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# A NOTE ON COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. In this note, several characterizations of commutativity of $\mathcal{A}$ are given. For instance, it is shown that $\mathcal{A}$ is commutative if

$$\|AB\| = \|BA\|$$

for all $A, B \in \mathcal{A}$, or if the spectral radius on $\mathcal{A}$ is a norm.

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Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. In this note, several characterizations of the commutativity of $\mathcal{A}$ are studied.

The following theorem is a simple characterization of commutativity in terms of norm inequalities, whose proof depends on complex analysis as the well-known one for the Fuglede-Putnum theorem, for instance, see [2, p. 278].

**Theorem 1.** *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|_0$. If there is a norm $\|\cdot\|$ on $\mathcal{A}$ and positive constants $\gamma, \kappa$ such that*

$$\|A\| \leqq \gamma \|A\|_0, \quad \|AB\| \leqq \kappa \|BA\|$$

*for all $A, B \in \mathcal{A}$, then $\mathcal{A}$ is commutative, that is, $AB = BA$ for all $A, B \in \mathcal{A}$.*

Before giving a proof, we recall the definition of $e^A$ for $A \in \mathcal{A}$:

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n \in \mathcal{A}.$$

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The assumption that $\mathcal{A}$ is a complete, unital normed algebra with a submultiplicative norm guarantees the convergence of this infinite series in $\mathcal{A}$ and implies

$$\frac{d}{dz}e^{zA} = Ae^{zA} \quad (z \in \mathbb{C}).$$

*Proof.* Let $A, B \in \mathcal{A}$. Let us consider the normed space $(\mathcal{A}, \|\cdot\|)$. For each bounded linear functional $\varphi$ on this normed space, we define a complex-valued function $f$ on $\mathbb{C}$ by

$$f(z) := \varphi(e^{zA}Be^{-zA}) \quad (z \in \mathbb{C}).$$

Then the first assumption of $\|\cdot\|$ guarantees that $f$ is an entire analytic function. $f$ is also bounded: in fact, by the second assumption

$$\begin{aligned} |f(z)| &\leqq \|\varphi\|\|e^{zA}Be^{-zA}\| \\ &\leqq \kappa\|\varphi\|\|Be^{-zA}\cdot e^{zA}\| \\ &= \kappa\|\varphi\|\|B\| < \infty \qquad (z \in \mathbb{C}). \end{aligned}$$

Thus, by the Liouville theorem, $f$ is constant. Hence,

$$0 = f'(z) = \varphi\left((Ae^{zA})Be^{-zA} + e^{zA}B(-Ae^{-zA})\right).$$

Putting $z = 0$ yields

$$\varphi(AB - BA) = 0$$

for each bounded linear functional $\varphi$ on $\mathcal{A}$. By the Hahn-Banach theorem, $AB = BA$ and the proof is completed. □

**Remark 2.**

(1) By considering completion, we find it sufficient to assume in Theorem 1 that $\mathcal{A}$ is a unital normed algebra over $\mathbb{C}$ with submultiplicative norm $\|\cdot\|_0$.

(2) The assumption that

$$\|AB\| \leqq \kappa\|BA\|$$

for all $A, B \in \mathcal{A}$ can be replaced with a weaker one

$$\|SAS^{-1}\| \leqq \kappa\|A\|$$

for all $A \in \mathcal{A}$ and all invertible $S \in \mathcal{A}$, or even further

$$\|e^{zA}Be^{-zA}\| \leqq \kappa\|B\|$$

for all $A, B \in \mathcal{A}$ and all $z \in \mathbb{C}$. In fact, it is essential to the proof of Theorem 1 that for given $A, B$

$$\sup\{\|e^{zA}Be^{-zA}\| : z \in \mathbb{C}\} < \infty.$$

Theorem 1 and Remark 1 (2) yield:

**Corollary 3.** *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. Suppose that there is a positive constant $\gamma$ such that*

$$\|AB\| \leqq \gamma\,\|BA\|$$

*for all $A, B \in \mathcal{A}$. Then $\mathcal{A}$ is commutative. In particular, if $\|AB\| = \|BA\|$ for all $A, B \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

**Corollary 4** ([1, Exercise IV 4.1])**.** *On the set of all complex $n$-square matrices for $n \geqq 2$ no norm is invariant under all similarity transformations.*

See [1, p.102] for similarity transformations.

**Corollary 5.** *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. If the spectral radius is a norm on $\mathcal{A}$, then $\mathcal{A}$ is commutative.*

This follows from Theorem 1 and the properties of the spectral radius $r(A)$ that $r(AB) = r(BA)$ and $r(A) \leqq \|A\|$ for $A, B \in \mathcal{A}$.

**Remark 6.** There is a unital Banach algebra whose spectral radius is not a norm but a semi-norm. This semi-norm condition is not sufficient for commutativity.

In fact, let $\mathcal{A}$ $(\subseteqq M_n(\mathbb{C}))$ be the set of upper triangular matrices whose diagonal entries are identical; $\mathcal{A}$ consists of $A := (a_{ij}) \in M_n(\mathbb{C})$ such that $a_{11} = a_{22} = \cdots = a_{nn} (=: \alpha)$ and $a_{ij} = 0 \quad (i > j)$. For this $A$, $r(A) = |\alpha|$ and the spectral radius on $\mathcal{A}$ is a semi-norm. Therefore, the unital Banach algebra $\mathcal{A}$ is a non-commutative example.

## REFERENCES

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