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COEFFICIENT ESTIMATES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. For some real α ($\alpha > 1$), two subclasses $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of analytic fuctions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in $\mathbb{U}$ are introduced. The object of the present paper is to discuss the coefficient estimates for functions $f(z)$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \qquad (z \in \mathbb{U})$$

for some α ($\alpha > 1$). And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \qquad (z \in \mathbb{U})$$

for some α ($\alpha > 1$). Then, we see that $f(z) \in \mathcal{N}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}(\alpha)$.

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Remark 1.1. For $1 < \alpha \leq \frac{4}{3}$, the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced by Uralegaddi et al. [3].

Remark 1.2. The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ correspond to the case $k = 2$ of the classes $\mathcal{M}_k(\alpha)$ and $\mathcal{N}_k(\alpha)$, respectively, which were investigated recently by Owa and Srivastava [1].

We easily see that

Example 1.1.

(i) $f(z) = z(1-z)^{2(\alpha-1)} \in \mathcal{M}(\alpha)$.

(ii) $g(z) = \frac{1}{2\alpha-1}\left\{1-(1-z)^{2\alpha-1}\right\} \in \mathcal{N}(\alpha)$.

2. INCLUSION THEOREMS INVOLVING COEFFICIENT INEQUALITIES

In this section we derive sufficient conditions for $f(z)$ to belong to the aforementioned function classes, which are obtained by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\left\{(n-k)+|n+k-2\alpha|\right\}|a_n| \leqq 2(\alpha-1)$$

for some k $(0 \leqq k \leqq 1)$ and some α $(\alpha > 1)$, then $f(z) \in \mathcal{M}(\alpha)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty}\left\{(n-k)+|n+k-2\alpha|\right\}|a_n| \leqq 2(\alpha-1) \tag{2.1}$$

for $f(z) \in \mathcal{A}$.

It suffices to show that

$$\left|\frac{\frac{zf'(z)}{f(z)}-k}{\frac{zf'(z)}{f(z)}-(2\alpha-k)}\right| < 1 \qquad (z \in \mathbb{U}).$$

We note that

$$\begin{aligned}\left|\frac{\frac{zf'(z)}{f(z)}-k}{\frac{zf'(z)}{f(z)}-(2\alpha-k)}\right| &= \left|\frac{1-k+\sum_{n=2}^{\infty}(n-k)a_nz^{n-1}}{1+k-2\alpha+\sum_{n=2}^{\infty}(n+k-2\alpha)a_nz^{n-1}}\right| \\ &\leqq \frac{1-k+\sum_{n=2}^{\infty}(n-k)|a_n||z|^{n-1}}{2\alpha-1-k-\sum_{n=2}^{\infty}|n+k-2\alpha|\,|a_n||z|^{n-1}} \\ &< \frac{1-k+\sum_{n=2}^{\infty}(n-k)|a_n|}{2\alpha-1-k-\sum_{n=2}^{\infty}|n+k-2\alpha|\,|a_n|}.\end{aligned}$$

The last expression is bounded above by 1 if

$$1-k+\sum_{n=2}^{\infty}(n-k)|a_n| \leqq 2\alpha-1-k-\sum_{n=2}^{\infty}|n+k-2\alpha|\,|a_n|$$

which is equivalent to our condition:

$$\sum_{n=2}^{\infty}\{(n-k)+|n+k-2\alpha|\}|a_n| \leqq 2(\alpha-1)$$

of the theorem. This completes the proof of the theorem. □

If we take $k = 1$ and some α $\left(1 < \alpha \leqq \frac{3}{2}\right)$ in Theorem 2.1, then we have

Corollary 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq \alpha - 1$$

for some α $\left(1 < \alpha \leqq \frac{3}{2}\right)$, then $f(z) \in \mathcal{M}(\alpha)$.

Example 2.1. The function $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha-1)}{n(n+1)(n-k+|n+k-2\alpha|)} z^n$$

belongs to the class $\mathcal{M}(\alpha)$.

For the class $\mathcal{N}(\alpha)$, we have

Theorem 2.3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n-k+1+|n+k-2\alpha|)|a_n| \leqq 2(\alpha-1) \tag{2.2}$$

for some k $(0 \leqq k \leqq 1)$ and some α $(\alpha > 1)$, then $f(z)$ belongs to the class $\mathcal{N}(\alpha)$.

Corollary 2.4. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq \alpha - 1$$

for some α $\left(1 < \alpha \leqq \frac{3}{2}\right)$, then $f(z) \in \mathcal{N}(\alpha)$.

Example 2.2. The function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha-1)}{n^2(n+1)(n-k+|n+k-2\alpha|)} z^n$$

belongs to the class $\mathcal{N}(\alpha)$.

Further, denoting by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ consisting of all starlike functions of order α, and of all convex functions of order α, respectively (see [2]), we derive

Theorem 2.5. *If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality (2.1) for some α $\left(1 < \alpha \leqq \frac{k+2}{2} \leqq \frac{3}{2}\right)$, then $f(z) \in \mathcal{S}^*\left(\frac{4-3\alpha}{3-2\alpha}\right)$. If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality (2.2) for some α $\left(1 < \alpha \leqq \frac{k-2}{2} \leqq \frac{3}{2}\right)$ then $f(z) \in \mathcal{K}\left(\frac{4-3\alpha}{3-2\alpha}\right)$.*

Proof. For some α $\left(1 < \alpha \leqq \frac{k+2}{2} \leqq \frac{3}{2}\right)$, we see that the coefficient inequality (2.1) implies that

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq \alpha - 1.$$

It is well-known that if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}|a_n| \leqq 1$$

for some β $(0 \leqq \beta < 1)$, then $f(z) \in \mathcal{S}^*(\beta)$ by Silverman [2]. Therefore, we have to find the smallest positive β such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta}|a_n| \leqq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha-1}|a_n| \leqq 1.$$

This gives that

$$\beta \leqq \frac{(2-\alpha)n-\alpha}{n-2\alpha+1} \tag{2.3}$$

for all $n = 2, 3, 4, \cdots$. Noting that the right-hand side of the inequality (2.3) is increasing for n, we conclude that

$$\beta \leqq \frac{4-3\alpha}{3-2\alpha},$$

which proves that $f(z) \in \mathcal{S}^*\left(\frac{4-3\alpha}{3-2\alpha}\right)$. Similarly, we can show that if $f(z) \in \mathcal{A}$ satisfies (2.2), then $f(z) \in \mathcal{K}\left(\frac{4-3\alpha}{3-2\alpha}\right)$. □

Our result for the coefficient estimates of functions $f(z) \in \mathcal{M}(\alpha)$ is contained in

Theorem 2.6. *If $f(z) \in \mathcal{M}(\alpha)$, then*

$$|a_n| \leqq \frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{(n-1)!} \qquad (n \geqq 2). \tag{2.4}$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1}$$

for $f(z) \in \mathcal{M}(\alpha)$. Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0\,(z \in \mathbb{U})$. Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then $|p_n| \leqq 2\,(n \geqq 1)$. Since

$$\alpha f(z) - zf'(z) = (\alpha - 1)p(z)f(z),$$

we obtain that

$$(1-n)a_n = (\alpha-1)(p_{n-1} + a_2 p_{n-2} + a_3 p_{n-3} + \cdots + a_{n-1}p_1).$$

If $n = 2$, then $-a_2 = (\alpha - 1)p_1$ implies that

$$|a_2| = (\alpha - 1)|p_1| \leqq 2\alpha - 2.$$

Thus the coefficient estimate (2.4) holds true for $n = 2$. Next, suppose that the coefficient estimate

$$|a_k| \leqq \frac{\prod_{j=2}^{k}(j+2\alpha-4)}{(k-1)!}$$

is true for all $k = 2, 3, 4, \cdots, n$. Then we have that

$$-na_{n+1} = (\alpha-1)(p_n + a_2 p_{n-1} + a_3 p_{n-2} + \cdots + a_n p_1),$$

so that

$$\begin{aligned} n|a_{n+1}| &\leqq (2\alpha-2)(1+|a_2|+|a_3|+\cdots+|a_n|) \\ &\leqq (2\alpha-2)\left(1+(2\alpha-2)+\frac{(2\alpha-2)(2\alpha-1)}{2!}+\cdots+\frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{(n-1)!}\right) \\ &= (2\alpha-2)\left(\frac{(2\alpha-1)2\alpha(2\alpha+1)\cdots(2\alpha+n-4)}{(n-2)!}\right. \\ &\qquad\qquad\left.+\frac{(2\alpha-2)(2\alpha-1)2\alpha\cdots(2\alpha+n-4)}{(n-1)!}\right) \\ &= \frac{\Pi_{j=2}^{n+1}(j+2\alpha-4)}{(n-1)!}. \end{aligned}$$

Thus, the coefficient estimate (2.4) holds true for the case of $k = n+1$. Applying the mathematical induction for the coefficient estimate (2.4), we complete the proof of Theorem 2.6. □

For the functions $f(z)$ belonging to the class $\mathcal{N}(\alpha)$, we also have

Theorem 2.7. *If* $f(z) \in \mathcal{N}(\alpha)$, *then*

$$|a_n| \leqq \frac{\prod_{j=2}^{n}(j+2\alpha-4)}{n!} \qquad (n \geqq 2).$$

Remark 2.8. We can not show that Theorem 2.6 and Theorem 2.7 are sharp. If we can prove that Theorem 2.6 is sharp, then the sharpness of Theorem 2.7 follows.

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