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THE RATIO BETWEEN THE TAIL OF A SERIES AND ITS APPROXIMATING INTEGRAL

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ABSTRACT. For a strictly positive function $f(x)$, let $S(n) = \sum_{k=n}^{\infty} f(k)$ and $I(x) = \int_x^{\infty} f(t)dt$, assumed convergent. If $f'(x)/f(x)$ is increasing, then $S(n)/I(n)$ is decreasing and $S(n+1)/I(n)$ is increasing. If $f''(x)/f(x)$ is increasing, then $S(n)/I(n-\frac{1}{2})$ is decreasing. Under suitable conditions, analogous results are obtained for the "continuous tail" defined by $S(x) = \sum_{n=0}^{\infty} f(x+n)$: these results apply, in particular, to the Hurwitz zeta function.

Key words and phrases: Series, Tail, Ratio, Monotonic, Zeta function.

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1. INTRODUCTION

Let f be a positive function with $\int_1^{\infty} f(t)dt$ convergent, and let

$$S(n) = \sum_{k=n}^{\infty} f(k), \qquad I(x) = \int_x^{\infty} f(t)dt.$$

The problem addressed in this article is to determine conditions ensuring that ratios of the type $S(n)/I(n)$ are either increasing or decreasing. For decreasing f, one has $I(n) \le S(n) \le I(n-1)$, and one might expect $S(n)/I(n)$ to decrease and $S(n)/I(n-1)$ to increase, but, as we show, the truth is not quite so simple. In general, $I\left(n-\frac{1}{2}\right)$ is a much better approximation to $S(n)$ than either $I(n)$ or $I(n-1)$, so we also consider the ratio $S(n)/I\left(n-\frac{1}{2}\right)$.

Questions of this type arise repeatedly in the context of generalizations of the discrete Hardy and Hilbert inequalities, often in the form of estimations of the norms and so-called "lower bounds" of matrix operators on weighted ℓ_p spaces or Lorentz sequence spaces. These topics

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have been studied in numerous papers, e.g. ([3], [4], [5], [7], [8]). Often, the problem equates to finding the supremum and infimum of a ratio like $S(n)/I(n)$ for a suitable function f. In many “natural" cases, the ratio is in fact monotonic, so the required bounds are simply the first term and the limit, one way round or the other.

Sporadic results on monotonicity have appeared for particular cases, especially $f(t) = 1/t^p$, in some of the papers mentioned, though not for ratios involving $I\left(n - \frac{1}{2}\right)$. However, the author is not aware of any previous work formulating general criteria. As we show, such criteria can, in fact, be given. Though the methods are essentially elementary, the criteria are far from transparent at the outset, indeed somewhat unexpected.

We show that the kernel of the problem is already contained in the corresponding question for ratios of integrals (on intervals of fixed length) to single values of the function. Indeed, write

$$J_1(x) = \int_{x-h}^{x} f(t)dt, \qquad J_2(x) = \int_{x}^{x+h} f(t)dt, \qquad J_3(x) = \int_{x-h}^{x+h} f(t)dt.$$

For both types of problem, the outcome is determined by monotonicity of f'/f or f''/f, as follows:

(1) If $f'(x)/f(x)$ is increasing, then $J_1(x)/f(x)$ is decreasing and $J_2(x)/f(x)$ is increasing. Further, $S(n)/I(n)$ is decreasing and $S(n)/I(n-1)$ is increasing.

(2) If $f''(x)/f(x)$ is increasing, then $J_3(x)/f(x)$ is increasing, and $S(n)/I\left(n - \frac{1}{2}\right)$ is decreasing. Opposite results apply to a second type of ratio relating to the trapezium rule.

If the hypotheses are reversed, so are the conclusions. When applied to x^p, the statements in (2) are stronger than those in (1).

By rather different methods, but still as a consequence of the earlier results on $J_r(x)/f(x)$, we then obtain analogous results for the “continuous tail" defined by

$$S(x) = \sum_{n=0}^{\infty} f(x+n).$$

When $f(t) = 1/t^p$, this defines the Hurwitz zeta function $\zeta(p, x)$, which has important applications in analytic number theory [2].

Other studies of tails of series include [9], [10] and further papers cited there. Typically, these studies describe relationships between $S(n-1)$, $S(n)$ and $S(n+1)$, and are specific to power series, whereas the natural context for our results is the situation where $S(n) \sim I(n)$ as $n \to \infty$, which occurs for series like $\sum 1/n^p$.

2. RATIOS BETWEEN INTEGRALS AND FUNCTIONAL VALUES

Let f be a strictly positive, differentiable function on a real interval E, and let $h \geq 0$, $k \geq 0$. On the suitably reduced interval E', define

$$J(x) = \int_{x-h}^{x+k} f(t)dt$$

We shall consider particularly the cases where one of h, k is 0 (so that x is an end point of the interval) or where $h = k$ (so that x is the mid-point). Our aim is to investigate monotonicity of $G(x)$, where

$$G(x) = \frac{J(x)}{f(x)}.$$

We shall work with the expression for the derivative $G'(x)$ given in the next lemma (we include the proof, though it is elementary, since this lemma underlies all our further results).

Lemma 2.1. *With the above notation, we have*

$$G'(x) = \frac{1}{f(x)^2} \int_{x-h}^{x+k} W(x,t)dt,$$

where

$$W(x,t) = f(x)f'(t) - f'(x)f(t).$$

Proof. We have

$$J'(x) = f(x+k) - f(x-h) = \int_{x-h}^{x+k} f'(t)dt,$$

and hence

$$G'(x) = \frac{1}{f(x)} \int_{x-h}^{x+k} f'(t)dt - \frac{f'(x)}{f(x)^2} \int_{x-h}^{x+k} f(t)dt,$$

which is equivalent to the statement. □

So our problem, in the various situations considered, will be to establish that

$$\int_{x-h}^{x+k} W(x,t)dt$$

is either positive or negative. The function W is, of course, a certain kind of Wronskian. Note that it satisfies $W(x,x) = 0$ and $W(y,x) = -W(x,y)$. Further, we have:

Lemma 2.2. *Let f be strictly positive and differentiable on an interval E, and let $W(x,y) = f(x)f'(y) - f'(x)f(y)$. Then the following statements are equivalent:*

(i) $f'(x)/f(x)$ *is increasing on* E,

(ii) $W(x,y) \geq 0$ *when* $x, y \in E$ *and* $x < y$.

Proof. Write $f'(x)/f(x) = q(x)$. Then

$$W(x,y) = f(x)f(y)\big(q(y) - q(x)\big).$$

The stated equivalence follows at once. □

Hence we have, very easily, the following solution of the end-point problems.

Proposition 2.3. *Let f be strictly positive and differentiable on an interval E. Fix $h > 0$, and define (on suitably reduced intervals)*

$$J_1(x) = \int_{x-h}^{x} f(t)dt, \qquad J_2(x) = \int_{x}^{x+h} f(t)dt.$$

If $f'(x)/f(x)$ is increasing, then $J_1(x)/f(x)$ is decreasing and $J_2(x)/f(x)$ is increasing. The opposite holds if $f'(x)/f(x)$ is decreasing.

Proof. Again write $f'(x)/f(x) = q(x)$. If $q(x)$ is increasing, then, by Lemma 2.2, $W(x,t)$ is positive for t in $[x, x+h]$ and negative for t in $[x-h, x]$. The statements follow, by Lemma 2.1. □

Corollary 2.4. *Fix $h > 0$. Let*

$$G_1(x) = \frac{1}{x^p} \int_{x-h}^{x} t^p dt, \qquad G_2(x) = \frac{1}{x^p} \int_{x}^{x+h} t^p dt.$$

If $p > 0$, then $G_1(x)$ is increasing on (h, ∞), and $G_2(x)$ is decreasing on $(0, \infty)$. The opposite conclusions hold when $p < 0$.

Proof. Then $q(x) = p/x$, which is decreasing on $(0, \infty)$ when $p > 0$, and increasing when $p < 0$. □

Remark 2.5. Neither the statement of Corollary 2.4, nor its proof, is improved by writing out the integrals explicitly.

Remark 2.6. Corollary 2.4 might lead one to suppose that monotonicity of $f(x)$ itself is significant, but this is not true. If $f(x) = x^2$, then Proposition 2.3 shows that $J_1(x)/f(x)$ is increasing both for $x < 0$ and for $x > h$.

Remark 2.7. Clearly, the case where $J_1(x)/f(x)$ and $J_2(x)/f(x)$ are *constant* is given by $f(x) = e^{cx}$.

Remark 2.8. Three equivalents to the statement that $f'(x)/f(x)$ is increasing (given that $f(x) > 0$) are:

(i) $f'(x)^2 \le f(x)f''(x)$,
(ii) $\log f(x)$ is convex,
(iii) $f(x+\delta)/f(x)$ is increasing for each $\delta > 0$.

Condition (iii) is implicitly used in [7, Corollary 3.3] to give an alternative proof of Corollary 2.4.

We now consider the symmetric ratios occurring when $h = k$. Let

$$J(x) = \int_{x-h}^{x+h} f(t)dt.$$

There are actually two symmetric ratios that arise naturally, both of which have applications to tails of series. The *mid-point* estimate for the integral $J(x)$ (describing the area below the tangent at the mid-point) is $2hf(x)$, while the *trapezium* estimate is $hf_h(x)$, where

$$f_h(x) = f(x-h) + f(x+h).$$

If f is *convex*, then it is geometrically obvious (and easily proved) that

$$2hf(x) \le J(x) \le hf_h(x),$$

with equality occuring when f is linear. So we consider monotonicity of the mid-point ratio $J(x)/f(x)$ and the two-end-point ratio $J(x)/f_h(x)$. The outcome is less transparent than in the end-point problem. We shall see that it is determined, in the opposite direction for the two cases, by monotonicity of $f''(x)/f(x)$. Both the statements and the proofs can be compared with Sturm's comparison theorem on solutions of differential equations of the form $y'' = r(x)y$ [11, section 25]. Where Sturm's theorem requires positivity or negativity of $r(x)$, we require monotonicity, and the proofs share the feature of considering the derivative of a Wronskian. The key lemma is the following, relating monotonicity of $f''(x)/f(x)$ to properties of $W(x, y)$.

Lemma 2.9. *Let f be strictly positive and twice differentiable on an interval (a, b). Then the following statements are equivalent:*

(i) *$f''(x)/f(x)$ is increasing on (a, b);*
(ii) *for each fixed u in $(0, b-a)$, the function $W(x, x+u)$ is increasing on $(a, b-u)$.*

Proof. Write $f''(x) = r(x)f(x)$ and

$$A(x) = W(x, x+u) = f(x)f'(x+u) - f'(x)f(x+u).$$

Then

$$\begin{aligned} A'(x) &= f(x)f''(x+u) - f''(x)f(x+u) \\ &= \big(r(x+u) - r(x)\big)f(x)f(x+u), \end{aligned}$$

from which the stated equivalence is clear. □

Lemma 2.10. *Let x be fixed and let w be a continuous function such that*

$$w(x+u)+w(x-u)\geq 0$$

for $0\leq u\leq h$. Then

$$\int_{x-h}^{x+h} w(t)dt\geq 0.$$

Proof. Clear, on substituting $t=x+u$ on $[x,x+h]$ and $t=x-u$ on $[x-h,x]$. □

We can now state our result on the mid-point ratio.

Proposition 2.11. *Let f be strictly positive and twice differentiable on an interval E. Fix $h>0$, and let*

$$J(x)=\int_{x-h}^{x+h} f(t)dt.$$

If $f''(x)/f(x)$ is increasing (or decreasing) on E, then $J(x)/f(x)$ is increasing (or decreasing) on the suitably reduced sub-interval.

Proof. Fix u with $0<u\leq h$. Assume that $f''(x)/f(x)$ is increasing. By Lemma 2.9, if x and $x+u$ are in E, then

$$W(x,x+u)\geq W(x-u,x)=-W(x,x-u).$$

The statement follows, by Lemmas 2.1 and 2.10. □

Corollary 2.12. *Fix $h>0$. Let*

$$G(x)=\frac{1}{x^p}\int_{x-h}^{x+h} t^p\,dt.$$

If $p\geq 1$ or $p\leq 0$, then $G(x)$ is decreasing on (h,∞). If $0\leq p\leq 1$, it is increasing there.

Proof. Let $f(x)=x^p$. Then

$$\frac{f''(x)}{f(x)}=\frac{p(p-1)}{x^2},$$

which is decreasing (for positive x) if $p(p-1)\geq 0$. (Alternatively, it is not hard to prove this corollary directly from Lemmas 2.1 and 2.10.) □

Note that Corollary 2.12 strengthens one or other statement in Corollary 2.4 in each case. For example, if $p>1$, then $\big(x/(x-h)\big)^p$ is decreasing, so Corollary 2.12 implies that $J(x)/(x-h)^p$ is decreasing (as stated by 2.4).

Corollary 2.13. *If f possesses a third derivative on E, then the following scheme applies:*

f'	f''	f'''	J/f
$+$	$-$	$+$	*incr*
$-$	$+$	$+$	*incr*
$+$	$+$	$-$	*decr*
$-$	$-$	$-$	*decr*

Proof. By differentiation, one sees that $f''(x)/f(x)$ is increasing if $f(x)f'''(x)\geq f'(x)f''(x)$. In each case, the hypotheses ensure that these two expressions have opposite signs. □

However, the signs of the first three derivatives do not determine monotonicity of f''/f in the other cases. Two specific examples of type $+++$ are x^3 for $x>0$ and x^{-2} for $x<0$. In both cases, $f''(x)/f(x)=6x^{-2}$, which is increasing for $x<0$ and decreasing for $x>0$.

Clearly, $J(x)/f(x)$ is constant when $f''(x)/f(x)$ is constant.

For the two-end-point problem, we need the following modification of Lemma 2.1.

Lemma 2.14. *Let $G(x) = J(x)/f_h(x)$, where $J(x)$ and $f_h(x)$ are as above. Then*

$$G'(x) = \frac{1}{f_h(x)^2} \int_{x-h}^{x+h} \big(W(x-h,t) + W(x+h,t)\big)\, dt,$$

where $W(x,t)$ is defined as before.

Proof. Elementary. □

Proposition 2.15. *Let f be strictly positive and twice differentiable on an interval E. Fix $h > 0$. Let $f_h(x) = f(x-h) + f(x+h)$ and*

$$J(x) = \int_{x-h}^{x+h} f(t)dt.$$

If $f''(x)/f(x)$ is increasing on E, then $J(x)/f_h(x)$ is decreasing on the suitably reduced subinterval (and similarly with "increasing" and "decreasing" interchanged).

Proof. By Lemmas 2.10 and 2.14, the statement will follow if we can show that

$$W(x-h,x-u) + W(x+h,x-u) + W(x-h,x+u) + W(x+h,x+u) \le 0$$

for $0 < u \le h$. With u fixed, let $A(x) = W(x+u,x+h)$. By Lemma 2.9, $A(x)$ is increasing, hence

$$\begin{aligned} 0 &\ge A(x-u-h) - A(x) \\ &= W(x-h,x-u) - W(x+u,x+h) \\ &= W(x-h,x-u) + W(x+h,x+u). \end{aligned}$$

Similarly, $B(x) = W(x-h,x+u)$ is increasing, hence

$$\begin{aligned} 0 &\ge B(x) - B(x+h-u) \\ &= W(x-h,x+u) - W(x-u,x+h) \\ &= W(x-h,x+u) + W(x+h,x-u). \end{aligned}$$

These two statements together give the required inequality. □

Corollary 2.16. *The expression*

$$\frac{(x+h)^{p+1} - (x-h)^{p+1}}{(x+h)^p + (x-h)^p}$$

is increasing if $p \ge 1$ or $-1 \le p \le 0$, decreasing in other cases.

3. TAILS OF SERIES: DISCRETE VERSION

Let f be a function satisfying the following conditions:

(A1) $f(x) > 0$ *for all* $x > 0$*;*
(A2) $f(x)$ *is decreasing on some interval* $[x_0, \infty)$*;*
(A3) $\int_1^\infty f(t)dt$ *is convergent.*

We will also assume, as appropriate, either

(A4) f *is differentiable on* $(0,\infty)$

or

(A4′) f *is twice differentiable on* $(0,\infty)$.

Clearly, under these assumptions, $\sum_{k=1}^{\infty} f(k)$ is convergent. Throughout the following, we write

$$S(n) = \sum_{k=n}^{\infty} f(k), \qquad I(x) = \int_x^{\infty} f(t)\, dt.$$

By simple integral comparison, $S(n+1) \le I(n) \le S(n)$ for $n \ge x_0$. Further, if $f(n)/I(n) \to 0$ as $n \to \infty$, then $S(n)/I(n)$ tends to 1. From these considerations, one might expect $S(n)/I(n)$ to decrease with n, and $S(n+1)/I(n)$ to increase.

Functions of the type now being considered will often be convex, at least for sufficiently large x. In this case, the mid-point and trapezium estimations mentioned in Section 2 come into play. Mid-point comparison, on successive intervals $\left[r - \frac{1}{2}, r + \frac{1}{2}\right]$, shows that $S(n) \le I\left(n - \frac{1}{2}\right)$, while trapezium comparison on intervals $[r, r+1]$ gives $S^*(n) \ge I(n)$, where

$$S^*(n) = \tfrac{1}{2} f(n) + S(n+1).$$

In general, both these estimations give a much closer approximation to the tail of the series than simple integral comparison. From the stated inequalities, we might expect $S(n)/I\left(n - \frac{1}{2}\right)$ to increase, and $S^*(n)/I(n)$ to decrease.

We show that statements of this sort do indeed hold, and can be derived from our earlier theorems. However, the correct hypotheses are those of the earlier theorems, not simply that $f(x)$ is decreasing or convex. Indeed, cases of the opposite, “unexpected" type can occur.

The link is provided by the following lemma. Given a convergent series $\sum_{n=1}^{\infty} a_n$, we write $A_{(n)} = \sum_{k=n}^{\infty} a_k$ (with similar notation for b_n, etc.).

Lemma 3.1. *Suppose that $a_n > 0,\ b_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent. If a_n/b_n increases (or decreases) for $n \ge n_0$, then so does $A_{(n)}/B_{(n)}$.*

Proof. Write $a_n = c_n b_n$ and $A_{(n)} = K_n B_{(n)}$. Assume that (c_n) is increasing. Then $A_{(n)} \ge c_n B_{(n)}$, so $K_n \ge c_n$. Writing

$$A_{(n)} = a_n + A_{(n)} = c_n b_n + K_{n+1} B_{(n+1)},$$

one deduces easily that $A_{(n)} \le K_{n+1} B_{(n)}$, so that $K_n \le K_{n+1}$. □

Theorem 3.2. *Suppose that f satisfies (A1), (A2), (A3), (A4) and, for some n_0, that $f'(x)/f(x)$ is increasing for $x \ge n_0$. Then $S(n)/I(n)$ is decreasing and $S(n+1)/I(n)$ is increasing for $n \ge n_0$. The opposite applies if $f'(x)/f(x)$ is decreasing.*

Proof. Let

$$b_n = \int_n^{n+1} f(t)\, dt,$$

so that $B_{(n)} = I(n)$. Assume that $f'(x)/f(x)$ is increasing. By Proposition 2.3, $b_n/f(n)$ is increasing and $b_n/f(n+1)$ is decreasing. So by Lemma 3.1, $I(n)/S(n)$ is increasing and $I(n)/S(n+1)$ decreasing. □

Corollary 3.3. *(*[5, Remark 4.10] *and* [7, Proposition 6]*) Let $f(x) = 1/x^{p+1}$, where $p > 0$. Then (with the same notation) $n^p S(n)$ decreases with n, and $n^p S(n+1)$ increases.*

Proof. Then $f'(x)/f(x) = -(p+1)/x$, which is increasing, and $I(n) = 1/px^p$. □

Here $S(n)$ is the tail of the series for $\zeta(p+1)$, and we deduce (for example) that $\sup_{n\ge 1} n^p S(n) = S(1) = \zeta(p+1)$. In [7, Theorem 7], this is exactly the computation needed to evaluate the norm of the averaging (alias Cesaro) operator on the space $\ell_1(w)$, with $w_n = 1/n^p$. In [5, sections 4, 10], it is an important step in establishing the “factorized" Hardy and Copson inequalities.

In the same way, one obtains the following result for the series $\sum_{n=1}^{\infty}(\log n)/n^{p+1} = -\zeta'(p+1)$; we omit the details.

Corollary 3.4. *Let* $f(x) = (\log x)/x^{p+1}$, *where* $p > 0$. *Let* $r = \max[1, 2/(p+1)]$. *For* $n \geq e^r$, $n^p S(n)/(1 + p\log n)$ *decreases with* n, *and* $n^p S(n+1)/(1+p\log n)$ *increases.*

We now formulate the theorems deriving from our earlier results on symmetric ratios.

Theorem 3.5. *Suppose that* f *satisfies (A1), (A2), (A3) and (A4′). If* $f''(x)/f(x)$ *is decreasing (or increasing) for* $x \geq n_0 - \frac{1}{2}$, *then* $S(n)/I\left(n - \frac{1}{2}\right)$ *increases (or decreases) for* $n \geq n_0$.

Proof. Let

$$b_n = \int_{n-1/2}^{n+1/2} f(t)\, dt.$$

Then $B_{(n)} = I\left(n - \frac{1}{2}\right)$. If $f''(x)/f(x)$ is decreasing, then, by Proposition 2.11, $b_n/f(n)$ is decreasing. By Lemma 3.1, it follows that $I\left(n - \frac{1}{2}\right)/S(n)$ is decreasing. □

Corollary 3.6. *Let* $f(x) = 1/x^{p+1}$, *where* $p > 0$. *Then* $\left(n - \frac{1}{2}\right)^p S(n)$ *increases with* n. *Further, we have*

$$S(n+1) \geq \frac{\left(n - \frac{1}{2}\right)^p}{n^{p+1}\left[\left(n + \frac{1}{2}\right)^p - \left(n - \frac{1}{2}\right)^p\right]}.$$

Proof. The first statement is a case of Theorem 3.5, and the second one is an algebraic rearrangement of $(n - \frac{1}{2})^p S(n) \leq (n + \frac{1}{2})^p S(n+1)$. □

This strengthens the second statement in Corollary 3.3.

Theorem 3.7. *Suppose that* f *satisfies (A1), (A2), (A3) and (A4′). Let* $S^*(n) = \frac{1}{2}f(n) + S(n+1)$. *If* $f''(x)/f(x)$ *is decreasing (or increasing) for* $x \geq n_0$, *then* $S^*(n)/I(n)$ *decreases (or increases) for* $n \geq n_0$.

Proof. Similar, with

$$a_n = \frac{1}{2}\big(f(n) + f(n+1)\big), \qquad b_n = \int_n^{n+1} f(t)\, dt,$$

and applying Proposition 2.15 instead of Proposition 2.11. □

For the case $f(x) = 1/x^{p+1}$, it is easy to show that $S(n)/S^*(n)$ is decreasing. Hence Theorem 3.7 strengthens the first statement in Corollary 3.3.

Remark 3.8. If $f(x) = 1/x^{p+1}$, then $f'(x)/f(x)$ is increasing and $f''(x)/f(x)$ is decreasing. A case of the opposite type is $f(x) = xe^{-x}$, for which $f'(x)/f(x) = 1/x - 1$ and $f''(x)/f(x) = 1 - 2/x$. Note that the corresponding series is the power series $\sum ny^n$, with $y = e^{-1}$. Of course, for series of this type, $I(n)$ is not asymptotically equivalent to $S(n)$; in this case, one finds that $S(n)/I(n) \to e/(e-1)$ and $S(n+1)/I(n) \to 1/(e-1)$ as $n \to \infty$.

Finite sums. Clearly, the same reasoning can be applied to finite sums. Write $A_n = \sum_{j=1}^n a_j$. The statement corresponding to Lemma 3.1 is: if a_n/b_n is increasing (or decreasing), then so is A_n/B_n. A typical conclusion is:

Proposition 3.9. *Let* f *be strictly positive and differentiable on* $(0, \infty)$. *Write*

$$F(n) = \sum_{j=1}^n f(j), \qquad J(n) = \int_0^n f(t)\, dt.$$

If $f'(x)/f(x)$ *is increasing (or decreasing), then so is* $F(n)/J(n)$.

Proof. Let $b_n = \int_{n-1}^{n} f$, so that $B_n = J(n)$. If $f'(x)/f(x)$ is increasing, then $b_n/f(n)$ is decreasing, so $J(n)/F(n)$ is decreasing. □

Corollary 3.10. *(*[4, p. 59], [6, Proposition 3]*) If $a_n = 1/n^p$, where $0 < p < 1$, then A_n/n^{1-p} is increasing.*

4. TAILS OF SERIES: CONTINUOUS VERSION

We continue to assume that f is a function satisfying (A1), (A2), (A3) and (A4), and to write $I(x) = \int_x^\infty f(t)dt$. The previous definition of $S(n)$ is extended to a real variable x by defining

$$S(x) = \sum_{n=0}^{\infty} f(x+n).$$

For any $x_0 > 0$, integral comparison ensures uniform convergence of this series for $x \geq x_0$. Clearly, $S(x)$ is decreasing and tends to 0 as $x \to \infty$. Also, $S(x) - S(x+1) = f(x)$.

When $f(x) = 1/x^p$, our $S(x)$ is the "Hurwitz zeta function" $\zeta(p, x)$, which has applications in analytic number theory [2, chapter 12]. Note that $\zeta(p, 1) = \zeta(p)$ and $\zeta'(p, x) = -p\zeta(p+1, x)$.

Under our assumptions, $f'(x) \leq 0$ for $x > x_0$ and $\int_x^\infty f'(t)dt = -f(x)$. We make the following further assumption:

(A5) $f'(x)$ *is increasing on some interval* $[x_1, \infty)$.

This ensures that $\sum_{n=0}^{\infty} f'(x+n)$ is uniformly convergent for $x \geq x_0$, and hence that $S'(x)$ exists and equals the sum of this series. (An alternative would be to assume that f is an analytic complex function on some open region containing the positive real axis.)

We shall establish results analogous to the theorems of Section 3, by somewhat different methods. Unlike the discrete case, there is a simple expression for $I(x)$ in terms of $S(x)$:

Lemma 4.1. *With notation as above, we have*

$$I(x) = \int_x^{x+1} S(t)dt.$$

Proof. Let $X > x+1$. Then

$$\begin{aligned}\int_x^X f(t)\,dt &= \int_x^X [S(t) - S(t+1)]dt \\ &= \int_x^X S(t)dt - \int_{x+1}^{X+1} S(t)dt \\ &= \int_x^{x+1} S(t)dt - \int_X^{X+1} S(t)dt \\ &\to \int_x^{x+1} S(t)dt \quad \text{as } X \to \infty\end{aligned}$$

since $S(t) \to 0$ as $t \to \infty$. □

So $I(x)/S(x)$ is already a ratio of the type considered in Section 2, with $S(x)$ as the integrand. There is no need (and indeed no obvious opportunity) to use Lemma 3.1 or its continuous analogue. Instead, we apply the ideas of Section 2 to $S(x)$ instead of $f(x)$. This will require some extra work. We continue to write

$$W(x, y) = f(x)f'(y) - f'(x)f(y).$$

We need to examine

$$W_S(x, y) = S(x)S'(y) - S'(x)S(y).$$

Lemma 4.2. *With this notation, we have*

$$W_S(x,y) = \sum_{n=0}^{\infty} W(x+n, y+n) + \sum_{m<n} \big(W(x+m, y+n) + W(x+n, y+m)\big).$$

Proof. We have

$$W_S(x,y) = \left(\sum_{m=0}^{\infty} f(x+m)\right)\left(\sum_{n=0}^{\infty} f'(y+n)\right) - \left(\sum_{m=0}^{\infty} f'(x+m)\right)\left(\sum_{n=0}^{\infty} f(y+n)\right).$$

Since the terms of each series are ultimately of one sign, we can multiply the series and rearrange. For fixed n, the terms with $m = n$ equate to $W(x+n, y+n)$. For fixed m, n with $m \neq n$, the corresponding terms equate to $W(x+m, y+n)$. ☐

Lemma 4.3. *If $f'(x)/f(x)$ is increasing for $x > 0$, then for $0 < t < c$,*

(i) $f(c-t)f(c+t)$ *increases with* t,
(ii) $W(c-t, c+t)$ *increases with* t.

Proof. Write $f'(x)/f(x) = q(x)$. Then

$$W(c-t, c+t) = f(c-t)f(c+t)\big(q(c+t) - q(c-t)\big).$$

This is non-negative when $t > 0$. Also, the derivative of $f(c-t)f(c+t)$ is $W(c-t, c+t)$, hence statement (i) holds. By the above expression, statement (ii) follows. ☐

Theorem 4.4. *Suppose that $f(x)$ satisfies (A1), (A2), (A3), (A4) and (A5), and that $f'(x)/f(x)$ is increasing for $x > 0$. Then:*

(i) $S'(x)/S(x)$ *is increasing for* $x > 0$,
(ii) $S(x)/I(x)$ *is decreasing and* $S(x)/I(x-1)$ *is increasing.*

Opposite conclusions hold if $f'(x)/f(x)$ is decreasing.

Proof. We show that $W_S(x,y) \geq 0$ when $x < y$. Then (i) follows, by the implication (ii) ⇒ (i) in Lemma 2.2, and (ii) follows in the same way as in Proposition 2.3. It is sufficient to prove the stated inequality when $y - x < 1$. By Lemma 2.2, $W(x+n, y+n) \geq 0$ for all n. Now fix $m < n$. Note that $y + m < x + n$, since $y - x < 1$. In Lemma 4.3, take

$$c = \tfrac{1}{2}(x+y+m+n), \qquad t = c - (x+m), \qquad t' = c - (y+m).$$

Then $0 < t' < t < c$, also $c + t = y + n$ and $c + t' = x + m$. We obtain

$$W(x+m,\, y+n) \geq W(y+m,\, x+n),$$

hence $W(x+m, y+n) + W(x+n, y+m) \geq 0$. The required inequality follows, by Lemma 4.2. ☐

Corollary 4.5. *Let $p > 1$, and let $\zeta(p,x) = \sum_{n=0}^{\infty}(x+n)^{-p}$. Then $x^{p-1}\zeta(p,x)$ decreases with x, and $(x-1)^{p-1}\zeta(p,x)$ increases. Also, $\zeta(p+1,x)/\zeta(p,x)$ decreases.*

We now establish the continuous analogue of Theorem 3.5, which will lead to a sharper version of the second statement in Corollary 4.5. First, another lemma.

Lemma 4.6. *Suppose that $f'(x)/f(x)$ is increasing and $f''(x)/f(x)$ is decreasing for $x > 0$. If $0 < b < a$, then*

$$W(x-a,\, x+a) - W(x-b,\, x+b)$$

decreases with x for $x > a$.

Proof. Write $f''(x)/f(x) = r(x)$. As in the proof of Lemma 2.9, we have

$$\frac{d}{dx}W(x-a,\, x+a) = f(x-a)f(x+a)\big(r(x+a) - r(x-a)\big),$$

and similarly for $W(x-b, x+b)$. Since $r(x)$ is decreasing, we have

$$r(x-a) - r(x+a) \geq r(x-b) - r(x+b) \geq 0.$$

Also, since $f'(x)/f(x)$ is increasing, Lemma 4.3 gives

$$f(x-a)f(x+a) \geq f(x-b)f(x+b).$$

The statement follows. □

Theorem 4.7. *Suppose that $f(x)$ satisfies (A1), (A2), (A3), (A4′) and (A5), and also that $f'(x)/f(x)$ is increasing and $f''(x)/f(x)$ is decreasing for $x > 0$. Then(i) $S''(x)/S(x)$ is decreasing for $x > 0$, and (ii) $S(x)/I\left(x - \frac{1}{2}\right)$ is increasing for $x > \frac{1}{2}$. The opposite holds if the hypotheses are reversed.*

Proof. Recall that, by Lemma 4.1,

$$I\left(x - \frac{1}{2}\right) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} S(t)\, dt.$$

The statements will follow, by Lemma 2.9 and Proposition 2.11, if we can show that $W_S(x, x+u)$ decreases with x for each fixed u in $\left(0, \frac{1}{2}\right)$. We use the expression in Lemma 4.2, with $y = x + u$. By Lemma 2.9, $W(x+n, x+n+u)$ decreases with x for each n. Now take $m < n$. We apply Lemma 4.6, with

$$z = x + \frac{1}{2}(m+n+u), \qquad a = \frac{1}{2}(n-m+u), \qquad b = \frac{1}{2}(n-m-u).$$

Then $0 < b < a$ (since $n - m \geq 1$), and

$$z - a = x + m, \quad z + a = x + n + u, \quad z - b = x + m + u, \quad z + b = x + n,$$

so the lemma shows that

$$W(x+m, x+n+u) + W(x+n, x+m+u)$$

decreases with x, as required. □

Corollary 4.8. *The function $\left(x - \frac{1}{2}\right)^{p-1} \zeta(p, x)$ is increasing for $x > \frac{1}{2}$.*

Remark 4.9. In Theorem 4.7, unlike Theorem 3.5, we assumed a hypothesis on $f'(x)/f(x)$ as well as $f''(x)/f(x)$. We leave it as an open problem whether this hypothesis can be removed.

Remark 4.10. Lemmas 4.3 and 4.6 both involve a symmetrical perturbation of the two variables. Our assumptions do not imply that $W(x, y)$ is a monotonic function of y for fixed x. For example, if $f(x) = 1/x^2$, then $W(1, y) = 2/y^2 - 2/y^3$, which increases for $0 < y \leq 3/2$ and then decreases.

Finally, the continuous analogue of Theorem 3.7:

Theorem 4.11. *Let*

$$S^*(x) = \frac{1}{2}f(x) + \sum_{n=1}^{\infty} f(x+n).$$

If f satisfies the hypotheses of Theorem 4.7, then $S^*(x)/I(x)$ is decreasing.

Proof. Note that $S^*(x) = \frac{1}{2}S(x) + \frac{1}{2}S(x+1)$. By Theorem 4.7, $S''(x)/S(x)$ is decreasing. By Lemma 4.1 and Proposition 2.15, it follows that $I(x)/S^*(x)$ is increasing. □

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