

# HEISENBERG-PAULI-WEYL UNCERTAINTY PRINCIPLE FOR THE RIEMANN-LIOUVILLE OPERATOR

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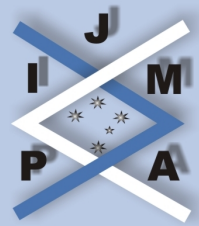
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*Key words:* Heisenberg-Pauli-Weyl Inequality, Riemann-Liouville operator, Fourier transform, local uncertainty principle.

*Abstract:* The Heisenberg-Pauli-Weyl inequality is established for the Fourier transform connected with the Riemann-Liouville operator. Also, a generalization of this inequality is proved. Lastly, a local uncertainty principle is studied.



Heisenberg-Pauli-Weyl  
Uncertainty Principle

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## 1. Introduction

Uncertainty principles play an important role in harmonic analysis and have been studied by many authors and from many points of view [8]. These principles state that a function  $f$  and its Fourier transform  $\widehat{f}$  cannot be simultaneously sharply localized. The theorems of Hardy, Morgan, Beurling, ... are established for several Fourier transforms in [4], [9], [13] and [14]. In this context, a remarkable Heisenberg uncertainty principle [10] states, according to Weyl [25] who assigned the result to Pauli, that for all square integrable functions  $f$  on  $\mathbb{R}^n$  with respect to the Lebesgue measure, we have

$$\left( \int_{\mathbb{R}^n} x_j^2 |f(x)|^2 dx \right) \left( \int_{\mathbb{R}^n} \xi_j^2 |\widehat{f}(\xi)|^2 d\xi \right) \geq \frac{1}{4} \left( \int_{\mathbb{R}^n} |f(x)|^2 dx \right)^2, \quad j \in \{1, \dots, n\}.$$

This inequality is called the Heisenberg-Pauli-Weyl inequality for the classical Fourier transform.

Recently, many works have been devoted to establishing the Heisenberg-Pauli-Weyl inequality for various Fourier transforms, Rösler [21] and Shimeno [22] have proved this inequality for the Dunkl transform, in [20] Rösler and Voit have established an analogue of the Heisenberg-Pauli-Weyl inequality for the generalized Hankel transform. In the same context, Battle [3] has proved this inequality for wavelet states, and Wolf [26], has studied this uncertainty principle for Gelfand pairs. We cite also De Bruijn [5] who has established the same result for the classical Fourier transform by using Hermite Polynomials, and Rassias [17, 18, 19] who gave several generalized forms for the Heisenberg-Pauli-Weyl inequality.

In [2], the second author with others considered the singular partial differential

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operators defined by

$$\begin{cases} \Delta_1 = \frac{\partial}{\partial x}, \\ \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}; \end{cases} (r, x) \in ]0, +\infty[ \times \mathbb{R}; \alpha \geq 0.$$

and they associated to  $\Delta_1$  and  $\Delta_2$  the following integral transform, called the Riemann-Liouville operator, defined on  $\mathcal{C}_*(\mathbb{R}^2)$  (the space of continuous functions on  $\mathbb{R}^2$ , even with respect to the first variable) by

$$\begin{aligned} & \mathcal{R}_\alpha(f)(r, x) \\ = & \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt) (1-t^2)^{\alpha-\frac{1}{2}} (1-s^2)^{\alpha-1} dt ds; & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r\sqrt{1-t^2}, x+rt) \frac{dt}{\sqrt{(1-t^2)}}; & \text{if } \alpha = 0. \end{cases} \end{aligned}$$

In addition, a convolution product and a Fourier transform  $\mathcal{F}_\alpha$  connected with the mapping  $\mathcal{R}_\alpha$  have been studied and many harmonic analysis results have been established for the Fourier transform  $\mathcal{F}_\alpha$  (Inversion formula, Plancherel formula, Paley-Winer and Plancherel theorems, ...).

Our purpose in this work is to study the Heisenberg-Pauli-Weyl uncertainty principle for the Fourier transform  $\mathcal{F}_\alpha$  connected with  $\mathcal{R}_\alpha$ . More precisely, using Laguerre and Hermite polynomials we establish firstly the Heisenberg-Pauli-Weyl inequality for the Fourier transform  $\mathcal{F}_\alpha$ , that is

- For all  $f \in L^2(d\nu_\alpha)$ , we have

$$\left( \int_0^{+\infty} \int_{\mathbb{R}} (r^2 + x^2) |f(r, x)|^2 d\nu_\alpha(r, x) \right)^{\frac{1}{2}}$$

$$\begin{aligned} & \times \left( \int \int_{\Gamma_+} (\mu^2 + 2\lambda^2) |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1}{2}} \\ & \geq \frac{2\alpha + 3}{2} \left( \int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)|^2 d\nu_\alpha(r, x) \right), \end{aligned}$$

with equality if and only if

$$f(r, x) = C e^{-\frac{r^2+x^2}{2t_0^2}}; \quad C \in \mathbb{C}, \quad t_0 > 0,$$

where

- $d\nu_\alpha(r, x)$  is the measure defined on  $\mathbb{R}_+ \times \mathbb{R}$  by

$$d\nu_\alpha(r, x) = \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha + 1) \sqrt{2\pi}} dr \otimes dx.$$

- $d\gamma_\alpha(\mu, \lambda)$  is the measure defined on the set

$$\Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \{(it, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}; t \leq |x|\},$$

by

$$\begin{aligned} & \int \int_{\Gamma_+} g(\mu, \lambda) d\gamma_\alpha(\mu, \lambda) \\ & = \frac{1}{2^\alpha \Gamma(\alpha + 1) \sqrt{2\pi}} \left( \int_0^{+\infty} \int_{\mathbb{R}} g(\mu, \lambda) (\mu^2 + \lambda^2)^\alpha \mu d\mu d\lambda \right. \\ & \quad \left. + \int_{\mathbb{R}} \int_0^{|\lambda|} g(i\mu, \lambda) (\lambda^2 - \mu^2)^\alpha \mu d\mu d\lambda \right). \end{aligned}$$



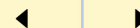
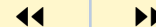
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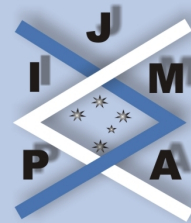
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Next, we give a generalization of the Heisenberg-Pauli-Weyl inequality, that is

- For all  $f \in L^2(d\nu_\alpha)$ ,  $a, b \in \mathbb{R}$ ;  $a, b \geq 1$  and  $\eta \in \mathbb{R}$  such that  $\eta a = (1 - \eta)b$ , we have

$$\begin{aligned} & \left( \int_0^{+\infty} \int_{\mathbb{R}} (r^2 + x^2)^a |f(r, x)|^2 d\nu_\alpha(r, x) \right)^{\frac{\eta}{2}} \\ & \quad \times \left( \int \int_{\Gamma_+} (\mu^2 + 2\lambda^2)^b |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1-\eta}{2}} \\ & \geq \left( \frac{2\alpha + 3}{2} \right)^{a\eta} \left( \int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)|^2 d\nu_\alpha(r, x) \right)^{\frac{1}{2}}, \end{aligned}$$

with equality if and only if

$$a = b = 1 \quad \text{and} \quad f(r, x) = C e^{-\frac{r^2+x^2}{2t_0^2}}; \quad C \in \mathbb{C}; \quad t_0 > 0.$$

In the last section of this paper, building on the ideas of Faris [7], and Price [15, 16], we develop a family of inequalities in their sharpest forms, which constitute the principle of local uncertainty.

Namely, we have established the following main results

- For all real numbers  $\xi$ ;  $0 < \xi < \frac{2\alpha+3}{2}$ , there exists a positive constant  $K_{\alpha,\xi}$  such that for all  $f \in L^2(d\nu_\alpha)$ , and for all measurable subsets  $E \subset \Gamma_+$ ;  $0 < \gamma_\alpha(E) < +\infty$ , we have

$$\begin{aligned} & \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \\ & < K_{\alpha,\xi} (\gamma_\alpha(E))^{\frac{2\xi}{2\alpha+3}} \int_0^{+\infty} \int_{\mathbb{R}} (r^2 + x^2)^\xi |f(r, x)|^2 d\nu_\alpha(r, x). \end{aligned}$$



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- For all real number  $\xi$ ;  $\xi > \frac{2\alpha+3}{2}$ , there exists a positive constant  $M_{\alpha,\xi}$  such that for all  $f \in L^2(d\nu_\alpha)$ , and for all measurable subsets  $E \subset \Gamma_+$ ;  $0 < \gamma_\alpha(E) < +\infty$ , we have

$$\begin{aligned} & \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \\ & < M_{\alpha,\xi} \gamma_\alpha(E) \left( \int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)|^2 d\nu_\alpha(r, x) \right)^{\frac{2\xi-2\alpha-3}{2\xi}} \\ & \quad \times \left( \int_0^{+\infty} \int_{\mathbb{R}} (r^2 + x^2)^\xi |f(r, x)|^2 d\nu_\alpha(r, x) \right)^{\frac{2\alpha+3}{2\xi}}, \end{aligned}$$

where  $M_{\alpha,\xi}$  is the best (the smallest) constant satisfying this inequality.

## 2. The Fourier Transform Associated with the Riemann-Liouville Operator

It is well known [2] that for all  $(\mu, \lambda) \in \mathbb{C}^2$ , the system

$$\begin{cases} \Delta_1 u(r, x) = -i\lambda u(r, x), \\ \Delta_2 u(r, x) = -\mu^2 u(r, x), \\ u(0, 0) = 1, \quad \frac{\partial u}{\partial r}(0, x) = 0, \quad \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution  $\varphi_{\mu, \lambda}$ , given by

$$(2.1) \quad \forall (r, x) \in \mathbb{R}^2; \quad \varphi_{\mu, \lambda}(r, x) = j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) e^{-i\lambda x},$$

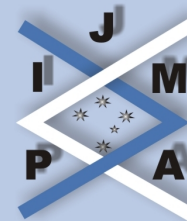
where

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(x)}{x^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(\alpha + n + 1)} \left( \frac{x}{2} \right)^{2n},$$

and  $J_\alpha$  is the Bessel function of the first kind and index  $\alpha$  [6, 11, 12, 24].

The modified Bessel function  $j_\alpha$  has the following integral representation [1, 11], for all  $\mu \in \mathbb{C}$ , and  $r \in \mathbb{R}$  we have

$$j_\alpha(r\mu) = \begin{cases} \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(r\mu t) dt, & \text{if } \alpha > -\frac{1}{2}; \\ \cos(r\mu), & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$



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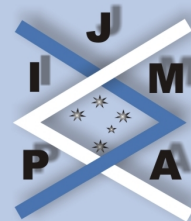
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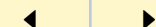
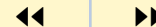
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In particular, for all  $r, s \in \mathbb{R}$ , we have

$$(2.2) \quad |j_\alpha(rs)| \leq \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} |\cos(rst)| dt \\ \leq \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} dt = 1.$$

From the properties of the Bessel function, we deduce that the eigenfunction  $\varphi_{\mu,\lambda}$  satisfies the following properties

•

$$(2.3) \quad \sup_{(r,x) \in \mathbb{R}^2} |\varphi_{\mu,\lambda}(r, x)| = 1,$$

if and only if  $(\mu, \lambda)$  belongs to the set

$$\Gamma = \mathbb{R}^2 \cup \{(it, x); (t, x) \in \mathbb{R}^2; |t| \leq |x|\}.$$

• The eigenfunction  $\varphi_{\mu,\lambda}$  has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 \cos(\mu r s \sqrt{1-t^2}) e^{-i\lambda(x+rt)} (1-t^2)^{\alpha-\frac{1}{2}} (1-s^2)^{\alpha-1} dt ds; & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 \cos(r\mu\sqrt{1-t^2}) e^{-i\lambda(x+rt)} \frac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha = 0. \end{cases}$$

In [2], using this integral representation, the authors have defined the Riemann-Liouville integral transform associated with  $\Delta_1, \Delta_2$  by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt) (1-t^2)^{\alpha-\frac{1}{2}} (1-s^2)^{\alpha-1} dt ds; & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r\sqrt{1-t^2}, x+rt) \frac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha = 0. \end{cases}$$



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where  $f$  is a continuous function on  $\mathbb{R}^2$ , even with respect to the first variable.

The transform  $\mathcal{R}_\alpha$  generalizes the "mean operator" defined by

$$\mathcal{R}_0(f)(r, x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \theta, x + r \cos \theta) d\theta.$$

In the following we denote by

- $d\nu_\alpha$  the measure defined on  $\mathbb{R}_+ \times \mathbb{R}$ , by

$$d\nu_\alpha(r, x) = \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha + 1) \sqrt{2\pi}} dr \otimes dx.$$

- $L^p(d\nu_\alpha)$  the space of measurable functions  $f$  on  $\mathbb{R}_+ \times \mathbb{R}$  such that

$$\|f\|_{p, \nu_\alpha} = \left( \int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)|^p d\nu_\alpha(r, x) \right)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1, +\infty[,$$

$$\|f\|_{\infty, \nu_\alpha} = \text{ess sup}_{(r, x) \in \mathbb{R}_+ \times \mathbb{R}} |f(r, x)| < \infty, \quad \text{if } p = +\infty.$$

- $\langle \ / \ \rangle_{\nu_\alpha}$  the inner product defined on  $L^2(d\nu_\alpha)$  by

$$\langle f/g \rangle_{\nu_\alpha} = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \overline{g(r, x)} d\nu_\alpha(r, x).$$

- $\Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \{(it, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}; t \leq |x|\}$ .

- $\mathcal{B}_{\Gamma_+}$  the  $\sigma$ -algebra defined on  $\Gamma_+$  by

$$\mathcal{B}_{\Gamma_+} = \{\theta^{-1}(B), B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\},$$

where  $\theta$  is the bijective function defined on the set  $\Gamma_+$  by

$$(2.4) \quad \theta(\mu, \lambda) = \left( \sqrt{\mu^2 + \lambda^2}, \lambda \right).$$



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- $d\gamma_\alpha$  the measure defined on  $\mathcal{B}_{\Gamma_+}$  by

$$(2.5) \quad \forall A \in \mathcal{B}_{\Gamma_+}; \gamma_\alpha(A) = \nu_\alpha(\theta(A))$$

- $L^p(d\gamma_\alpha)$  the space of measurable functions  $f$  on  $\Gamma_+$ , such that

$$\|f\|_{p,\gamma_\alpha} = \left( \int \int_{\Gamma_+} |f(\mu, \lambda)|^p d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1, +\infty[,$$

$$\|f\|_{\infty,\gamma_\alpha} = \text{ess sup}_{(\mu,\lambda) \in \Gamma_+} |f(\mu, \lambda)| < \infty, \quad \text{if } p = +\infty.$$

- $\langle \ / \ \rangle_{\gamma_\alpha}$  the inner product defined on  $L^2(d\gamma_\alpha)$  by

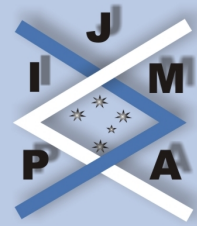
$$\langle f/g \rangle_{\gamma_\alpha} = \int \int_{\Gamma_+} f(\mu, \lambda) \overline{g(\mu, \lambda)} d\gamma_\alpha(\mu, \lambda).$$

Then, we have the following properties.

**Proposition 2.1.**

i) For all non negative measurable functions  $g$  on  $\Gamma_+$ , we have

$$(2.6) \quad \int \int_{\Gamma_+} g(\mu, \lambda) d\gamma_\alpha(\mu, \lambda) \\
= \frac{1}{2^\alpha \Gamma(\alpha + 1) \sqrt{2\pi}} \left( \int_0^{+\infty} \int_{\mathbb{R}} g(\mu, \lambda) (\mu^2 + \lambda^2)^\alpha \mu d\mu d\lambda \right. \\
\left. + \int_{\mathbb{R}} \int_0^{|\lambda|} g(i\mu, \lambda) (\lambda^2 - \mu^2)^\alpha \mu d\mu d\lambda \right).$$



In particular

$$(2.7) \quad d\gamma_{\alpha+1}(\mu, \lambda) = \frac{\mu^2 + \lambda^2}{2(\alpha + 1)} d\gamma_{\alpha}(\mu, \lambda).$$

ii) For all measurable functions  $f$  on  $\mathbb{R}_+ \times \mathbb{R}$ , the function  $f \circ \theta$  is measurable on  $\Gamma_+$ . Furthermore, if  $f$  is non negative or an integrable function on  $\mathbb{R}_+ \times \mathbb{R}$  with respect to the measure  $d\nu_{\alpha}$ , then we have

$$(2.8) \quad \int \int_{\Gamma_+} (f \circ \theta)(\mu, \lambda) d\gamma_{\alpha}(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) d\nu_{\alpha}(r, x).$$

In the following, we shall define the Fourier transform  $\mathcal{F}_{\alpha}$  associated with the operator  $\mathcal{R}_{\alpha}$  and we give some properties that we use in the sequel.

**Definition 2.2.** The Fourier transform  $\mathcal{F}_{\alpha}$  associated with the Riemann-liouville operator  $\mathcal{R}_{\alpha}$  is defined on  $L^1(d\nu_{\alpha})$  by

$$(2.9) \quad \forall (\mu, \lambda) \in \Gamma; \mathcal{F}_{\alpha}(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu_{\alpha}(r, x).$$

By the relation (2.3), we deduce that the Fourier transform  $\mathcal{F}_{\alpha}$  is a bounded linear operator from  $L^1(d\nu_{\alpha})$  into  $L^{\infty}(d\gamma_{\alpha})$ , and that for all  $f \in L^1(d\nu_{\alpha})$ , we have

$$(2.10) \quad \|\mathcal{F}_{\alpha}(f)\|_{\infty, \gamma_{\alpha}} \leq \|f\|_{1, \nu_{\alpha}}.$$

**Theorem 2.3 (Inversion formula).** Let  $f \in L^1(d\nu_{\alpha})$  such that  $\mathcal{F}_{\alpha}(f) \in L^1(d\gamma_{\alpha})$ , then for almost every  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$ , we have

$$f(r, x) = \int \int_{\Gamma_+} \mathcal{F}_{\alpha}(f)(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_{\alpha}(\mu, \lambda).$$

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**Theorem 2.4 (Plancherel).** *The Fourier transform  $\mathcal{F}_\alpha$  can be extended to an isometric isomorphism from  $L^2(d\nu_\alpha)$  onto  $L^2(d\gamma_\alpha)$ .*

In particular, for all  $f, g \in L^2(d\nu_\alpha)$ , we have the following Parseval's equality

$$(2.11) \quad \int \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu, \lambda) \overline{\mathcal{F}_\alpha(g)(\mu, \lambda)} d\gamma_\alpha(\mu, \lambda) \\ = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \overline{g(r, x)} d\nu_\alpha(r, x).$$

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### 3. Hilbert Basis of the Spaces $L^2(d\nu_\alpha)$ , and $L^2(d\gamma_\alpha)$

In this section, using Laguerre and Hermite polynomials, we construct a Hilbert basis of the spaces  $L^2(d\nu_\alpha)$  and  $L^2(d\gamma_\alpha)$ , and establish some intermediate results that we need in the next section.

It is well known [11, 23] that for every  $\alpha \geq 0$ , the Laguerre polynomials  $L_m^\alpha$  are defined by the following Rodriguez formula

$$L_m^\alpha(r) = \frac{1}{m!} e^r r^{-\alpha} \frac{d^m}{dr^m} (r^{m+\alpha} e^{-r}); \quad m \in \mathbb{N}.$$

Also, the Hermite polynomials are defined by the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}); \quad n \in \mathbb{N}.$$

Moreover, the families

$$\left\{ \sqrt{\frac{m!}{\Gamma(\alpha + m + 1)}} L_m^\alpha \right\}_{m \in \mathbb{N}} \quad \text{and} \quad \left\{ \sqrt{\frac{1}{2^n n! \sqrt{\pi}}} H_n \right\}_{n \in \mathbb{N}}$$

are respectively a Hilbert basis of the Hilbert spaces  $L^2(\mathbb{R}_+, e^{-r} r^\alpha dr)$  and  $L^2(\mathbb{R}, e^{-x^2} dx)$ .

Therefore the families

$$\left\{ \sqrt{\frac{2^{\alpha+1} \Gamma(\alpha + 1) m!}{\Gamma(\alpha + m + 1)}} e^{-\frac{r^2}{2}} L_m^\alpha(r^2) \right\}_{m \in \mathbb{N}} \quad \text{and} \quad \left\{ \sqrt{\frac{1}{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_n \right\}_{n \in \mathbb{N}}$$

are respectively a Hilbert basis of the Hilbert spaces  $L^2\left(\mathbb{R}_+, \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dr\right)$  and  $L^2\left(\mathbb{R}, \frac{dx}{\sqrt{2\pi}}\right)$ ,



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hence the family  $\left\{ e_{m,n}^\alpha \right\}_{(m,n) \in \mathbb{N}^2}$  defined by

$$e_{m,n}^\alpha(r, x) = \left( \frac{2^{\alpha+1} \Gamma(\alpha + 1) m!}{2^{n-\frac{1}{2}} n! \Gamma(m + \alpha + 1)} \right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} L_m^\alpha(r^2) H_n(x),$$

is a Hilbert basis of the space  $L^2(d\nu_\alpha)$ .

Using the relation (2.8), we deduce that the family  $\left\{ \xi_{m,n}^\alpha \right\}_{(m,n) \in \mathbb{N}^2}$ , defined by

$$\begin{aligned} \xi_{m,n}^\alpha(\mu, \lambda) &= (e_{m,n}^\alpha \circ \theta)(\mu, \lambda) \\ &= \left( \frac{2^{\alpha+1} \Gamma(\alpha + 1) m!}{2^{n-\frac{1}{2}} n! \Gamma(m + \alpha + 1)} \right)^{\frac{1}{2}} e^{-\frac{\mu^2+\lambda^2}{2}} L_m^\alpha(\mu^2 + \lambda^2) H_n(\lambda), \end{aligned}$$

is a Hilbert basis of the space  $L^2(d\gamma_\alpha)$ , where  $\theta$  is the function defined by the relation (2.4).

In the following, we agree that the Laguerre and Hermite polynomials with negative index are zero.

**Proposition 3.1.** For all  $(m, n) \in \mathbb{N}^2$ ,  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$  and  $(\mu, \lambda) \in \Gamma_+$ , we have

$$(3.1) \quad x e_{m,n}^\alpha(r, x) = \sqrt{\frac{n+1}{2}} e_{m,n+1}^\alpha(r, x) + \sqrt{\frac{n}{2}} e_{m,n-1}^\alpha(r, x).$$

$$(3.2) \quad \lambda \xi_{m,n}^\alpha(\mu, \lambda) = \sqrt{\frac{n+1}{2}} \xi_{m,n+1}^\alpha(\mu, \lambda) + \sqrt{\frac{n}{2}} \xi_{m,n-1}^\alpha(\mu, \lambda).$$

$$(3.3) \quad r^2 e_{m,n}^{\alpha+1}(r, x) = \sqrt{2(\alpha+1)(\alpha+m+1)} e_{m,n}^\alpha(r, x) - \sqrt{2(\alpha+1)(m+1)} e_{m+1,n}^\alpha(r, x).$$



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$$(3.4) \quad (\mu^2 + \lambda^2)\xi_{m,n}^{\alpha+1}(\mu, \lambda) = \sqrt{2(\alpha+1)(\alpha+m+1)}\xi_{m,n}^{\alpha}(\mu, \lambda) \\ - \sqrt{2(\alpha+1)(m+1)}\xi_{m+1,n}^{\alpha}(\mu, \lambda).$$

*Proof.* We know [11] that the Hermite polynomials satisfy the following recurrence formula

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0; \quad n \in \mathbb{N},$$

Therefore, for all  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$ , we have

$$xe_{m,n}^{\alpha}(r, x) = \left( \frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+1)} \right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} L_m^{\alpha}(r^2)xH_n(x) \\ = \sqrt{\frac{n+1}{2}} \left( \frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n+\frac{1}{2}}(n+1)!\Gamma(m+\alpha+1)} \right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} L_m^{\alpha}(r^2)H_{n+1}(x) \\ + \sqrt{\frac{n}{2}} \left( \frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{3}{2}}(n-1)!\Gamma(m+\alpha+1)} \right)^{\frac{1}{2}} e^{-\frac{|(r,x)|^2}{2}} L_m^{\alpha}(r^2)H_{n-1}(x) \\ = \sqrt{\frac{n+1}{2}}e_{m,n+1}^{\alpha}(r, x) + \sqrt{\frac{n}{2}}e_{m,n-1}^{\alpha}(r, x)$$

and it is obvious that the same relation holds for the elements  $\xi_{m,n}^{\alpha}$ .

On the other hand

$$r^2e_{m,n}^{\alpha+1}(r, x) = \left( \frac{2^{\alpha+2}\Gamma(\alpha+2)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+2)} \right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} r^2L_m^{\alpha+1}(r^2)H_n(x).$$

However, the Laguerre polynomials satisfy the following recurrence formulas

$$(m+1)L_{m+1}^{\alpha}(r) + (r-\alpha-2m-1)L_m^{\alpha}(r) + (m+\alpha)L_{m-1}^{\alpha}(r) = 0; \quad m \in \mathbb{N},$$



and

$$L_m^{\alpha+1}(r) - L_{m-1}^{\alpha+1}(r) = L_m^\alpha(r); \quad m \in \mathbb{N}.$$

Hence, we deduce that

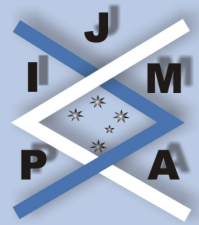
$$\begin{aligned} & r^2 e_{m,n}^{\alpha+1}(r, x) \\ &= \left( \frac{2^{\alpha+2} \Gamma(\alpha+2) m!}{2^{n-\frac{1}{2}} n! \Gamma(m+\alpha+2)} \right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} H_n(x) \\ & \quad \times \left( (\alpha+2m+2) L_m^{\alpha+1}(r^2) - (m+1) L_{m+1}^{\alpha+1}(r^2) - (\alpha+m+1) L_{m-1}^{\alpha+1}(r^2) \right) \\ &= \left( \frac{2^{\alpha+2} \Gamma(\alpha+2) m!}{2^{n-\frac{1}{2}} n! \Gamma(m+\alpha+2)} \right)^{\frac{1}{2}} (\alpha+m+1) e^{-\frac{r^2+x^2}{2}} L_m^\alpha(r^2) H_n(x) \\ & \quad - \left( \frac{2^{\alpha+2} \Gamma(\alpha+2) m!}{2^{n-\frac{1}{2}} n! \Gamma(m+\alpha+2)} \right)^{\frac{1}{2}} (m+1) e^{-\frac{r^2+x^2}{2}} L_{m+1}^\alpha(r^2) H_n(x) \\ &= \sqrt{2(\alpha+1)(\alpha+m+1)} e_{m,n}^\alpha(r, x) - \sqrt{2(\alpha+1)(m+1)} e_{m+1,n}^\alpha(r, x). \end{aligned}$$

□

**Proposition 3.2.** For all  $(m, n) \in \mathbb{N}^2$ , and  $(\mu, \lambda) \in \Gamma_+$ , we have

$$(3.5) \quad \mathcal{F}_\alpha(e_{m,n}^\alpha)(\mu, \lambda) = (-i)^{2m+n} \xi_{m,n}^\alpha(\mu, \lambda).$$

*Proof.* It is clear that for all  $(m, n) \in \mathbb{N}^2$ , the function  $e_{m,n}^\alpha$  belongs to the space



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$L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$ , hence by using Fubini's theorem, we get

$$\begin{aligned} \mathcal{F}_\alpha(e_{m,n}^\alpha)(\mu, \lambda) &= \left( \frac{2^{\alpha+1} \Gamma(\alpha+1) m!}{2^{n-\frac{1}{2}} n! \Gamma(m+\alpha+1)} \right)^{\frac{1}{2}} \\ &\times \left( \int_0^{+\infty} e^{-\frac{r^2}{2}} L_m^\alpha(r^2) j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dr \right) \\ &\times \left( \int_{\mathbb{R}} e^{-\frac{x^2}{2} - i\lambda x} H_n(x) \frac{dx}{\sqrt{2\pi}} \right), \end{aligned}$$

and then the required result follows from the following equalities [11]:

$$\forall m \in \mathbb{N}; \int_0^{+\infty} e^{-\frac{r^2}{2}} L_m^\alpha(r) J_\alpha(\sqrt{ry}) r^{\frac{\alpha}{2}} dr = (-1)^m 2 e^{-\frac{y}{2}} y^{\frac{\alpha}{2}} L_m^\alpha(y),$$

and

$$\forall n \in \mathbb{N}; \int_{\mathbb{R}} e^{ixy} e^{-\frac{x^2}{2}} H_n(x) dx = i^n \sqrt{2\pi} e^{-\frac{y^2}{2}} H_n(y),$$

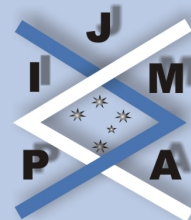
where  $J_\alpha$  denotes the Bessel function of the first kind and index  $\alpha$  defined for all  $x > 0$  by

$$J_\alpha(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(\alpha+n+1)} \left( \frac{x}{2} \right)^{2n+\alpha}.$$

□

**Proposition 3.3.** *Let  $f \in L^2(d\nu_\alpha) \cap L^2(d\nu_{\alpha+1})$  such that  $\mathcal{F}_\alpha(f) \in L^2(d\gamma_{\alpha+1})$ , then for all  $(m, n) \in \mathbb{N}^2$ , we have*

$$(3.6) \quad \langle f/e_{m,n}^{\alpha+1} \rangle_{\nu_{\alpha+1}} = \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}} \langle f/e_{m,n}^\alpha \rangle_{\nu_\alpha} - \sqrt{\frac{m+1}{2(\alpha+1)}} \langle f/e_{m+1,n}^\alpha \rangle_{\nu_\alpha},$$



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and

$$(3.7) \quad \langle \mathcal{F}_\alpha(f)/\xi_{m,n}^{\alpha+1} \rangle_{\gamma_{\alpha+1}} = \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}} (-i)^{2m+n} \langle f/e_{m,n}^\alpha \rangle_{\nu_\alpha} \\ + \sqrt{\frac{m+1}{2(\alpha+1)}} (-i)^{2m+n} \langle f/e_{m+1,n}^\alpha \rangle_{\nu_\alpha}.$$

*Proof.* We have

$$\langle f/e_{m,n}^{\alpha+1} \rangle_{\nu_{\alpha+1}} = \int_0^{+\infty} \int_{\mathbb{R}} f(r,x) e_{m,n}^{\alpha+1}(r,x) d\nu_{\alpha+1}(r,x) \\ = \frac{1}{2(\alpha+1)} \int_0^{+\infty} \int_{\mathbb{R}} f(r,x) r^2 e_{m,n}^{\alpha+1}(r,x) d\nu_\alpha(r,x) \\ = \frac{1}{2(\alpha+1)} \langle f/r^2 e_{m,n}^{\alpha+1} \rangle_{\nu_\alpha},$$

hence by using the relation (3.3), we deduce that

$$\langle f/e_{m,n}^{\alpha+1} \rangle_{\nu_{\alpha+1}} = \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}} \langle f/e_{m,n}^\alpha \rangle_{\nu_\alpha} - \sqrt{\frac{m+1}{2(\alpha+1)}} \langle f/e_{m+1,n}^\alpha \rangle_{\nu_\alpha}.$$

In the same manner, and by virtue of the relation (2.7), we have

$$\langle \mathcal{F}_\alpha(f)/\xi_{m,n}^{\alpha+1} \rangle_{\gamma_{\alpha+1}} = \int \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu, \lambda) \xi_{m,n}^{\alpha+1}(\mu, \lambda) d\gamma_{\alpha+1}(\mu, \lambda) \\ = \frac{1}{2(\alpha+1)} \int \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu, \lambda) (\mu^2 + \lambda^2) \xi_{m,n}^{\alpha+1}(\mu, \lambda) d\gamma_\alpha(\mu, \lambda),$$



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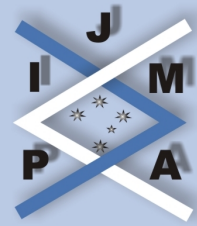
using the relations (3.4) and (3.5), we deduce that

$$\begin{aligned} \langle \mathcal{F}_\alpha(f)/\xi_{m,n}^{\alpha+1} \rangle_{\gamma_{\alpha+1}} &= \frac{1}{\sqrt{2(\alpha+1)}} \int \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu, \lambda) \left( \sqrt{\alpha+m+1} \xi_{m,n}^\alpha(\mu, \lambda) \right. \\ &\quad \left. - \sqrt{m+1} \xi_{m+1,n}^\alpha(\mu, \lambda) \right) d\gamma_\alpha(\mu, \lambda) \\ &= \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}} \langle \mathcal{F}_\alpha(f)/(i)^{2m+n} \mathcal{F}_\alpha(e_{m,n}^\alpha) \rangle_{\gamma_\alpha} \\ &\quad - \sqrt{\frac{m+1}{2(\alpha+1)}} \langle \mathcal{F}_\alpha(f)/(i)^{2m+2+n} \mathcal{F}_\alpha(e_{m+1,n}^\alpha) \rangle_{\gamma_\alpha}, \end{aligned}$$

hence, according to the Parseval's equality (2.11), we obtain

$$\begin{aligned} \langle \mathcal{F}_\alpha(f)/\xi_{m,n}^{\alpha+1} \rangle_{\gamma_{\alpha+1}} &= \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}} (-i)^{2m+n} \langle f/e_{m,n}^\alpha \rangle_{\nu_\alpha} \\ &\quad + \sqrt{\frac{m+1}{2(\alpha+1)}} (-i)^{2m+n} \langle f/e_{m+1,n}^\alpha \rangle_{\nu_\alpha}. \end{aligned}$$

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## 4. Heisenberg-Pauli-Weyl Inequality for the Fourier Transform

$\mathcal{F}_\alpha$

In this section, we will prove the main result of this work, that is the Heisenberg-Pauli-Weyl inequality for the Fourier transform  $\mathcal{F}_\alpha$  connected with the Riemann-Liouville operator  $\mathcal{R}_\alpha$ . Next we give a generalization of this result, for this we need the following important lemma.

**Lemma 4.1.** *Let  $f \in L^2(d\nu_\alpha)$ , such that*

$$\| |(r, x)|f \|_{2, \nu_\alpha} < +\infty \quad \text{and} \quad \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} < +\infty,$$

then

$$(4.1) \quad \| |(r, x)|f \|_{2, \nu_\alpha}^2 + \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2 = \sum_{m, n=0}^{+\infty} (2\alpha + 4m + 2n + 3) |a_{m, n}|^2,$$

where  $a_{m, n} = \langle f / e_{m, n}^\alpha \rangle_{\nu_\alpha}$ ;  $(m, n) \in \mathbb{N}^2$ .

*Proof.* Let  $f \in L^2(d\nu_\alpha)$ , such that

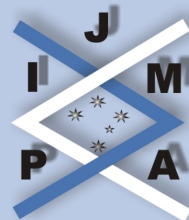
$$\forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = \sum_{m, n=0}^{+\infty} a_{m, n} e_{m, n}^\alpha(r, x),$$

and assume that

$$\| |(r, x)|f \|_{2, \nu_\alpha} < +\infty \quad \text{and} \quad \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} < +\infty,$$

then the functions  $(r, x) \mapsto rf(r, x)$  and  $(r, x) \mapsto xf(r, x)$  belong to the space  $L^2(d\nu_\alpha)$ , in particular  $f \in L^2(d\nu_\alpha) \cap L^2(d\nu_{\alpha+1})$ . In the same manner, the functions

$$(\mu, \lambda) \mapsto (\mu^2 + \lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)(\mu, \lambda), \quad \text{and} \quad (\mu, \lambda) \mapsto \lambda \mathcal{F}_\alpha(f)(\mu, \lambda)$$



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belong to the space  $L^2(d\gamma_\alpha)$ . In particular, by the relation (2.7), we deduce that  $\mathcal{F}_\alpha(f) \in L^2(d\gamma_\alpha) \cap L^2(d\gamma_{\alpha+1})$ , and we have

$$\begin{aligned} \|rf\|_{2,\nu_\alpha}^2 &= \int_0^{+\infty} \int_{\mathbb{R}} r^2 |f(r,x)|^2 d\nu_\alpha(r,x) \\ &= 2(\alpha+1) \|f\|_{2,\nu_{\alpha+1}}^2 \\ &= 2(\alpha+1) \sum_{m,n=0}^{+\infty} |\langle f/e_{m,n}^{\alpha+1} \rangle_{\nu_{\alpha+1}}|^2, \end{aligned}$$

hence, according to the relation (3.6), we obtain

$$(4.2) \quad \|rf\|_{2,\nu_\alpha}^2 = \sum_{m,n=0}^{+\infty} |\sqrt{\alpha+m+1}a_{m,n} - \sqrt{m+1}a_{m+1,n}|^2.$$

Similarly, we have

$$\begin{aligned} \|xf\|_{2,\nu_\alpha}^2 &= \int_0^{+\infty} \int_{\mathbb{R}} x^2 |f(r,x)|^2 d\nu_\alpha(r,x) \\ &= \sum_{m,n=0}^{+\infty} |\langle xf/e_{m,n}^\alpha \rangle_{\nu_\alpha}|^2 = \sum_{m,n=0}^{+\infty} |\langle f/x e_{m,n}^\alpha \rangle_{\nu_\alpha}|^2, \end{aligned}$$

and by the relation (3.1), we get

$$(4.3) \quad \|xf\|_{2,\nu_\alpha}^2 = \sum_{m,n=0}^{+\infty} \left| \sqrt{\frac{n+1}{2}} a_{m,n+1} + \sqrt{\frac{n}{2}} a_{m,n-1} \right|^2.$$

By the same arguments, and using the relations (3.2), (3.7) and the Parseval's equality (2.11), we obtain



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$$(4.4) \quad \|(\mu^2 + \lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^2 = \sum_{m,n=0}^{+\infty} |\sqrt{\alpha + m + 1}a_{m,n} + \sqrt{m + 1}a_{m+1,n}|^2,$$

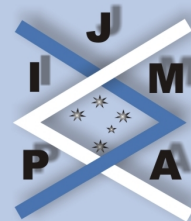
and

$$(4.5) \quad \|\lambda \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^2 = \sum_{m,n=0}^{+\infty} \left| \sqrt{\frac{n+1}{2}}a_{m,n+1} - \sqrt{\frac{n}{2}}a_{m,n-1} \right|^2.$$

Combining now the relations (4.2), (4.3), (4.4) and (4.5), we deduce that

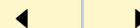
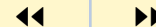
$$\begin{aligned} & \| |(r, x)|f \|_{2,\nu_\alpha}^2 + \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^2 \\ &= \|rf\|_{2,\nu_\alpha}^2 + \|(\mu^2 + \lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^2 + \|xf\|_{2,\nu_\alpha}^2 + \|\lambda \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^2 \\ &= 2 \sum_{m,n=0}^{+\infty} \left( (\alpha + m + 1)|a_{m,n}|^2 + (m + 1)|a_{m+1,n}|^2 \right) \\ &\quad + 2 \sum_{m,n=0}^{+\infty} \left( \frac{n+1}{2}|a_{m,n+1}|^2 + \frac{n}{2}|a_{m,n-1}|^2 \right) \\ &= 2 \sum_{m,n=0}^{+\infty} (\alpha + m + 1)|a_{m,n}|^2 + 2 \sum_{m,n=0}^{+\infty} m|a_{m,n}|^2 \\ &\quad + 2 \sum_{m,n=0}^{+\infty} \frac{n}{2}|a_{m,n}|^2 + 2 \sum_{m,n=0}^{+\infty} \frac{n+1}{2}|a_{m,n}|^2 \\ &= \sum_{m,n=0}^{+\infty} (2\alpha + 4m + 2n + 3)|a_{m,n}|^2. \end{aligned}$$

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*Remark 1.* From the relation (4.1), we deduce that for all  $f \in L^2(d\nu_\alpha)$ , we have

$$(4.6) \quad \| |(r, x)|f \|_{2, \nu_\alpha}^2 + \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2 \geq (2\alpha + 3) \| f \|_{2, \nu_\alpha}^2,$$

with equality if and only if

$$\forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = C e^{-\frac{r^2 + x^2}{2}}; \quad C \in \mathbb{C}.$$

**Lemma 4.2.** Let  $f \in L^2(d\nu_\alpha)$  such that,

$$\| |(r, x)|f \|_{2, \nu_\alpha} < +\infty \quad \text{and} \quad \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} < +\infty,$$

then

1) For all  $t > 0$ ,

$$\frac{1}{t^2} \| |(r, x)|f \|_{2, \nu_\alpha}^2 + t^2 \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2 \geq (2\alpha + 3) \| f \|_{2, \nu_\alpha}^2.$$

2) The following assertions are equivalent

i)  $\| |(r, x)|f \|_{2, \nu_\alpha} \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} = \frac{2\alpha + 3}{2} \| f \|_{2, \nu_\alpha}^2.$

ii) There exists  $t_0 > 0$ , such that

$$\| |(r, x)|f_{t_0} \|_{2, \nu_\alpha}^2 + \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f_{t_0}) \|_{2, \gamma_\alpha}^2 = (2\alpha + 3) \| f_{t_0} \|_{2, \nu_\alpha}^2,$$

where  $f_{t_0}(r, x) = f(t_0 r, t_0 x)$ .

*Proof.* 1) Let  $f \in L^2(d\nu_\alpha)$  satisfy the hypothesis. For all  $t > 0$  we put  $f_t(r, x) = f(tr, tx)$ , and then by a simple change of variables, we get

$$(4.7) \quad \| f_t \|_{2, \nu_\alpha}^2 = \frac{1}{t^{2\alpha+3}} \| f \|_{2, \nu_\alpha}^2,$$





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and

$$(4.8) \quad \|(r, x)|f_t\|_{2, \nu_\alpha}^2 = \frac{1}{t^{2\alpha+5}} \|(r, x)|f\|_{2, \nu_\alpha}^2.$$

For all  $(\mu, \lambda) \in \Gamma$ ,

$$(4.9) \quad \mathcal{F}_\alpha(f_t)(\mu, \lambda) = \frac{1}{t^{2\alpha+3}} \mathcal{F}_\alpha(f) \left( \frac{\mu}{t}, \frac{\lambda}{t} \right),$$

and by using the relation (2.6), we deduce that

$$(4.10) \quad \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f_t)\|_{2, \gamma_\alpha}^2 = \frac{1}{t^{2\alpha+1}} \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha}^2.$$

Then, the desired result follows by replacing  $f$  by  $f_t$  in the relation (4.6).

2) Let  $f \in L^2(d\nu_\alpha)$ ;  $f \neq 0$ .

- Assume that

$$\|(r, x)|f\|_{2, \nu_\alpha} \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha} = \frac{2\alpha + 3}{2} \|f\|_{2, \nu_\alpha}^2.$$

By Theorem 2.3, we have  $\|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha} \neq 0$ , then for

$$t_0 = \sqrt{\frac{\|(r, x)|f\|_{2, \nu_\alpha}}{\|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha}}},$$

we have

$$\frac{1}{t_0^2} \|(r, x)|f\|_{2, \nu_\alpha}^2 + t_0^2 \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha}^2 = (2\alpha + 3) \|f\|_{2, \nu_\alpha}^2,$$

and this is equivalent to

$$\|(r, x)|f_{t_0}\|_{2, \nu_\alpha}^2 + \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f_{t_0})\|_{2, \gamma_\alpha}^2 = (2\alpha + 3) \|f_{t_0}\|_{2, \nu_\alpha}^2.$$



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- Conversely, suppose that there exists  $t_1 > 0$ , such that

$$\| |(r, x)|f_{t_1} \|_{2, \nu_\alpha}^2 + \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f_{t_1}) \|_{2, \gamma_\alpha}^2 = (2\alpha + 3) \| f_{t_1} \|_{2, \nu_\alpha}^2.$$

This is equivalent to

$$(4.11) \quad \frac{1}{t_1^2} \| |(r, x)|f \|_{2, \nu_\alpha}^2 + t_1^2 \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2 = (2\alpha + 3) \| f \|_{2, \nu_\alpha}^2.$$

However, let  $h$  be the function defined on  $]0, +\infty[$ , by

$$h(t) = \frac{1}{t^2} \| |(r, x)|f \|_{2, \nu_\alpha}^2 + t^2 \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2,$$

then, the minimum of the function  $h$  is attained at the point

$$t_0 = \sqrt{\frac{\| |(r, x)|f \|_{2, \nu_\alpha}^2}{\| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2}}$$

and

$$h(t_0) = 2 \| |(r, x)|f \|_{2, \nu_\alpha} \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}.$$

Thus by 1) of this lemma, we have

$$h(t_1) \geq h(t_0) = 2 \| |(r, x)|f \|_{2, \nu_\alpha} \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} \geq (2\alpha + 3) \| f \|_{2, \nu_\alpha}^2.$$

According to the relation (4.11), we deduce that

$$h(t_1) = h(t_0) = 2 \| |(r, x)|f \|_{2, \nu_\alpha} \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} = (2\alpha + 3) \| f \|_{2, \nu_\alpha}^2.$$

□

**Theorem 4.3 (Heisenberg-Pauli-Weyl inequality).** For all  $f \in L^2(d\nu_\alpha)$ , we have

$$(4.12) \quad \| |(r, x)|f \|_{2, \nu_\alpha} \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} \geq \frac{(2\alpha + 3)}{2} \| f \|_{2, \nu_\alpha}^2$$



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with equality if and only if

$$\forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = C e^{-\frac{r^2+x^2}{2t_0^2}}; \quad t_0 > 0, C \in \mathbb{C}.$$

*Proof.* It is obvious that if  $f = 0$ , or if  $\| |(r, x)|f \|_{2, \nu_\alpha} = +\infty$ , or  $\| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} = +\infty$ , then the inequality (4.12) holds.

Let us suppose that  $\| |(r, x)|f \|_{2, \nu_\alpha} + \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} < +\infty$ , and  $f \neq 0$ .

By 1) of Lemma 4.2, we have for all  $t > 0$

$$\frac{1}{t^2} \| |(r, x)|f \|_{2, \nu_\alpha}^2 + t^2 \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^2 \geq (2\alpha + 3) \| f \|_{2, \nu_\alpha}^2,$$

and the result follows if we pick

$$t = \sqrt{\frac{\| |(r, x)|f \|_{2, \nu_\alpha}}{\| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}}}.$$

By 2) of Lemma 4.2, we have

$$\| |(r, x)|f \|_{2, \nu_\alpha} \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} = \frac{2\alpha + 3}{2} \| f \|_{2, \nu_\alpha}^2,$$

if and only if there exists  $t_0$ , such that

$$\| |(r, x)|f_{t_0} \|_{2, \nu_\alpha}^2 + \| (\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f_{t_0}) \|_{2, \gamma_\alpha}^2 = (2\alpha + 3) \| f_{t_0} \|_{2, \nu_\alpha}^2,$$

and according to Remark 1, this is equivalent to

$$f_{t_0}(r, x) = C e^{-\frac{r^2+x^2}{2}}; \quad C \in \mathbb{C},$$

which means that

$$f(r, x) = C e^{-\frac{r^2+x^2}{2t_0^2}}; \quad C \in \mathbb{C}.$$

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The following gives a generalization of the Heisenberg-Pauli-Weyl inequality.

**Theorem 4.4.** Let  $a, b \geq 1$  and  $\eta \in \mathbb{R}$  such that  $\eta a = (1 - \eta)b$ , then for all  $f \in L^2(d\nu_\alpha)$  we have

$$\| |(r, x)|^a f \|_{2, \nu_\alpha}^\eta \| (\mu^2 + 2\lambda^2)^{\frac{b}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha}^{1-\eta} \geq \left( \frac{2\alpha + 3}{2} \right)^{a\eta} \| f \|_{2, \nu_\alpha}$$

with equality if and only if  $a = b = 1$  and

$$\forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = C e^{-\frac{r^2 + x^2}{2t_0^2}}; \quad t_0 > 0, C \in \mathbb{C}.$$

*Proof.* Let  $f \in L^2(d\nu_\alpha)$ ,  $f \neq 0$ , such that

$$\| |(r, x)|^a f \|_{2, \nu_\alpha} + \| (\mu^2 + 2\lambda^2)^{\frac{b}{2}} \mathcal{F}_\alpha(f) \|_{2, \gamma_\alpha} < +\infty.$$

Then for all  $a > 1$ , we have

$$\| |(r, x)|^a f \|_{2, \nu_\alpha}^{\frac{1}{a'}} \| f \|_{2, \nu_\alpha}^{\frac{1}{a'}} = \| |(r, x)|^2 |f|^{\frac{2}{a}} \|_{a, \nu_\alpha}^{\frac{1}{2}} \| |f|^{\frac{2}{a'}} \|_{a', \nu_\alpha}^{\frac{1}{2}},$$

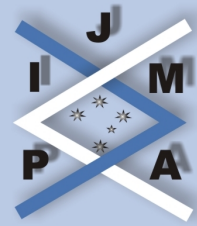
where  $a'$  is defined as usual by  $a' = \frac{a}{a-1}$ . By Hölder's inequality we get

$$\| |(r, x)|^a f \|_{2, \nu_\alpha}^{\frac{1}{a}} \| f \|_{2, \nu_\alpha}^{\frac{1}{a'}} > \| |(r, x)| f \|_{2, \nu_\alpha}.$$

The strict inequality here is justified by the fact that if  $f \neq 0$ , then the functions  $| (r, x) |^{2a} |f|^2$  and  $|f|^2$  cannot be proportional. Thus for all  $a \geq 1$ , we have

$$(4.13) \quad \| |(r, x)|^a f \|_{2, \nu_\alpha}^{\frac{1}{a}} \geq \frac{\| |(r, x)| f \|_{2, \nu_\alpha}}{\| f \|_{2, \nu_\alpha}^{\frac{1}{a'}}}.$$

with equality if and only if  $a = 1$ .



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In the same manner and using Plancherel's Theorem 2.4, we have for all  $b \geq 1$

$$(4.14) \quad \begin{aligned} \|(\mu^2 + 2\lambda^2)^{\frac{b}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^{\frac{1}{b}} &\geq \frac{\|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}}{\|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^{\frac{1}{b}}} \\ &\geq \frac{\|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}}{\|f\|_{2,\nu_\alpha}^{\frac{1}{b}}}. \end{aligned}$$

with equality if and only if  $b = 1$ .

Let  $\eta = \frac{b}{a+b}$ , then by the relations (4.13), (4.14) and for all  $a, b \geq 1$ , we have

$$\| |(r, x)|^a f \|_{2,\nu_\alpha}^\eta \|(\mu^2 + 2\lambda^2)^{\frac{b}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^{1-\eta} \geq \left( \frac{\| |(r, x)| f \|_{2,\nu_\alpha} \|(\mu^2 + 2\lambda^2)^{\frac{1}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}}{\|f\|_{2,\nu_\alpha}^{\frac{1}{a} + \frac{1}{b}}} \right)^{\eta a},$$

with equality if and only if  $a = b = 1$ .

Applying Theorem 4.3, we obtain

$$\| |(r, x)|^a f \|_{2,\nu_\alpha}^\eta \|(\mu^2 + 2\lambda^2)^{\frac{b}{2}} \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}^{1-\eta} \geq \left( \frac{2\alpha + 3}{2} \right)^{\eta a} \|f\|_{2,\nu_\alpha},$$

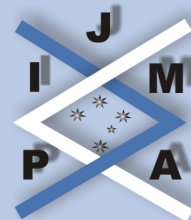
with equality if and only if  $a = b = 1$  and

$$\forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = C e^{-\frac{r^2 + x^2}{2t_0^2}}; \quad t_0 > 0, C \in \mathbb{C}.$$

□

*Remark 2.* In the particular case when  $a = b = 2$ , the previous result gives us the Heisenberg-Pauli-Weyl inequality for the fourth moment of Heisenberg

$$\| |(r, x)|^2 f \|_{2,\nu_\alpha} \|(\mu^2 + 2\lambda^2) \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha} > \left( \frac{2\alpha + 3}{2} \right)^2 \|f\|_{2,\nu_\alpha}^2.$$



## 5. The Local Uncertainty Principle

**Theorem 5.1.** Let  $\xi$  be a real number such that  $0 < \xi < \frac{2\alpha+3}{2}$ , then for all  $f \in L^2(d\nu_\alpha)$ ,  $f \neq 0$ , and for all measurable subsets  $E \subset \Gamma_+$ ;  $0 < \gamma_\alpha(E) < +\infty$ , we have

$$(5.1) \quad \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) < K_{\alpha, \xi} (\gamma_\alpha(E))^{\frac{2\xi}{2\alpha+3}} \| |f|^\xi \|_{2, \nu_\alpha}^2,$$

where

$$K_{\alpha, \xi} = \left( \frac{2\alpha + 3 - 2\xi}{\xi^{2\alpha + \frac{5}{2}} \Gamma(\alpha + \frac{3}{2})} \right)^{\frac{2\xi}{2\alpha+3}} \left( \frac{2\alpha + 3}{2\alpha + 3 - 2\xi} \right)^2.$$

*Proof.* For all  $s > 0$ , we put

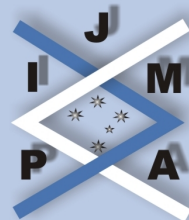
$$B_s = \{(r, x) \in \mathbb{R}_+ \times \mathbb{R}; r^2 + x^2 < s^2\}.$$

Let  $f \in L^2(d\nu_\alpha)$ . By Minkowski's inequality, we have

$$(5.2) \quad \left( \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1}{2}} \\ = \|\mathcal{F}_\alpha(f)\mathbf{1}_E\|_{2, \gamma_\alpha} \\ \leq \|\mathcal{F}_\alpha(f\mathbf{1}_{B_s})\mathbf{1}_E\|_{2, \gamma_\alpha} + \|\mathcal{F}_\alpha(f\mathbf{1}_{B_s^c})\mathbf{1}_E\|_{2, \gamma_\alpha} \\ \leq (\gamma_\alpha(E))^{\frac{1}{2}} \|\mathcal{F}_\alpha(f\mathbf{1}_{B_s})\|_{\infty, \gamma_\alpha} + \|\mathcal{F}_\alpha(f\mathbf{1}_{B_s^c})\|_{2, \gamma_\alpha}.$$

Applying the relation (2.10), we deduce that for every  $s > 0$ , we have

$$(5.3) \quad \left( \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1}{2}} \\ \leq (\gamma_\alpha(E))^{\frac{1}{2}} \|f\mathbf{1}_{B_s}\|_{1, \nu_\alpha} + \|\mathcal{F}_\alpha(f\mathbf{1}_{B_s^c})\|_{2, \gamma_\alpha}.$$



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On the other hand, by Hölder's inequality we have

$$(5.4) \quad \|f \mathbf{1}_{B_s}\|_{1, \nu_\alpha} \leq \| |(r, x)|^\xi f \|_{2, \nu_\alpha} \| |(r, x)|^{-\xi} \mathbf{1}_{B_s} \|_{2, \nu_\alpha}$$

$$(5.5) \quad = \| |(r, x)|^\xi f(r, x) \|_{2, \nu_\alpha} \frac{s^{\frac{2\alpha+3-2\xi}{2}}}{\left(2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) (2\alpha + 3 - 2\xi)\right)^{\frac{1}{2}}}.$$

By Plancherel's theorem 2.4, we have also

$$(5.6) \quad \begin{aligned} \|\mathcal{F}_\alpha(f \mathbf{1}_{B_s^c})\|_{2, \gamma_\alpha} &= \|f \mathbf{1}_{B_s^c}\|_{2, \nu_\alpha} \\ &\leq \| |(r, x)|^\xi f \|_{2, \nu_\alpha} \| |(r, x)|^{-\xi} \mathbf{1}_{B_s^c} \|_{\infty, \nu_\alpha} \\ &= s^{-\xi} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}. \end{aligned}$$

Combining the relations (5.3), (5.5) and (5.6), we deduce that for all  $s > 0$  we have

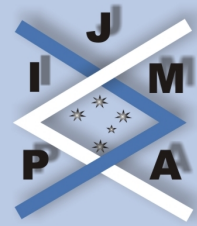
$$(5.7) \quad \left( \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1}{2}} \leq g_{\alpha, \xi}(s) \| |(r, x)|^\xi f(r, x) \|_{2, \nu_\alpha},$$

where  $g_{\alpha, \xi}$  is the function defined on  $]0, +\infty[$  by

$$g_{\alpha, \xi}(s) = s^{-\xi} + \left( \frac{\gamma_\alpha(E)}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) (2\alpha + 3 - 2\xi)} \right)^{\frac{1}{2}} s^{\frac{2\alpha+3-2\xi}{2}}.$$

Thus, the inequality (5.7) holds for

$$s_0 = \left( \frac{\xi^2 2^{\alpha+\frac{5}{2}} \Gamma\left(\alpha + \frac{3}{2}\right)}{\gamma_\alpha(E) (2\alpha + 3 - 2\xi)} \right)^{\frac{1}{2\alpha+3}},$$



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however

$$g_{\alpha,\xi}(s_0) = (\gamma_\alpha(E))^{\frac{\xi}{2\alpha+3}} K_{\alpha,\xi}^{\frac{1}{2}},$$

where

$$K_{\alpha,\xi} = \left( \frac{2\alpha + 3 - 2\xi}{\xi^2 2^{\alpha+\frac{5}{2}} \Gamma(\alpha + \frac{3}{2})} \right)^{\frac{2\xi}{2\alpha+3}} \left( \frac{2\alpha + 3}{2\alpha + 3 - 2\xi} \right)^2.$$

Let us prove that the equality in (5.1) cannot hold. Indeed, suppose that

$$\int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) = K_{\alpha,\xi}(\gamma_\alpha(E))^{\frac{2\xi}{2\alpha+3}} \| |(r, x)|^\xi f \|_{2,\nu_\alpha}^2.$$

Then

$$\int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) = g_{\alpha,\xi}(s_0)^2 \| |(r, x)|^\xi f \|_{2,\nu_\alpha}^2,$$

and therefore by the relations (5.2), (5.3) and (5.4), we get

$$(5.8) \quad \| \mathcal{F}_\alpha(f \mathbf{1}_{B_{s_0}}) \mathbf{1}_E \|_{2,\gamma_\alpha} = (\gamma_\alpha(E))^{\frac{1}{2}} \| \mathcal{F}_\alpha(f \mathbf{1}_{B_{s_0}}) \|_{\infty,\gamma_\alpha},$$

$$(5.9) \quad \| f \mathbf{1}_{B_{s_0}} \|_{1,\nu_\alpha} = \| \mathcal{F}_\alpha(f \mathbf{1}_{B_{s_0}}) \|_{\infty,\gamma_\alpha},$$

and

$$(5.10) \quad \| f \mathbf{1}_{B_{s_0}} \|_{1,\nu_\alpha} = \| |(r, x)|^\xi f \|_{2,\nu_\alpha} \| |(r, x)|^{-\xi} \mathbf{1}_{B_{s_0}} \|_{2,\nu_\alpha}.$$

However, if  $f$  satisfies the equality (5.10), then there exists  $C > 0$ , such that

$$|(r, x)|^{2\xi} f(r, x)^2 = C |(r, x)|^{-2\xi} \mathbf{1}_{B_{s_0}},$$





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hence

$$(5.11) \quad \forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = C e^{i\Phi(r,x)} |(r, x)|^{-2\xi} \mathbf{1}_{B_{s_0}},$$

where  $\Phi$  is a real measurable function on  $\mathbb{R}_+ \times \mathbb{R}$ .

But if  $f$  satisfies the relation (5.9), then there exists  $(\mu_0, \lambda_0) \in \Gamma_+$ , such that

$$\|f\|_{1,\nu_\alpha} = \|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = |\mathcal{F}_\alpha(f)(\mu_0, \lambda_0)|.$$

So, there exists  $\theta_0 \in \mathbb{R}$  satisfying

$$\mathcal{F}_\alpha(f)(\mu_0, \lambda_0) = e^{i\theta_0} \|f\|_{1,\nu_\alpha},$$

and therefore

$$C e^{i\theta_0} \int_0^{+\infty} \int_{\mathbb{R}} |(r, x)|^{-2\xi} \mathbf{1}_{B_{s_0}}(r, x) \left( e^{i\Phi(r,x) - i\lambda_0 x - i\theta_0} j_\alpha \left( r \sqrt{\mu_0^2 + \lambda_0^2} \right) - 1 \right) d\nu_\alpha(r, x) = 0.$$

This implies that for almost every  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$e^{i\Phi(r,x) - i\lambda_0 x - i\theta_0} j_\alpha \left( r \sqrt{\mu_0^2 + \lambda_0^2} \right) = 1.$$

Hence, we deduce that for all  $r \in \mathbb{R}_+$ ,

$$\left| j_\alpha \left( r \sqrt{\mu_0^2 + \lambda_0^2} \right) \right| = 1.$$

Using the relation (2.2), it follows that  $\mu_0^2 + \lambda_0^2 = 0$ , or  $\mu_0 = i|\lambda_0|$ , and then

$$e^{i\Phi(r,x)} = e^{i\theta_0 + i\lambda_0 x}.$$

Replacing in (5.11), we get

$$f(r, x) = C^{i\lambda_0 x} |(r, x)|^{-2\xi} \mathbf{1}_{B_{s_0}}(r, x).$$



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Now, the relation (5.8) means that for almost every  $(\mu, \lambda) \in E$ , we have

$$|\mathcal{F}_\alpha(f)(\mu, \lambda)| = \|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha} = |\mathcal{F}_\alpha(f)(\mu_0, \lambda_0)|,$$

which implies that for almost every  $(\mu, \lambda) \in E$ , we have

$$\mathcal{F}_\alpha(f)(\mu, \lambda) = e^{i\psi(\mu, \lambda)} \mathcal{F}_\alpha(f)(\mu_0, \lambda_0),$$

where  $\psi$  is a real measurable function on  $E$ , and therefore

$$C^{i\psi(\mu, \lambda)} \int_0^{+\infty} \int_{\mathbb{R}} |(r, x)|^{-2\xi} \mathbf{1}_{B_{s_0}}(r, x) \left( e^{-i\lambda x + i\lambda_0 x - i\psi(\mu, \lambda)} j_\alpha r \sqrt{\mu^2 + \lambda^2} - 1 \right) d\nu_\alpha(r, x) = 0.$$

Consequently for all  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$e^{-i\lambda x + i\lambda_0 x - i\psi(\mu, \lambda)} j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) = 1,$$

which implies that  $\lambda = \lambda_0$  and  $\mu = \mu_0$ .

However, since  $\gamma_\alpha(E) > 0$ , this contradicts the fact that for almost every  $(\mu, \lambda) \in E$ ,

$$|\mathcal{F}_\alpha(f)(\mu, \lambda)| = |\mathcal{F}_\alpha(f)(\mu_0, \lambda_0)|,$$

and shows that the inequality in (5.1) is strictly satisfied.  $\square$

**Lemma 5.2.** *Let  $\xi$  be a real number such that  $\xi > \frac{2\alpha+3}{2}$ , then for all measurable function  $f$  on  $\mathbb{R}_+ \times \mathbb{R}$  we have*

$$(5.12) \quad \|f\|_{1, \nu_\alpha}^2 \leq M_{\alpha, \xi} \|f\|_{2, \nu_\alpha}^{\frac{2-(2\alpha+3)}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{(2\alpha+3)}{\xi}},$$

where

$$M_{\alpha, \xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) (2\xi - 2\alpha - 3)^{\frac{2\xi-2\alpha+3}{2\xi}} (2\alpha+3)^{\frac{2\alpha+3}{2\xi}} \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$



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with equality in (5.12) if and only if there exist  $a, b > 0$  such that

$$|f(r, x)| = (a + b|(r, x)|^{2\xi})^{-1}.$$

*Proof.* We suppose naturally that  $f \neq 0$ . It is obvious that the inequality (5.12) holds if  $\|f\|_{2, \nu_\alpha} = +\infty$  or  $\| |(r, x)|^\xi f \|_{2, \nu_\alpha} = +\infty$ .

Assume that  $\|f\|_{2, \nu_\alpha} + \| |(r, x)|^\xi f \|_{2, \nu_\alpha} < +\infty$ . From the hypothesis  $2\xi > 2\alpha + 3$ , we deduce that for all  $a, b > 0$ , the function

$$(r, x) \longmapsto (a + b|(r, x)|^{2\xi})^{-1}$$

belongs to  $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$  and by Hölder's inequality, we have

$$(5.13) \quad \|f\|_{1, \nu_\alpha}^2 \leq \left\| (1 + |(r, x)|^{2\xi})^{\frac{1}{2}} f \right\|_{2, \nu_\alpha}^2 \left\| (1 + |(r, x)|^{2\xi})^{-\frac{1}{2}} \right\|_{2, \nu_\alpha}^2 \\ \leq (\|f\|_{2, \nu_\alpha}^2 + \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^2) \left\| (1 + |(r, x)|^{2\xi})^{-\frac{1}{2}} \right\|_{2, \nu_\alpha}^2.$$

However, by standard calculus, we have

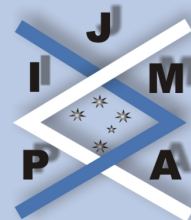
$$\left\| (1 + |(r, x)|^{2\xi})^{-\frac{1}{2}} \right\|_{2, \nu_\alpha}^2 = \frac{\pi}{\xi 2^{\alpha + \frac{3}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \sin\left(\pi\left(\frac{2\alpha + 3}{2\xi}\right)\right)}.$$

Thus

$$(5.14) \quad \|f\|_{1, \nu_\alpha}^2 \leq \frac{\pi}{\xi 2^{\alpha + \frac{3}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \sin\left(\pi\left(\frac{2\alpha + 3}{2\xi}\right)\right)} (\|f\|_{2, \nu_\alpha}^2 + \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^2),$$

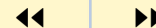
with equality in (5.14) if and only if we have equality in (5.13), that is there exists  $C > 0$  satisfying

$$(1 + |(r, x)|^{2\xi})^{\frac{1}{2}} |f(r, x)| = C(1 + |(r, x)|^{2\xi})^{-\frac{1}{2}},$$



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or

$$(5.15) \quad |f(r, x)| = C(1 + |(r, x)|^{2\xi})^{-1}.$$

For  $t > 0$ , we put as above,  $f_t(r, x) = f(tr, tx)$ , then we have

$$(5.16) \quad \|f_t\|_{1, \nu_\alpha}^2 = \frac{1}{t^{4\alpha+6}} \|f\|_{1, \nu_\alpha}^2,$$

and

$$(5.17) \quad \|||(r, x)|^\xi f_t\|_{2, \nu_\alpha}^2 = \frac{1}{t^{2\xi+2\alpha+3}} \|||(r, x)|^\xi f\|_{2, \nu_\alpha}^2.$$

Replacing  $f$  by  $f_t$  in the relation (5.14), we deduce that for all  $t > 0$ , we have

$$\|f\|_{1, \nu_\alpha}^2 \leq \frac{\pi}{\xi 2^{\alpha+\frac{3}{2}} \Gamma(\alpha + \frac{3}{2}) \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)} \left( t^{2\alpha+3} \|f\|_{2, \nu_\alpha}^2 + t^{2\alpha+3-2\xi} \|||(r, x)|^\xi f\|_{2, \nu_\alpha}^2 \right).$$

In particular, for

$$t = t_0 = \left( \frac{(2\xi - 2\alpha - 3) \|||(r, x)|^\xi f\|_{2, \nu_\alpha}^2}{(2\alpha + 3) \|f\|_{2, \nu_\alpha}^2} \right)^{\frac{1}{2\xi}},$$

we get

$$\|f\|_{1, \nu_\alpha}^2 \leq M_{\alpha, \xi} \|f\|_{2, \nu_\alpha}^{2-\frac{(2\alpha+3)}{\xi}} \|||(r, x)|^\xi f\|_{2, \nu_\alpha}^{\frac{(2\alpha+3)}{\xi}},$$

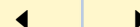
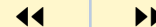
where

$$M_{\alpha, \xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma(\alpha + \frac{3}{2}) (2\xi - 2\alpha - 3)^{\frac{2\xi-2\alpha+3}{2\xi}} (2\alpha + 3)^{\frac{2\alpha+3}{2\xi}} \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$



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Now suppose that we have equality in the last inequality. Then we have equality in (5.14) for  $f_{t_0}$  and by means of (5.15), we obtain

$$|f_{t_0}(r, x)| = C(1 + |(r, x)|^{2\xi})^{-1},$$

and then

$$|f(r, x)| = (a + b|(r, x)|^{2\xi})^{-1}.$$

□

**Theorem 5.3.** Let  $\xi$  be a real number such that  $\xi > \frac{2\alpha+3}{2}$ . Then for all  $f \in L^2(d\nu_\alpha)$ ,  $f \neq 0$ , and for all measurable subsets  $E \subset \Gamma_+$ ;  $0 < \gamma_\alpha(E) < +\infty$ , we have

$$(5.18) \quad \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) < M_{\alpha, \xi} \gamma_\alpha(E) \|f\|_{2, \nu_\alpha}^{2 - \frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{2\alpha+3}{\xi}},$$

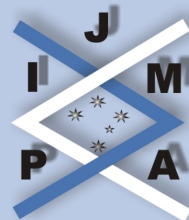
where

$$M_{\alpha, \xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) (2\xi - 2\alpha - 3)^{\frac{2\xi - 2\alpha + 3}{2\xi}} (2\alpha + 3)^{\frac{2\alpha + 3}{2\xi}} \sin\left(\pi \left(\frac{2\alpha + 3}{2\xi}\right)\right)}.$$

Moreover,  $M_{\alpha, \xi}$  is the best (the smallest) constant satisfying (5.18).

*Proof.* • Suppose that the right-hand side of (5.18) is finite. Then, according to Lemma 5.2, the function  $f$  belongs to  $L^1(d\nu_\alpha)$  and we have

$$\begin{aligned} \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) &\leq \gamma_\alpha(E) \|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha}^2 \\ &\leq \gamma_\alpha(E) \|f\|_{1, \nu_\alpha}^2 \\ &\leq \gamma_\alpha(E) M_{\alpha, \xi} \|f\|_{2, \nu_\alpha}^{2 - \frac{(2\alpha+3)}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{(2\alpha+3)}{\xi}}, \end{aligned}$$



where

$$M_{\alpha,\xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) (2\xi - 2\alpha - 3)^{\frac{2\xi-2\alpha+3}{2\xi}} (2\alpha + 3)^{\frac{2\alpha+3}{2\xi}} \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

• Let us prove that the equality in (5.18) cannot hold.

Indeed, suppose that

$$\iint_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) = M_{\alpha,\xi} \gamma_\alpha(E) \|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Then

$$(5.19) \quad \iint_{\Gamma_+} |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) = \gamma_\alpha(E) \|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha}^2,$$

$$(5.20) \quad \|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = \|f\|_{1,\nu_\alpha},$$

and

$$(5.21) \quad \|f\|_{1,\nu_\alpha}^2 = M_{\alpha,\xi} \|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Applying Lemma 5.2 and the relation (5.21), we deduce that

$$(5.22) \quad \forall (r, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r, x) = \varphi(r, x) (a + b |(r, x)|^{2\xi})^{-1},$$

with  $|\varphi(r, x)| = 1$ ;  $a, b > 0$ .

On the other hand, there exists  $(\mu_0, \lambda_0) \in \Gamma_+$  such that

$$(5.23) \quad \|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = |\mathcal{F}_\alpha(f)(\mu_0, \lambda_0)| = e^{i\theta_0} \mathcal{F}_\alpha(f)(\mu_0, \lambda_0); \quad \theta_0 \in \mathbb{R}.$$



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Combining now the relations (5.20), (5.22) and (5.23), we get

$$\int_0^{+\infty} \int_{\mathbb{R}} \left[ 1 - e^{i\theta_0} \varphi(r, x) j_\alpha \left( r \sqrt{\mu_0^2 + \lambda_0^2} \right) e^{-i\lambda_0 x} \right] (a + b |(r, x)|^{2\xi})^{-1} d\nu_\alpha(r, x) = 0.$$

This implies that for almost every  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$e^{-i\lambda_0 x} e^{i\theta_0} \varphi(r, x) j_\alpha \left( r \sqrt{\mu_0^2 + \lambda_0^2} \right) = 1.$$

Since  $|\varphi(r, x)| = 1$ , we deduce that for all  $r \in \mathbb{R}_+$ ,

$$\left| j_\alpha \left( r \sqrt{\mu_0^2 + \lambda_0^2} \right) \right| = 1.$$

Using the relation (2.2), it follows that  $\mu_0^2 + \lambda_0^2 = 0$ , or  $\mu_0 = i|\lambda_0|$ , and therefore

$$\varphi(r, x) = e^{-i\theta_0} e^{i\lambda_0 x}.$$

Replacing in (5.22), we get

$$f(r, x) = C e^{i\lambda_0 x} (a + b |(r, x)|^{2\xi})^{-1}; \quad |C| = 1.$$

Now, the relation (5.19) means that

$$\int \int_E (\|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha}^2 - |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2) d\gamma_\alpha(\mu, \lambda) = 0.$$

Hence, for almost every  $(\mu, \lambda) \in E$ ,

$$(5.24) \quad |\mathcal{F}_\alpha(f)(\mu, \lambda)| = \|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha} = e^{i\theta_0} \mathcal{F}_\alpha(f)(\mu_0, \lambda_0).$$



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Let  $\Psi(\mu, \lambda) \in \mathbb{R}$ , such that

$$|\mathcal{F}_\alpha(f)(\mu, \lambda)| = e^{i\Psi(\mu, \lambda)} \mathcal{F}_\alpha(f)(\mu, \lambda).$$

Then from (5.24), for almost every  $(\mu, \lambda) \in E$ ,

$$e^{i\Psi(\mu, \lambda)} \mathcal{F}_\alpha(f)(\mu, \lambda) = e^{i\theta_0} \mathcal{F}_\alpha(f)(\mu_0, \lambda_0),$$

and therefore

$$\int_0^{+\infty} \int_{\mathbb{R}} (a + b|(r, x)|^{2\xi})^{-1} \left[ 1 - e^{i(\lambda_0 - \lambda)x} e^{i\Psi(\mu, \lambda)} j_\alpha \left( r\sqrt{\mu^2 + \lambda^2} \right) \right] d\nu_\alpha(r, x) = 0.$$

Consequently, for all  $(r, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$e^{i(\lambda_0 - \lambda)x} e^{i\Psi(\mu, \lambda)} j_\alpha \left( r\sqrt{\mu^2 + \lambda^2} \right) = 1,$$

which implies that  $\lambda = \lambda_0$  and  $\mu = \mu_0$ .

However, since  $\gamma_\alpha(E) > 0$ , this contradicts the fact that for almost every  $(\mu, \lambda) \in E$ ,

$$e^{i\Psi(\mu, \lambda)} \mathcal{F}_\alpha(f)(\mu, \lambda) = e^{i\theta_0} \mathcal{F}_\alpha(f)(\mu_0, \lambda_0),$$

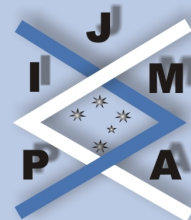
and shows that the inequality in (5.18) is strictly satisfied.

- Let us prove that the constant  $M_{\alpha, \xi}$  is the best one satisfying (5.18).

Let  $A$  be a positive constant, such that for all  $f \in L^2(d\nu_\alpha)$ ,

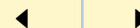
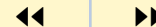
$$(5.25) \quad \int \int_E |\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \leq A \gamma_\alpha(E) \|f\|_{2, \nu_\alpha}^{2 - \frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$





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We assume that  $\| |(r, x)|^\xi f \|_{2, \nu_\alpha} < +\infty$ . Then by Lemma 5.2,  $f$  belongs to  $L^1(d\nu_\alpha)$ . Replacing  $f$  by  $f_t(r, x) = f(tr, tx)$  in (5.25), and using the relations (4.7), (4.9) and (5.17), we deduce that for all  $t > 0$

$$\int \int_E \left| \mathcal{F}_\alpha(f) \left( \frac{\mu}{t}, \frac{\lambda}{t} \right) \right|^2 d\gamma_\alpha(\mu, \lambda) \leq A \|f\|_{2, \nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Using the dominate convergence theorem and the relations (2.1), (2.3), (2.9) and (2.10), we deduce that

$$\lim_{t \rightarrow +\infty} \int \int_E \left| \mathcal{F}_\alpha(f) \left( \frac{\mu}{t}, \frac{\lambda}{t} \right) \right|^2 d\gamma_\alpha(\mu, \lambda) = |\mathcal{F}_\alpha(f)(0, 0)|^2 \gamma_\alpha(E).$$

Consequently, for all  $f \in L^2(d\nu_\alpha)$ , such that  $\| |(r, x)|^\xi f \|_{2, \nu_\alpha} < +\infty$ ,

$$(5.26) \quad |\mathcal{F}_\alpha(f)(0, 0)|^2 \leq A \|f\|_{2, \nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Now, let  $f \in L^2(d\nu_\alpha)$ , such that  $\| |(r, x)|^\xi f \|_{2, \nu_\alpha} < +\infty$ , and let  $(\mu, \lambda) \in \Gamma$ . Putting

$$g(r, x) = j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) e^{-i\lambda x} f(r, x),$$

then  $g \in L^2(d\nu_\alpha)$ , and  $\| |(r, x)|^\xi g \|_{2, \nu_\alpha} < +\infty$ . Moreover,  $\mathcal{F}_\alpha(g)(0, 0) = \mathcal{F}_\alpha(f)(\mu, \lambda)$ , and by (5.26), it follows that

$$|\mathcal{F}_\alpha(f)(\mu, \lambda)|^2 \leq A \|f\|_{2, \nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

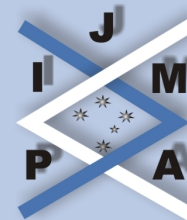
Thus, for all  $f \in L^2(d\nu_\alpha)$  such that  $\| |(r, x)|^\xi f \|_{2, \nu_\alpha} < +\infty$ , we have

$$(5.27) \quad \|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha}^2 \leq A \|f\|_{2, \nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r, x)|^\xi f \|_{2, \nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Taking  $f_0(r, x) = (1 + |(r, x)|^{2\xi})^{-1}$ , we have

$$\begin{aligned} \|\mathcal{F}_\alpha(f_0)\|_{\infty, \gamma_\alpha}^2 &= \|f_0\|_{1, \nu_\alpha}^2 = \left( \frac{\pi}{\xi 2^{\alpha+\frac{3}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)} \right)^2, \\ \|f_0\|_{2, \nu_\alpha}^2 &= \frac{(2\xi - 2\alpha - 3)\pi}{\xi^2 2^{\alpha+\frac{5}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}, \\ \| |(r, x)|^\xi f_0 \|_{2, \nu_\alpha}^2 &= \frac{(2\alpha + 3)\pi}{\xi^2 2^{\alpha+\frac{5}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}. \end{aligned}$$

Replacing  $f$  by  $f_0$  in the relation (5.27), we obtain  $A \geq M_{\alpha, \xi}$ .  
This completes the proof of Theorem 5.3. □



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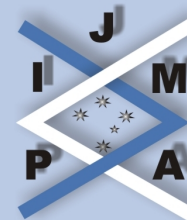
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Heisenberg-Pauli-Weyl  
Uncertainty Principle

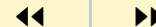
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