journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 3, Article 82, 7 pp.

THE REFINEMENT AND REVERSE OF A GEOMETRIC INEQUALITY

YU-LIN WU

DEPARTMENT OF MATHEMATICS
BEIJING UNIVERSITY OF TECHNOLOGY
100 PINGLEYUAN, CHAOYANG DISTRICT
BEIJING 100124, PEOPLE'S REPUBLIC OF CHINA.
wuyulin2007@emails.bjut.edu.cn

Received 15 April, 2009; accepted 27 August, 2009
Communicated by S.S. Dragomir

ABSTRACT. In this paper, we give a refinement and a reverse of a geometric inequality in a triangle posed by Jiang [2] by making use of the equivalent form of a fundamental inequality [6] and classic analysis.

Key words and phrases: Geometric inequality; Best constant; Triangle.

2000 *Mathematics Subject Classification.* Primary 51M16; Secondary 51M25, 52A40.

1. INTRODUCTION AND MAIN RESULT

For $\triangle ABC$, let a, b, c be the side-lengths, A, B, C the angles, s the semi-perimeter, R the circumradius and r the inradius, respectively. Moreover, we will customarily use the cyclic sum symbols, that is: $\sum f(a) = f(a) + f(b) + f(c)$, $\sum f(b,c) = f(a,b) + f(b,c) + f(c,a)$ and $\prod f(a) = f(a)f(b)f(c)$ etc.

In 2008, Jiang [2] posed the following geometric inequality problem.

Problem 1.1. In $\triangle ABC$, prove that

$$\sum \frac{a}{b+c}\left(\tan^4\frac{B}{2} + \tan^4\frac{C}{2}\right) \geq \frac{1}{3}. \tag{1.1}$$

In the same year, Manh Dung Nguyen and Duy Khanh Nguyen [4] proved inequality (1.1).

In this paper, we give a refinement and a reverse of inequality (1.1).

Theorem 1.1. *In $\triangle ABC$, the best constant k for the following inequality*

$$\sum \frac{a}{b+c}\left(\tan^4\frac{B}{2} + \tan^4\frac{C}{2}\right) \geq \frac{1}{3} + k\left(1 - \frac{2r}{R}\right). \tag{1.2}$$

is $\lambda_0 \approx 1.330090721$ which is the positive real root of:

$$3564\lambda^6 + 114588\lambda^5 - 246261\lambda^4 + 137484\lambda^3 - 29712\lambda^2 + 2336\lambda - 60 = 0 \tag{1.3}$$

The author would like to thank the anonymous referee for his valuable suggestions.
101-09

It is easy to see that inequality (1.1) follows from Theorem 1.1 and **Euler**'s inequality $R \geq 2r$ immediately.

Theorem 1.2. *In* $\triangle ABC$*, we have*

$$\sum \frac{a}{b+c}\left(\tan^4 \frac{B}{2} + \tan^4 \frac{C}{2}\right) \leq \frac{1}{3} + \frac{8}{3}\left[\left(\frac{R}{2r}\right)^2 - 1\right]. \tag{1.4}$$

2. PRELIMINARY RESULTS

In order to prove Theorem 1.1 and Theorem 1.2, we shall require the following five lemmas.

Lemma 2.1 ([6])**.** *For any triangle* ABC*, the following inequalities hold true:*

$$\frac{1}{4}\delta(4-\delta)^3 \leq \frac{s^2}{R^2} \leq \frac{1}{4}(2-\delta)(2+\delta)^3, \tag{2.1}$$

where $\delta = 1 - \sqrt{1 - \frac{2r}{R}} \in (0, 1]$*. Equality on the left hand side of the double inequality* (2.1) *is valid if and only if triangle* ABC *is an isosceles triangle with top-angle greater than or equal to* $\frac{\pi}{3}$*, and equality on the right hand side of the double inequality* (2.1) *is valid if and only if triangle* ABC *is an isosceles triangle with top-angle less than or equal to* $\frac{\pi}{3}$*.*

Lemma 2.2. *In* $\triangle ABC$*, we have*

$$\begin{aligned} \sum \frac{a}{b+c}\left(\tan^4 \frac{B}{2} + \tan^4 \frac{C}{2}\right) &= \frac{1}{s^4(s^2+2Rr+r^2)} \cdot [2s^6 - 2(32R^2+24Rr+r^2)s^4 \\ &\quad + 2(4R+r)(32R^3+72R^2r+28Rr^2+r^3)s^2 - 2r(2R+r)(4R+r)^4]. \end{aligned} \tag{2.2}$$

Proof. From the law of cosines, we get

$$\tan^2 \frac{A}{2} = \frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}} = \frac{1-\cos A}{1+\cos A} = \frac{1 - \frac{b^2+c^2-a^2}{2bc}}{1 + \frac{b^2+c^2-a^2}{2bc}} = \frac{(a+b-c)(c+a-b)}{(a+b+c)(b+c-a)}.$$

In the same manner, we can also obtain

$$\tan^2 \frac{B}{2} = \frac{(b+c-a)(a+b-c)}{(a+b+c)(c+a-b)}, \qquad \tan^2 \frac{C}{2} = \frac{(c+a-b)(b+c-a)}{(a+b+c)(a+b-c)}.$$

Hence,

$$\begin{aligned} \sum \frac{a}{b+c}\left(\tan^4 \frac{B}{2} + \tan^4 \frac{C}{2}\right) &= \sum \frac{a}{b+c}\left[\frac{(b+c-a)^2(a+b-c)^2}{(a+b+c)^2(c+a-b)^2} + \frac{(c+a-b)^2(b+c-a)^2}{(a+b+c)^2(a+b-c)^2}\right] \\ &= \frac{\sum a(c+a)(a+b)(b+c-a)^4[(a+b-c)^4+(c+a-b)^4]}{(a+b+c)^2 \cdot \prod(b+c-a)^2 \cdot \prod(b+c)}. \end{aligned} \tag{2.3}$$

And it is not difficult to verify the following three identities.

$$\prod (b+c) = (ab+bc+ca)(a+b+c) - abc, \tag{2.4}$$

$$\prod (b+c-a) = -(a+b+c)^3 + 4(ab+bc+ca)(a+b+c) - 8abc, \tag{2.5}$$

$$\begin{aligned}(2.6)\qquad &\sum a(c+a)(a+b)(b+c-a)^4[(a+b-c)^4+(c+a-b)^4]\\ &= 2(a+b+c)^{11} - 28(ab+bc+ca)(a+b+c)^9 - 18abc(a+b+c)^8\\ &\quad + 160(ab+bc+ca)^2(a+b+c)^7 + 224abc(ab+bc+ca)(a+b+c)^6\\ &\quad - 400a^2b^2c^2(a+b+c)^5 - 480(ab+bc+ca)^3(a+b+c)^5\\ &\quad - 768abc(ab+bc+ca)^2(a+b+c)^4\\ &\quad + 2560a^2b^2c^2(ab+bc+ca)(a+b+c)^3\\ &\quad + 768(ab+bc+ca)^4(a+b+c)^3 - 1280a^3b^3c^3(a+b+c)^2\\ &\quad + 512abc(ab+bc+ca)^3(a+b+c)^2 - 512(ab+bc+ca)^5(a+b+c)\\ &\quad - 3328a^2b^2c^2(ab+bc+ca)^2(a+b+c) + 2048a^3b^3c^3(ab+bc+ca)\\ &\quad + 512abc(ab+bc+ca)^4,\end{aligned}$$

Identity (2.2) follows directly from identities (2.3) – (2.6) and the following known identities:

$$a+b+c=2s, \qquad ab+bc+ca=s^2+4Rr+r^2, \qquad abc=4Rrs.$$

□

Lemma 2.3 ([9]). *In $\triangle ABC$, we have*

$$s^4-(20Rr-r^2)s^2+4r^2(4R+r)^2\geq 0. \tag{2.7}$$

Lemma 2.4. *The function*

$$\begin{aligned}f(s)=\frac{1}{s^4(s^2+2Rr+r^2)}&\cdot[2s^6-2(32R^2+24Rr+r^2)s^4\\ &+2(4R+r)(32R^3+72R^2r+28Rr^2+r^3)s^2-2r(2R+r)(4R+r)^4]\end{aligned}$$

is strictly monotone decreasing on the interval $[s_1, s_2]$, where

$$\begin{aligned}s_1&=\sqrt{2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}}\\ &=\frac{1}{2}\sqrt{\delta(4-\delta)^3}R\end{aligned}$$

and

$$\begin{aligned}s_2&=\sqrt{2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}}\\ &=\frac{1}{2}\sqrt{(2-\delta)(2+\delta)^3}R.\end{aligned}$$

Proof. Calculating the derivative for $f(s)$, we get

$$\begin{aligned}f'(s)=\frac{-8}{s^5(s^2+2Rr+r^2)^2}&\cdot\{(16R^2+13Rr+r^2)[s^4-(20Rr-r^2)s^2+4r^2(4R+r)^2]\\ &\cdot(4R^2+4Rr+3r^2-s^2)+64R^4[s^4-(20Rr-r^2)s^2+4r^2(4R+r)^2]\\ &+(116R^3r+164R^2r^2+18Rr^3+r^4)[-s^4+(4R^2+20Rr-2r^2)s^2\\ &-r(4R+r)^3]+[1024488r^6+3177399r^5(R-2r)+4148540r^4(R-2r)^2\\ &+2913136r^3(R-2r)^3+1156192r^2(R-2r)^4+244816r(R-2r)^5\\ &+21504(R-2r)^6]r^2\}.\end{aligned}$$

From inequality (2.7), **Euler**'s inequality $R \geq 2r$, **Gerretsen**'s inequality (see [1, page 45]) $s^2 \leq 4R^4 + 4Rr + 3r^2$ and the fundamental inequality (see [3, page 2])

$$-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R+r)^3 \geq 0,$$

we can conclude that $f'(s) < 0$. Therefore, $f(s)$ is strictly monotone decreasing on the interval $[s_1, s_2]$. □

Lemma 2.5 ([10]). *Denote*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m.$$

If $a_0 \neq 0$ or $b_0 \neq 0$, then the polynomials $f(x)$ and $g(x)$ have common roots if and only if

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{vmatrix} = 0,$$

where $R(f,g)$ is the Sylvester Resultant of $f(x)$ and $g(x)$.

3. THE PROOF OF THEOREM 1.1

Proof. By Lemma 2.2 and Lemma 2.4, we get

$$\text{(3.1)} \qquad \sum \frac{a}{b+c}\left(\tan^4\frac{B}{2} + \tan^4\frac{C}{2}\right) \geq f(s_2) = \frac{\delta^6 - 7\delta^5 + 20\delta^4 - 24\delta^3 + 32\delta^2 - 48\delta + 32}{(\delta+1)(\delta-2)^2(2+\delta)^2}.$$

Now we consider the best constant for the following inequality.

$$\text{(3.2)} \qquad \frac{\delta^6 - 7\delta^5 + 20\delta^4 - 24\delta^3 + 32\delta^2 - 48\delta + 32}{(\delta+1)(\delta-2)^2(2+\delta)^2} \geq \frac{1}{3} + k\left(1 - \frac{2r}{R}\right) = \frac{1}{3} + k(1-\delta)^2 \qquad (0 < \delta \leq 1).$$

(i) In the case of $\delta = 1$, the inequality (3.2) obviously holds.
(ii) In the case of $0 < \delta < 1$, the inequality (3.2) is equivalent to

$$k \leq g(\delta) := \frac{3\delta^4 - 16\delta^3 + 24\delta^2 + 80}{3(\delta+1)(\delta-2)^2(\delta+2)^2} \qquad (0 < \delta < 1).$$

Calculating the derivative for $g(\delta)$, we get

$$g'(\delta) = \frac{3\delta^6 - 32\delta^5 + 92\delta^4 - 32\delta^3 + 304\delta^2 + 512\delta - 320}{3(\delta+1)^2(2-\delta)^3(\delta+2)^3}.$$

Letting $g'(\delta) = 0$, we get

$$\text{(3.3)} \qquad 3\delta^6 - 32\delta^5 + 92\delta^4 - 32\delta^3 + 304\delta^2 + 512\delta - 320 = 0, \qquad (0 < \delta < 1).$$

It is not difficult to see that the equation (3.3) has only one positive root on the open interval $(0,1)$. Denote δ_0 to be the root of the equation (3.3). Then

$$g(\delta)_{min} = g(\delta_0) = \frac{3\delta_0^4 - 16\delta_0^3 + 24\delta_0^2 + 80}{(\delta_0+1)(\delta_0-2)^2(\delta_0+2)^2}. \tag{3.4}$$

It is easy to see that $g(\delta_0)$ is a root of the following nonlinear algebraic equation system.

$$\begin{cases} F(\delta_0) = 0, \\ G(\delta_0) = 0, \end{cases} \tag{3.5}$$

where

$$F(\delta_0) = 3(\delta_0+1)(\delta_0-2)^2(\delta_0+2)^2\lambda - (3\delta_0^4 - 16\delta_0^3 + 24\delta_0^2 + 80)$$

and

$$G(\delta_0) = 3\delta_0^6 - 32\delta_0^5 + 92\delta_0^4 - 32\delta_0^3 + 304\delta_0^2 + 512\delta_0 - 320.$$

Then,

$$\begin{aligned} R_{\delta_0}(F,G) &= \begin{vmatrix} 3\lambda & 16-24\lambda & 3\lambda-3 & \cdots & 48\lambda-80 & 0 & \cdots & 0 \\ 0 & 3\lambda & 16-24\lambda & \cdots & 48\lambda & 48\lambda-80 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 3\lambda & \cdots & \cdots & \cdots & 48\lambda-80 \\ 3 & -32 & 92 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 3 & -32 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -32 & \cdots & -320 \end{vmatrix} \\ &= -3177213868376064(3564\lambda^6 + 114588\lambda^5 - 246261\lambda^4 + 137484\lambda^3 \\ &\qquad - 29712\lambda^2 + 2336\lambda - 60). \end{aligned}$$

With Lemma 2.5, we can conclude that $g(\delta_0)$ is the real root of (1.3). And the equation (1.3) has only one positive real root, hence, $g(\delta_0)$ is the positive real root of (1.3). Namely, the best constant for inequality (3.2) is the real positive root of (1.3).

From (3.1) and above, we find that Theorem 1.1 holds.

Now we consider when we have equality in

$$\sum \frac{a}{b+c}\left(\tan^4\frac{B}{2} + \tan^4\frac{C}{2}\right) \geq \frac{1}{3} + \lambda_0\left(1 - \frac{2r}{R}\right). \tag{3.6}$$

It is easy to see that equality in (3.6) holds when $\triangle ABC$ is an equilateral triangle. We consider another case: From the process of seeking $g_{min}(\delta)$ and Lemma 2.1, we can find the equality of inequality (3.6) holds when $\triangle ABC$ is an isosceles triangle with top-angle less than or equal to $\frac{\pi}{3}$ and $\delta = \delta_0$ or $\frac{2r}{R} = 2\delta_0 - \delta_0^2$, there is no harm in supposing $b = c = 1 (0 < a < 1)$, then

$$2\delta_0 - \delta_0^2 = \frac{2r}{R} = \frac{(a+b-c)(b+c-a)(c+a-b)}{abc} = a(2-a),$$

Thus $a = \delta_0$, namely, the equality of inequality (3.6) holds when $\triangle ABC$ is isosceles and the ratio of its side-lengths is $\delta_0 : 1 : 1$. □

4. THE PROOF OF THEOREM 1.2

Proof. By Lemma 2.2 and Lemma 2.4,

$$\sum \frac{a}{b+c}\left(\tan^4\frac{B}{2}+\tan^4\frac{C}{2}\right) \le f(s_1) = \frac{\delta^8+5\delta^7-11\delta^6-123\delta^5+64\delta^4+1168\delta^3-2176\delta^2+512\delta+512}{(\delta^4-5\delta^3+12\delta^2-40\delta+64)(\delta-4)^2}. \tag{4.1}$$

Now we prove

$$\frac{-2(\delta^8+5\delta^7-11\delta^6-123\delta^5+64\delta^4+1168\delta^3-2176\delta^2+512\delta+512)}{(\delta^4-5\delta^3+12\delta^2-40\delta+64)(\delta-4)^2} \le \frac{1}{3}+\frac{8}{3}\left[\left(\frac{R}{2r}\right)^2-1\right] = \frac{1}{3}+\frac{8}{3}\left[\left(\frac{1}{2\delta-\delta^2}\right)^2-1\right]. \tag{4.2}$$

Inequality (4.2) is equivalent to

$$\frac{(\delta-1)X}{3\delta^2(\delta-2)^2(\delta-4)^2(\delta^4-5\delta^3+12\delta^2-40\delta+64)} \ge 0, \tag{4.3}$$

where

$$X = 6\delta^{11}+12\delta^{10}-157\delta^9-392\delta^8+1812\delta^7+8112\delta^6-43416\delta^5 + 70048\delta^4-46400\delta^3+12800\delta^2+1024\delta-8192. \tag{4.4}$$

From $0<\delta\le 1$, it is easy to see that $t=\frac{1}{\delta}-1\ge 0$, hence, we can easily obtain the following two inequalities

$$\delta^4-5\delta^3+12\delta^2-40\delta+64 = (1-\delta)^4+(1-\delta)^3+3(1-\delta)^2+27(1-\delta)+32>0 \tag{4.5}$$

and

$$X = \delta^{11}(-8192t^{11}-89088t^{10}-427520t^9-1236800t^8-2420832t^7-3346744t^6 - 3293632t^5-2280708t^4-1080664t^3-332453t^2-59702t-4743)<0. \tag{4.6}$$

For $0<\delta\le 1$, together with (4.4) – (4.6), we can conclude that inequality (4.3) holds, so inequality (4.2) holds. Inequality (1.4) immediately follows from (4.1) and (4.2). □

REFERENCES

[1] O. BOTTEMA, R.Ž. DJORDEVIĆ, R.R. JANIĆ, D.S. MITRINOVIĆ AND P.M.VASIĆ, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[2] W.-D. JIANG, A triangle inequality, Problem 4, (2008), *RGMIA Problem Corner*. [ONLINE `http://eureka.vu.edu.au/~rgmia/probcorner/2008/problem4-08.pdf`]

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Acad. Publ., Dordrecht, Boston, London, 1989.

[4] MANH DUNG NGUYEN AND DUY KHANH NGUYEN, Problem 4, (2008), Solution No. 1, *RGMIA Problem Corner*.[ONLINE `http://eureka.vu.edu.au/~rgmia/probcorner/2008/problem4-08-sol1.pdf`].

[5] S.-H. WU, A sharpened version of the fundamental triangle inequality, *Math. Inequal. Appl.*, **11**(3) (2008), 477–482.

[6] S.-H. WU AND M. BENCZE, An equivalent form of the fundamental triangle inequality and its applications, *J. Inequal. Pure Appl. Math.*, **10**(1) (2009), Art. 16. [ONLINE http://jipam.vu.edu.au/article.php?sid=1072].

[7] S.-H. WU AND Z.-H. ZHANG, A class of inequalities related to the angle bisectors and the sides of a triangle, *J. Inequal. Pure Appl. Math.*, **7**(3) (2006), Art. 108. [ONLINE http://jipam.vu.edu.au/article.php?sid=698].

[8] S.-H. WU AND Z.-H. ZHANG, Some strengthened results on Gerretsen's inequality, *RGMIA Res. Rep. Coll.*, **6**(3) (2003), Art. 16. [ONLINE http://www.staff.vu.edu.au/RGMIA/v6n3.asp].

[9] Y.-D. WU, On the open Question OQ.1285, *Octogon Mathematical Magazine*, **12**(2) (2004), 1041–1045.

[10] L. YANG, J.-Z. ZHANG AND X.-R. HOU, *Nonlinear Algebraic Equation System and Automated Theorem Proving*, Shanghai Scientific and Technological Education Press (1996), 23–25. (in Chinese)