(x) = e^{-\lambda(x,Ax)}$ and $G(x) = e^{-(1-\lambda)(x,Bx)}$. If $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$, according to (3.4) and (2.7) we have

$$\begin{aligned} &\frac{\pi^{n/2}}{|\lambda A + (1-\lambda) B|^{\frac{1}{2}}} \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(x) G(x)\, dx \\ &\le \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} G^q(x)\, dx \right\}^{\frac{1}{q}} (1-r)^m \\ &= \frac{\pi^{n/2}(1-r)^m}{\left(|A|^{\lambda} |B|^{1-\lambda}\right)^{\frac{1}{2}}}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} r &= \left( S_{\frac{1}{\lambda}}(F,H) - S_{\frac{1}{1-\lambda}}(G,H) \right)^2 \\ &= \left\{ \left( F^{\frac{1}{2\lambda}}, H \right) \|F\|_{\frac{1}{\lambda}}^{-\frac{1}{2\lambda}} - \left( G^{\frac{1}{2(1-\lambda)}}, H \right) \|G\|_{\frac{1}{1-\lambda}}^{-\frac{1}{2(1-\lambda)}} \right\}, \end{aligned}$$

where $H = e^{-\frac{1}{2}(x,Cx)}$, $C$ is a positive definite matrix of order $n$, and

$$\|H\| = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H^2(x)\, dx \right\}^{\frac{1}{2}} = 1.$$

By the definition of the variable unit-vector $H$, it is easy to deduce that $|C| = \pi^n$. Hence we have

$$\left(F^{\frac{1}{2\lambda}}, H\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^{\frac{1}{2\lambda}}(x)\, H(x)\, dx$$
$$= \frac{\pi^{n/2}}{\left|\frac{1}{2}(A+C)\right|^{\frac{1}{2}}} = \left\{\frac{|C|}{\left|\frac{1}{2}(A+C)\right|}\right\}^{\frac{1}{2}}$$

and

$$\|F\|_{1/\lambda}^{1/2\lambda} = \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^{1/\lambda}(x)\, dx\right\}^{\frac{1}{2}} = \left\{\frac{\pi^{n/2}}{|A|^{1/2}}\right\}^{\frac{1}{2}} = \left\{\frac{|C|}{|A|}\right\}^{\frac{1}{4}},$$

whence

$$S_{1/\lambda}(F,H) = \frac{|AC|^{\frac{1}{4}}}{\left|\frac{1}{2}(A+C)\right|^{\frac{1}{2}}}.$$

Similarly,

$$S_{1/(1-\lambda)}(G,H) = \frac{|BC|^{\frac{1}{4}}}{\left|\frac{1}{2}(B+C)\right|^{\frac{1}{2}}},$$

therefore we obtain

$$r = \left(\frac{|AC|^{\frac{1}{4}}}{\left|\frac{1}{2}(A+C)\right|^{\frac{1}{2}}} - \frac{|BC|^{\frac{1}{4}}}{\left|\frac{1}{2}(B+C)\right|^{\frac{1}{2}}}\right)^2. \tag{3.9}$$

It follows from (3.8) and (3.9) that the inequality (3.3) is valid. □

### 3.3. An Improvement of Hardy's Inequality.

We give firstly a refinement of Hardy's inequality for the discrete form.

**Theorem 3.5.** *Let* $a_n \geq 0$, $\beta_n = \frac{1}{n}\sum_{k=1}^{n} a_k$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \|a\|_p < +\infty$, *then*

$$\|\beta\|_p \leq \left(\frac{p}{p-1}\right)\|a\|_p (1-r)^m, \tag{3.10}$$

*where*

$$r = \left(\frac{(a^{p/2}, c)}{\|a\|_p^{p/2}} - \frac{(\beta^{p/2}, c)}{\|\beta\|_p^{p/2}}\right)^2,$$

$c$ *is a variable unit-vector and* $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

*Proof.* Firstly, we estimate the difference of the following two terms:

$$\beta_n^p - \frac{p}{p-1}\beta_n^{p-1}a_n = \beta_n^p - \frac{p}{p-1}\left(n\beta_n - (n-1)\beta_{n-1}\right)\beta_n^{p-1} \tag{3.11}$$
$$= \beta_n^p\left(1 - \frac{np}{p-1}\right) + \frac{(n-1)p}{p-1}\left((\beta_n^p)^{p-1}\beta_{n-1}^p\right)^{\frac{1}{p}}.$$

Applying the arithmetic-geometric mean inequality to the second term on the right-hand side of (3.11) we get

$$\left((\beta_n^p)^{p-1}\beta_{n-1}^p\right)^{\frac{1}{p}} \leq \frac{1}{p}\left((p-1)\beta_n^p + \beta_{n-1}^p\right). \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\beta_n^p - \frac{p}{p-1}\beta_n^{p-1}a_n \le \beta_n^p\left(1-\frac{np}{p-1}\right) + \frac{(n-1)}{p-1}\left((p-1)\beta_n^p + \beta_{n-1}^p\right)$$
$$= \frac{1}{p-1}\left((n-1)\beta_{n-1}^p - n\beta_n^p\right).$$

Summing the above inequality with respect to $n$, we have

$$\sum_{n=1}^{N}\beta_n^p - \frac{p}{p-1}\sum_{n=1}^{N}\beta_n^{p-1}a_n \le -\frac{1}{p-1}\left(N\beta_N^p\right) \le 0.$$

Hence

$$\sum_{n=1}^{N}\beta_n^p \le \frac{p}{p-1}\sum_{n=1}^{N}\beta_n^{p-1}a_n.$$

Letting $N \to \infty$, we get

$$\sum_{n=1}^{\infty}\beta_n^p \le \frac{p}{p-1}\sum_{n=1}^{\infty}\beta_n^{p-1}a_n. \tag{3.13}$$

Applying the inequality (2.2) to the right-hand side of (3.13) we obtain

$$\frac{p}{p-1}\sum_{n=1}^{\infty}a_n\beta_n^{p-1} \le \frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\beta_n^{(p-1)q}\right)^{\frac{1}{q}}(1-r)^m \tag{3.14}$$
$$= \frac{p}{p-1}\|a\|_p\left(\|\beta\|_p^p\right)^{\frac{1}{q}}(1-r)^m,$$

where $r = \left(S_p(a,c) - S_q(\beta^{p-1},c)\right)^2$, $c$ is a variable unit-vector and $m = \min\left\{\frac{1}{p},\frac{1}{q}\right\}$.

We obtain from (3.13) and (3.14) after simplification

$$\|\beta\|_p \le \left(\frac{p}{p-1}\right)\|a\|_p(1-r)^m. \tag{3.15}$$

It is easy to deduce that

$$S_p(a,c) = \frac{\left(a^{p/2},c\right)}{\|a\|_p^{p/2}} \quad \text{and} \quad S_q(\beta^{p-1},c) = \frac{\left(\beta^{(p-1)q/2},c\right)}{\|\beta^{p-1}\|_q^{q/2}} = \frac{\left(\beta^{p/2},c\right)}{\|\beta\|_p^{p/2}}.$$

Hence

$$r = \left(\left(a^{p/2},c\right)\|a\|_p^{-p/2} - \left(\beta^{p/2},c\right)\|\beta\|_p^{-p/2}\right)^2,$$

where $c$ is a variable unit-vector. The proof of the theorem is completed. □

A variable unit-vector $c$ can be chosen in accordance with our requirements. For example, we may choose $c \in \mathbb{R}^\infty$ such that $c = (1,0,0,\dots)$. Obviously, $\|c\| = 1$ and

$$r = a_1^p\left(\|a\|_p^{-p/2} - \|\beta\|_p^{-p/2}\right)^2.$$

Similarly, we can establish a refinement of Hardy's integral inequality.

**Theorem 3.6.** *Let* $f(x) \ge 0$, $g(x) = \frac{1}{x}\int_0^x f(t)dt$, $\frac{1}{p}+\frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \int_0^\infty f(t)dt < +\infty$, *then*

$$\|g\|_p < \frac{p}{p-1}\|f\|_p(1-r)^m, \tag{3.16}$$

*where*

$$r = \left( \frac{(f^{p/2}, h)}{\|f\|_p^{p/2}} - \frac{(g^{p/2}, h)}{\|g\|_p^{p/2}} \right)^2,$$

*$h$ is a variable unit-vector, i.e.*

$$\|h\| = \left( \int_0^\infty h^2(t)dt \right)^{\frac{1}{2}} = 1 \quad \textit{and} \quad m = \min\left\{ \frac{1}{p}, \frac{1}{q} \right\}.$$

*Proof.* Using integration by parts and then applying (2.2) we obtain that

$$\begin{aligned} \|g\|_p^p = \int_0^\infty g^p(t)dt &= \frac{p}{p-1}\left(f, g^{p-1}\right) \\ &\le \frac{p}{p-1}\|f\|_p \left\|g^{p-1}\right\|_q (1-r)^m \\ &= \frac{p}{p-1}\|f\|_p \|g\|_p^{p-1} (1-r)^m, \end{aligned} \tag{3.17}$$

where $r = (S_p(f,h) - S_q(g^{p-1},h))^2$, $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $h$ is a variable unit-vector. It is easy to deduce that

$$S_p(f,h) = \frac{\left(f^{p/2}, h\right)}{\|f\|_p^{p/2}} \qquad \text{and} \qquad S_q\left(g^{p-1}, h\right) = \frac{\left(g^{p/2}, h\right)}{\|g\|_p^{p/2}}.$$

It follows that the inequality (3.16) is valid. The theorem is thus proved. □

A variable unit-vector $h$ can be chosen in accordance with our requirements. For example, we may choose $h$ such that $h(x) = e^{-x/2}$. Obviously, we then have

$$\|h\| = \left( \int_0^\infty h^2(t)dt \right)^{\frac{1}{2}} = 1.$$

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