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PARTIAL SUMS OF CERTAIN MEROMORPHIC $p-$VALENT FUNCTIONS

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ABSTRACT. In this paper we establish some results concerning the partial sums of meromorphic p-valent starlike functions and meromorphic p-valent convex functions.

Key words and phrases: Partial sums, Meromorphic p-valent starlike functions, Meromorphic p-valent convex functions.

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1. INTRODUCTION

Let $\sum(p)$ $(p \in \mathbb{N} = \{1, 2, \dots\})$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p-1} z^{k+p-1} \qquad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic and $p-$valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. A function $f(z)$ in $\sum(p)$ is said to belong to $\sum^*(p,\alpha)$, the class of meromorphically p-valent starlike functions of order α $(0 \le \alpha < p)$, if and only if

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (0 \le \alpha < p; z \in U = U^* \cup \{0\}). \tag{1.2}$$

A function $f(z)$ in $\sum(p)$ is said to belong to $\sum_k(p,\alpha)$, the class of $p-$valent convex functions of order $\alpha(0 \le \alpha < p)$, if and only if

$$-\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (0 \le \alpha < p; z \in U). \tag{1.3}$$

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It follows from (1.2) and (1.3) that

$$f(z) \in \sum\nolimits_k (p,\alpha) \Longleftrightarrow -\frac{zf'(z)}{p} \in \sum\nolimits^* (p,\alpha). \tag{1.4}$$

The classes $\sum^*(p,\alpha)$and $\sum_k(p,\alpha)$ were studied by Kumar and Shukla [6]. A sufficient condition for a function $f(z)$ of the form (1.1) to be in $\sum^*(p,\alpha)$ is that

$$\sum_{k=1}^{\infty} (k+p-1+\alpha)\,|a_{k+p-1}| \leq (p-\alpha) \tag{1.5}$$

and to be in $\sum_k(p,\alpha)$ is that

$$\sum_{k=1}^{\infty} \left(\frac{k+p-1}{p}\right)(k+p-1+\alpha)\,|a_{k+p-1}| \leq (p-\alpha). \tag{1.6}$$

Further, we note that these sufficient conditions are also necessary for functions of the form (1.1) with positive or negative coefficients (see [1], [2], [5], [9], [14] and [15]). Recently , Silverman [11] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Also, Li and Owa [7] obtained the sharp radius which for the normalized univalent functions in U, the partial sums of the well known Libera integral operator [8] imply starlikeness. Further , for various other interesting developments concerning partial sums of analytic univalent functions (see [3], [10], [12], [13] and [16]).

Recently , Cho and Owa [4] have investigated the ratio of a function of the form (1.1) (with $p=1$) to its sequence of partial sums $f_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k$ when the coefficients are sufficiently small to satisfy either condition (1.5) or (1.6) with $p=1$. Also Cho and Owa [4] have determined sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}$, $\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\}$, and $\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\}$.

In this paper, applying methods used by Silverman [11] and Cho and Owa [4], we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums

$$f_{n+p-1}(z) = \frac{1}{z^p} + \sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+p-1}$$

when the coefficients are sufficiently small to satisfy either condition (1.5) or (1.6). More precisely, we will determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)}{f_{n+p-1}(z)}\right\}$, $\operatorname{Re}\left\{\frac{f_{n+p-1}(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f'(z)}{f_{n+p-1}'(z)}\right\}$, and $\operatorname{Re}\left\{\frac{f_{n+p-1}'(z)}{f'(z)}\right\}$.

In the sequel, we will make use of the well-known result that $\operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0$ $(z \in U)$ if and only if $w(z) = \sum_{k=1}^{\infty} c_k z^k$ satisfies the inequality $|w(z)| \leq |z|$. Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by

$$f_{n+p-1}(z) = \frac{1}{z^p} + \sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+p-1}.$$

2. MAIN RESULTS

Theorem 2.1. *If f of the form (1.1) satisfies condition* (1.5), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_{n+p-1}(z)}\right\} \geq \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \qquad (z \in U). \tag{2.1}$$

The result is sharp for every n and p, with extremal function

$$f(z) = \frac{1}{z^p} + \frac{p-\alpha}{n+2p-1+\alpha} z^{n+2p-1} \qquad (n \geq 0; p \in \mathbb{N}). \tag{2.2}$$

Proof. We may write

$$\begin{aligned}
\frac{n+2p-1+\alpha}{p-\alpha} &\left[\frac{f(z)}{f_{n+p-1}(z)} - \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \right] \\
&= \frac{1 + \sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1} + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}{1 + \sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1}} \\
&= \frac{1+A(z)}{1+B(z)}.
\end{aligned}$$

Set $\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)}$, so that $w(z) = \frac{A(z)-B(z)}{2+A(z)+B(z)}$. Then

$$w(z) = \frac{\left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}{2 + 2\sum_{k=1}^{n+p-1} a_{k+p-1} z^{k+2p-1} + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} a_{k+p-1} z^{k+2p-1}}$$

and

$$|w(z)| \leq \frac{\left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} |a_{k+p-1}|}{2 - 2\sum_{k=1}^{n+p-1} |a_{k+p-1}| - \left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} |a_{k+p-1}|}.$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} |a_{k+p-1}| \leq 2 - 2\sum_{k=1}^{n+p-1} |a_{k+p-1}|,$$

which is equivalent to

$$\sum_{k=1}^{n+p-1} |a_{k+p-1}| + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right) \sum_{k=n+p}^{\infty} |a_{k+p-1}| \leq 1. \tag{2.3}$$

It suffices to show that the left hand side of (2.3) is bounded above by $\sum_{k=1}^{\infty} \left(\frac{k+p-1+\alpha}{p-\alpha}\right) |a_{k+p-1}|$, which is equivalent to

$$\sum_{k=1}^{n+p-1} \left(\frac{k+2\alpha-1}{p-\alpha}\right) |a_{k+p-1}| + \sum_{k=n+p}^{\infty} \left(\frac{k-n-p}{p-\alpha}\right) |a_{k+p-1}| \geq 0.$$

To see that the function f given by (2.2) gives the sharp result, we observe for $z = re^{\pi i/(n+3p-1)}$ that

$$\begin{aligned}
\frac{f(z)}{f_{n+p-1}(z)} &= 1 + \frac{p-\alpha}{n+2p-1+\alpha} z^{n+3p-1} \to 1 - \frac{p-\alpha}{n+2p-1+\alpha} \\
&= \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \qquad \text{when } r \to 1^-.
\end{aligned}$$

Therefore we complete the proof of Theorem 2.1. □

Theorem 2.2. *If f of the form (1.1) satisfies condition (1.6), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_{n+p-1}(z)}\right\} \geq \frac{(n+2p)(n+2p-2+\alpha)+(1-p)(1+p-\alpha)}{(n+2p-1)(n+2p-1+\alpha)} \qquad (z\in U). \tag{2.4}$$

The result is sharp for every n *and* p, *with extremal function*

$$f(z) = \frac{1}{z^p} + \frac{p(p-\alpha)}{(n+2p-1)(n+2p-1+\alpha)} z^{n+2p-1} \qquad (n\geq 0; p\in\mathbb{N}). \tag{2.5}$$

Proof. We write

$$\begin{aligned}
&\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \\
&\qquad\qquad \times\left[\frac{f(z)}{f_{n+p-1}(z)} - \frac{(n+2p)(n+2p-2+\alpha)+(1-p)(1+p-\alpha)}{(n+2p-1)(n+2p-1+\alpha)}\right] \\
&\quad = \frac{1+\sum_{k=1}^{n+p-1} a_{k+p-1}z^{k+2p-1} + \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)}\sum_{k=n+p}^{\infty} a_{k+p-1}z^{k+2p-1}}{1+\sum_{k=1}^{n+p-1} a_{k+p-1}z^{k+2p-1}} \\
&\quad = \frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$w(z) = \frac{\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)}\sum_{k=n+p}^{\infty} a_{k+p-1}z^{k+2p-1}}{2+2\sum_{k=1}^{n+p-1} a_{k+p-1}z^{k+2p-1} + \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)}\sum_{k=n+p}^{\infty} a_{k+p-1}z^{k+2p-1}}.$$

Now

$$|w(z)| \leq \frac{\frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)}\sum_{k=n+p}^{\infty} |a_{k+p-1}|}{2-2\sum_{k=1}^{n+p-1} |a_{k+p-1}| - \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)}\sum_{k=n+p}^{\infty} |a_{k+p-1}|} \leq 1,$$

if

$$\sum_{k=1}^{n+p-1} |a_{k+p-1}| + \frac{(n+2p-1)(n+2p-1+\alpha)}{p(p-\alpha)} \sum_{k=n+p}^{\infty} |a_{k+p-1}| \leq 1. \tag{2.6}$$

The left hand side of (2.6) is bounded above by

$$\sum_{k=1}^{\infty} \frac{(k+p-1)(k+p-1+\alpha)}{p(p-\alpha)} |a_{k+p-1}|$$

if

$$\begin{aligned}
&\frac{1}{p(p-\alpha)}\left\{\sum_{k=1}^{n+p-1} [(k+p-1)(k+p-1+\alpha)-p(p-\alpha)]\,|a_{k+p-1}|\right. \\
&\quad \left. + \sum_{k=n+p}^{\infty} [(k+p-1)(k+p-1+\alpha)-(n+2p-1)(n+2p-1+\alpha)]\,|a_{k+p-1}|\right\} \geq 0,
\end{aligned}$$

and the proof is completed. □

We next determine bounds for $\operatorname{Re}\left\{\frac{f_{n+p-1}(z)}{f(z)}\right\}$.

Theorem 2.3. (a) *If f of the form* (1.1) *satisfies condition* (1.5), *then*

$$\mathrm{Re}\left\{\frac{f_{n+p-1}(z)}{f(z)}\right\} \geq \frac{n+2p-1+\alpha}{n+3p-1} \qquad (z \in U). \tag{2.7}$$

(b) *If f of the form* (1.1) *satisfies condition* (1.6), *then*

$$\mathrm{Re}\left\{\frac{f_{n+p-1)}(z)}{f(z)}\right\} \geq \frac{(n+2p-1)(n+2p-1+\alpha)}{(n+2p-1)(n+2p)-n(1-\alpha)+(1-p)(1-p-\alpha)} \qquad (z \in U). \tag{2.8}$$

Equalities hold in (a) and (b) for the functions given by (2.2) *and* (2.5), *respectively.*

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted. We write

$$\begin{aligned}
\frac{(n+2p-1)}{(p-\alpha)}&\left[\frac{f_{n+p-1)}(z)}{f(z)} - \frac{n+2p-1+\alpha}{n+3p-1}\right] \\
&= \frac{1+\sum_{k=1}^{n+p-1} a_{k+p-1}z^{k+2p-1} - \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty} a_{k+p-1}z^{k+2p-1}}{1+\sum_{k=1}^{\infty} a_{k+p-1}z^{k+2p-1}} \\
&= \frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$|w(z)| \leq \frac{\left(\frac{n+3p-1}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}|}{2-2\sum_{k=1}^{n+p-1}|a_{k+p-1}| - \left(\frac{n+p-1+2\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{k=1}^{n+p-1}|a_{k+p-1}| + \left(\frac{n+2p-1+\alpha}{p-\alpha}\right)\sum_{k=n+p}^{\infty}|a_{k+p-1}| \leq 1. \tag{2.9}$$

Since the left hand side of (2.9) is bounded above by $\sum_{k=1}^{\infty}\frac{(n+p-1+\alpha)}{(p-\alpha)}|a_{k+p-1}|$, the proof is completed. □

We next turn to ratios involving derivatives.

Theorem 2.4. *If f of the form (1.1) satisfies condition (1.5), then*

$$\mathrm{Re}\left\{\frac{f'(z)}{f'_{n+p-1}(z)}\right\} \geq \frac{2p(n+2p-1)-\alpha(n+p-1)}{p(n+2p-1+\alpha)} \qquad (z \in U), \tag{2.10}$$

$$\mathrm{Re}\left\{\frac{f'_{n+p-1}(z)}{f'(z)}\right\} \geq \frac{p(n+2p-1+\alpha)}{\alpha(n+3p-1)} \qquad (z \in U; \alpha \neq 0). \tag{2.11}$$

The extremal function for the case (2.10) *is given by* (2.2) *and the extremal function for the case* (2.11) *is given by* (2.2) *with* $\alpha \neq 0$.

The proof of Theorem 2.4 follows the pattern of those in Theorem 2.1 and (a) of Theorem 2.3 and so the details may be omitted.

Remark 2.5. Putting $p = 1$ in Theorem 2.4, we obtain the following corollary:

Corollary 2.6. *If f of the form (1.1) (with $p=1$) satisfies condition (1.5) (with $p=1$), then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{2(n+1)-\alpha n}{n+1+\alpha} \qquad (z \in U), \tag{2.12}$$

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1+\alpha}{\alpha(n+2)} \qquad (z \in U; \alpha \neq 0). \tag{2.13}$$

The extremal function for the case (2.12) is given by (2.2) (with $p=1$) and the extremal function for the case (2.13) is given by (2.2) (with $p=1$ and $\alpha \neq 0$).

Remark 2.7. We note that Corollary 2.6 corrects the result obtained by Cho and Owa [4, Theorem 5].

Theorem 2.8. *If f of the form* (1.1) *satisfies condition* (1.6)*, then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f_{n+p-1}'(z)}\right\} \geq \frac{n+p-1+2\alpha}{n+2p-1+\alpha} \qquad (z \in U), \tag{2.14}$$

$$\operatorname{Re}\left\{\frac{f_{n+p-1}'(z)}{f'(z)}\right\} \geq \frac{n+2p-1+\alpha}{n+3p-1} \qquad (z \in U). \tag{2.15}$$

In both cases, the extremal function is given by (2.5).

Proof. It is well known that $f \in \sum_k(p,\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \sum^*(p,\alpha)$. In particular, f satisfies condition (1.6) if and only if $-\frac{zf'(z)}{p}$ satisfies condition (1.5). Thus, (2.14) is an immediate consequence of Theorem 2.1 and (2.15) follows directly from Theorem 2.3 (a). □

Remark 2.9. Putting $p=1$ in the above results we get the results obtained by Cho and Owa [4].

REFERENCES

[1] M.K. AOUF, On a class of meromorphic multivalent functions with positive coefficients, *Math. Japon.*, **35** (1990), 603–608.

[2] M.K. AOUF AND A.E. SHAMMAKY, A certain subclass of meromorphically $p-$valent convex functions with negative coefficients, *J. Approx. Theory and Appl.*, **1**(2) (2005), 157–177.

[3] L. BRICKMAN, D.J. HALLENBECK, T.H. MACGREGOR AND D. WILKEN, Convex hulls and extreme points of families of starlike and convex mappings, *Trans. Amer. Math. Soc.*, **185** (1973), 413–428.

[4] N.E. CHO AND S. OWA, Partial sums of certain meromorphic functions, *J. Ineq. Pure and Appl. Math.*, **5**(2) (2004), Art. 30. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=377`].

[5] H.E. DARWISH , M.K. AOUF AND G.S. SĂLĂGEAN, On Some classes of meromorphically p-valent starlike functions with positive coefficients, *Libertas Math.*, **20** (2000), 49–54.

[6] V. KUMAR AND S.L. SHUKLA, Certain integrals for classes of p-valent meromorphic functions, *Bull. Austral. Math. Soc.*, **25** (1982), 85–97.

[7] J.L. LI AND S. OWA, On partial sums of the Libera integral operator, *J. Math. Anal. Appl.*, **213** (1997), 444–454.

[8] R.J. LIBERA, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, **16** (1965), 755–758.

[9] M.L. MOGRA, Meromorphic multivalent functions with positive coefficients. I, *Math. Japon.*, **35**(1) (1990), 1–11.

[10] T. SHELL-SMALL, A note on partial sums of convex schlicht functions, *Bull. London Math. Soc.*, **2** (1970), 165–168.

[11] H. SILVERMAN, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, **209** (1997), 221–227.

[12] E.M. SILVIA, On partial sums of convex functions of order α, *Houston J. Math.*, **11** (1985), 397–404.

[13] R. SINGH AND S. SINGH , Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.*, **106** (1989), 145–152.

[14] H.M. SRIVASTAVA, H.M. HOSSEN AND M.K. AOUF, A unified presentation of some classes of meromorphically multivalent functions, *Comput. Math. Appl.*, **38** (1999), 63–70.

[15] B.A. URALEGADDI AND M.D. GANIGI, Meromorphic multivalent functions with positive coefficients, *Nep. Math. Sci. Rep.*, **11** (1986), 95–102.

[16] D. YANG AND S. OWA, Subclasses of certain analytic functions, *Hokkaido Math. J.*, **32** (2003), 127–136.