

# Journal of Inequalities in Pure and Applied Mathematics

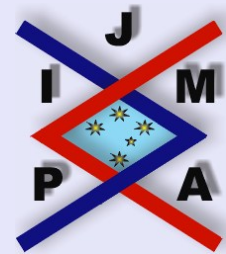
## NOTE ON INEQUALITIES INVOLVING INTEGRAL TAYLOR'S REMAINDER

ZHENG LIU

Institute of Applied Mathematics, Faculty of Science  
Anshan University of Science and Technology  
Anshan 114044, Liaoning, China

*EMail:* [lewzheng@163.net](mailto:lewzheng@163.net)

©2000 Victoria University  
ISSN (electronic): 1443-5756  
111-05



---

volume 6, issue 3, article 72,  
2005.

*Received 08 April, 2005;*  
*accepted 02 June, 2005.*

*Communicated by: H. Gauchman*

---

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

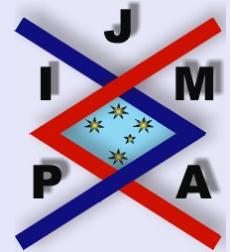
In this paper, some inequalities involving the integral Taylor's remainder are obtained by using various well-known methods.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Taylor's remainder, Leibniz formula, Variant of Grüss inequality, Taylor's formula, Steffensen inequality.

## Contents

1	Introduction .....	3
2	Results Obtained via the Leibniz Formula .....	5
3	Results Obtained by a Variant of the Grüss Inequality .....	8
4	Results Obtained via the Steffensen Inequality .....	10
	References	



---

### Note On Inequalities Involving Integral Taylor's Remainder

Zheng Liu

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 14

# 1. Introduction

In [4] – [5], H. Gauchman has derived some new types of inequalities involving Taylor’s remainder.

In [1], L. Bougoffa continued to create several integral inequalities involving Taylor’s remainder.

The purpose of this paper is to give some supplements and improvements for the results obtained in [1] – [3].

In [1], two notations  $R_{n,f}(c, x)$  and  $r_{n,f}(a, b)$  have been adopted to denote the  $n$ th Taylor’s remainder of function  $f$  with center  $c$  and the integral Taylor’s remainder respectively, i.e.,

$$R_{n,f}(c, x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k,$$

and

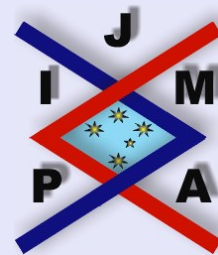
$$r_{n,f}(a, b) = \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x) dx.$$

However, it is evident that

$$R_{n,f}(a, b) = \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x) dx = r_{n,f}(a, b),$$

and

$$(-1)^n R_{n,f}(b, a) = \int_a^b \frac{(x-a)^n}{n!} f^{(n+1)}(x) dx = (-1)^n r_{n,f}(b, a).$$



---

**Note On Inequalities Involving  
Integral Taylor’s Remainder**

Zheng Liu

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 14

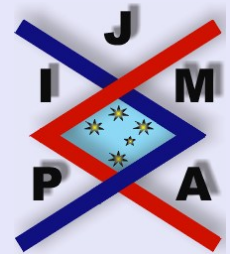
So, we would like only to keep the notation  $R_{n,f}(\cdot, \cdot)$  in what follows.

We start by changing the order of integration to give a simple different proof of Lemma 1.1 and Lemma 1.2 in [5] and [1]. i.e.,

$$\begin{aligned} \int_a^b R_{n,f}(a, x) dx &= \int_a^b \left( \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \right) dx \\ &= \int_a^b \left( \int_t^b \frac{(x-t)^n}{n!} f^{(n+1)}(t) dx \right) dt \\ &= \int_a^b \frac{(b-t)^{n+1}}{(n+1)!} f^{(n+1)}(t) dt. \end{aligned}$$

and

$$\begin{aligned} (-1)^{n+1} \int_a^b R_{n,f}(b, x) dx &= \int_a^b \left( \int_x^b \frac{(t-x)^n}{n!} f^{(n+1)}(t) dt \right) dx \\ &= \int_a^b \left( \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(t) dx \right) dt \\ &= \int_a^b \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(t) dt. \end{aligned}$$




---

**Note On Inequalities Involving  
Integral Taylor's Remainder**

Zheng Liu

---

Title Page

Contents



Go Back

Close

Quit

Page 4 of 14

## 2. Results Obtained via the Leibniz Formula

We prove the following theorem by using the Leibniz formula.

**Theorem 2.1.** Let  $f$  be a function defined on  $[a, b]$ . Assume that  $f \in C^{n+1}([a, b])$ . Then

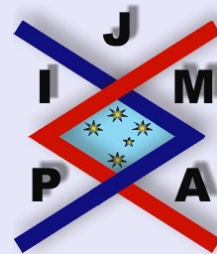
$$(2.1) \quad \left| \sum_{k=0}^p (-1)^k C_p^k R_{n-k, f}(a, b) \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(a)| \frac{(b-a)^{n-k}}{(n-k)!},$$

$$(2.2) \quad \left| \sum_{k=0}^p (-1)^{n-k+1} C_p^k R_{n-k, f}(b, a) \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(b)| \frac{(b-a)^{n-k}}{(n-k)!},$$

$$(2.3) \quad \left| \sum_{k=0}^p (-1)^k C_p^k \int_a^b R_{n-k, f}(a, x) dx \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(a)| \frac{(b-a)^{n-k+1}}{(n-k+1)!},$$

$$(2.4) \quad \left| \sum_{k=0}^p (-1)^{n-k+1} C_p^k \int_a^b R_{n-k, f}(b, x) dx \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(b)| \frac{(b-a)^{n-k+1}}{(n-k+1)!},$$

where  $C_p^k = \frac{p!}{(p-k)!k!}$ .



Note On Inequalities Involving  
Integral Taylor's Remainder

Zheng Liu

Title Page

Contents

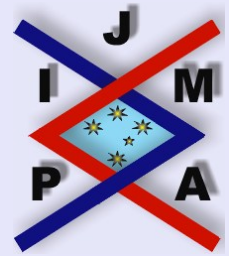


Go Back

Close

Quit

Page 5 of 14



Title Page

Contents



Go Back

Close

Quit

Page 6 of 14

*Proof.* We apply the following Leibniz formula

$$(FG)^{(p)} = F^{(p)}G + C_p^1 F^{(p-1)}G^{(1)} + \dots + C_p^{p-1} F^{(1)}G^{(p-1)} + FG^{(p)},$$

provided the functions  $F, G \in C^p([a, b])$ .

Let  $F(x) = f^{(n-p+1)}(x)$ ,  $G(x) = \frac{(b-x)^n}{n!}$ . Then

$$\left( f^{(n-p+1)}(x) \frac{(b-x)^n}{n!} \right)^{(p)} = \sum_{k=0}^p (-1)^k C_p^k f^{(n-k+1)}(x) \frac{(b-x)^{n-k}}{(n-k)!}.$$

Integrating both sides of the preceding equation with respect to  $x$  from  $a$  to  $b$  gives us

$$\begin{aligned} & \left[ \left( f^{(n-p+1)}(x) \frac{(b-x)^n}{n!} \right)^{(p-1)} \right]_{x=a}^{x=b} \\ &= \sum_{k=0}^p (-1)^k C_p^k \int_a^b f^{(n-k+1)}(x) \frac{(b-x)^{n-k}}{(n-k)!} dx. \end{aligned}$$

The integral on the right is  $R_{n-k,f}(a, x)$ , and to evaluate the term on the left hand side, we must again apply the Leibniz formula, obtaining

$$-\sum_{k=0}^{p-1} (-1)^k C_{p-1}^k f^{(n-k)}(a) \frac{(b-a)^{n-k}}{(n-k)!} = \sum_{k=0}^p (-1)^k C_p^k R_{n-k,f}(a, b).$$

Consequently,

$$\left| \sum_{k=0}^p (-1)^k C_p^k R_{n-k,f}(a, b) \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(a)| \frac{(b-a)^{n-k}}{(n-k)!},$$

which proves (2.1).

For the proof of (2.2), we take

$$F(x) = f^{(n-p+1)}(x), \quad G(x) = \frac{(x-a)^n}{n!}.$$

For the proof of (2.3), we take

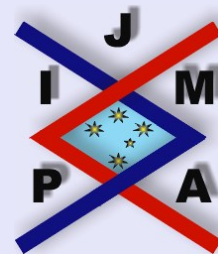
$$F(x) = f^{(n-p+1)}(x), \quad G(x) = \frac{(b-x)^{n+1}}{(n+1)!}.$$

For the proof of (2.4), we take

$$F(x) = f^{(n-p+1)}(x), \quad G(x) = \frac{(x-a)^{n+1}}{(n+1)!}.$$

□

**Remark 1.** *It should be noticed that (2.3) and (2.4) have been mentioned and proved in [1] with some misprints in the conclusion.*



---

**Note On Inequalities Involving  
Integral Taylor's Remainder**

Zheng Liu

---

Title Page

Contents



Go Back

Close

Quit

Page 7 of 14

### 3. Results Obtained by a Variant of the Grüss Inequality

The following is a variant of the Grüss inequality which has been proved almost at the same time by X.L. Cheng and J. Sun in [3] as well as M. Matić in [6] respectively.

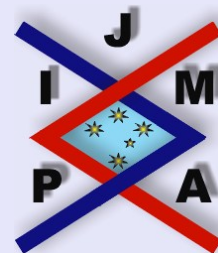
Let  $h, g : [a, b] \rightarrow \mathbb{R}$  be two integrable functions such that  $\gamma \leq g(x) \leq \Gamma$  for some constants  $\gamma, \Gamma$  for all  $x \in [a, b]$ . Then

$$(3.1) \quad \left| \int_a^b h(x)g(x) dx - \frac{1}{b-a} \int_a^b h(x) dx \int_a^b g(x) dx \right| \leq \frac{1}{2} \left( \int_a^b \left| h(x) - \frac{1}{b-a} \int_a^b h(y) dy \right| dx \right) (\Gamma - \gamma).$$

**Theorem 3.1.** Let  $f(x)$  be a function defined on  $[a, b]$  such that  $f \in C^{n+1}([a, b])$  and  $m \leq f^{(n+1)}(x) \leq M$  for each  $x \in [a, b]$ , where  $m$  and  $M$  are constants. Then

$$(3.2) \quad \left| R_{n,f}(a, b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!} (b-a)^n \right| \leq \frac{n(b-a)^{n+1}(M-m)}{(n+1)!(n+1)\sqrt[n+1]{n+1}},$$

$$(3.3) \quad \left| (-1)^{n+1} R_{n,f}(b, a) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!} (b-a)^n \right| \leq \frac{n(b-a)^{n+1}(M-m)}{(n+1)!(n+1)\sqrt[n+1]{n+1}},$$



Note On Inequalities Involving  
Integral Taylor's Remainder

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 8 of 14



$$(3.4) \quad \left| \int_a^b R_{n,f}(a, x) dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!} (b-a)^{n+1} \right| \leq \frac{(n+1)(b-a)^{n+2}(M-m)}{(n+2)!(n+2)^{n+1}\sqrt[n+2]{n+2}}$$

and

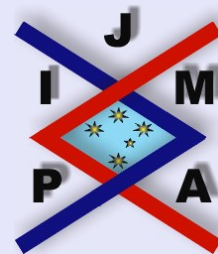
$$(3.5) \quad \left| (-1)^{n+1} \int_a^b R_{n,f}(b, x) dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!} (b-a)^{n+1} \right| \leq \frac{(n+1)(b-a)^{n+2}(M-m)}{(n+2)!(n+2)^{n+1}\sqrt[n+2]{n+2}}.$$

*Proof.* To prove (3.2), setting  $g(x) = f^{(n+1)}(x)$  and  $h(x) = \frac{(b-x)^n}{n!}$  in (3.1), we obtain

$$\begin{aligned} & \left| R_{n,f}(a, b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!} (b-a)^n \right| \\ & \leq \frac{M-m}{2} \int_a^b \left| \frac{(b-x)^n}{n!} - \frac{(b-a)^n}{(n+1)!} \right| dx \\ & = \frac{n(b-a)^{n+1}(M-m)}{(n+1)!(n+1)\sqrt[n+1]{n+1}}. \end{aligned}$$

The proofs of (3.3), (3.4) and (3.5) are similar and so are omitted.  $\square$

**Remark 2.** It should be noticed that Theorem 3.1 improves Theorem 3.1 in [1] and Theorem 2.1 in [5].



Note On Inequalities Involving  
Integral Taylor's Remainder

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 9 of 14

## 4. Results Obtained via the Steffensen Inequality

In [2] we can find a general version of the well-known Steffensen inequality as follows: Let  $h : [a, b] \rightarrow \mathbb{R}$  be a nonincreasing mapping on  $[a, b]$  and  $g : [a, b] \rightarrow \mathbb{R}$  be an integrable mapping on  $[a, b]$  with

$$\phi \leq g(x) \leq \Phi, \text{ for all } x \in [a, b],$$

then

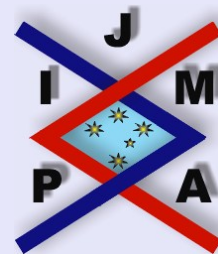
$$\begin{aligned} (4.1) \quad \phi \int_a^{b-\lambda} h(x)dx + \Phi \int_{b-\lambda}^b h(x)dx &\leq \int_a^b h(x)g(x)dx \\ &\leq \Phi \int_a^{a+\lambda} h(x)dx + \phi \int_{a+\lambda}^b h(x)dx, \end{aligned}$$

where

$$(4.2) \quad \lambda = \int_a^b G(x) dx, \quad G(x) = \frac{g(x) - \phi}{\Phi - \phi}, \quad \Phi \neq \phi.$$

**Theorem 4.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping such that  $f(x) \in C^{n+1}([a, b])$  and  $m \leq f^{(n+1)}(x) \leq M$  for each  $x \in [a, b]$ , where  $m$  and  $M$  are constants. Then

$$\begin{aligned} (4.3) \quad &\frac{m(b-a)^{n+1} + (M-m)\lambda^{n+1}}{(n+1)!} \\ &\leq R_{n,f}(a, b) \\ &\leq \frac{M(b-a)^{n+1} - (M-m)(b-a-\lambda)^{n+1}}{(n+1)!}, \end{aligned}$$



Note On Inequalities Involving  
Integral Taylor's Remainder

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 10 of 14

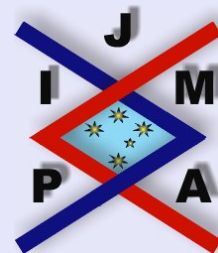
$$\begin{aligned}
 (4.4) \quad & \frac{m(b-a)^{n+1} + (M-m)\lambda^{n+1}}{(n+1)!} \\
 & \leq (-1)^{n+1} R_{n,f}(b, a) \\
 & \leq \frac{M(b-a)^{n+1} - (M-m)(b-a-\lambda)^{n+1}}{(n+1)!},
 \end{aligned}$$

$$\begin{aligned}
 (4.5) \quad & \frac{m(b-a)^{n+2} + (M-m)\lambda^{n+2}}{(n+2)!} \\
 & \leq \int_a^b R_{n,f}(a, x) dx \\
 & \leq \frac{M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}}{(n+2)!},
 \end{aligned}$$

and

$$\begin{aligned}
 (4.6) \quad & \frac{m(b-a)^{n+2} + (M-m)\lambda^{n+2}}{(n+2)!} \\
 & \leq (-1)^{n+1} \int_a^b R_{n,f}(b, x) dx \\
 & \leq \frac{M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}}{(n+2)!},
 \end{aligned}$$

where  $\lambda = \frac{f(b)-f(a)-m(b-a)}{M-m}$ .




---

**Note On Inequalities Involving  
Integral Taylor's Remainder**

Zheng Liu

---

Title Page

Contents



Go Back

Close

Quit

Page 11 of 14

*Proof.* Observe that  $\frac{(b-x)^n}{n!}$  is a decreasing function of  $x$  on  $[a, b]$ , then by (4.1) and (4.2) we have

$$\begin{aligned} m \int_a^{b-\lambda} \frac{(b-x)^n}{n!} dx + M \int_{b-\lambda}^b \frac{(b-x)^n}{n!} dx \\ \leq \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x) dx \\ \leq M \int_a^{a+\lambda} \frac{(b-x)^n}{n!} dx + m \int_{a+\lambda}^b \frac{(b-x)^n}{n!} dx \end{aligned}$$

with

$$\lambda = \int_a^b \frac{f^{(n+1)}(x) - m}{M - m} dx = \frac{f^{(n)}(b) - f^{(n)}(a) - m(b-a)}{M - m},$$

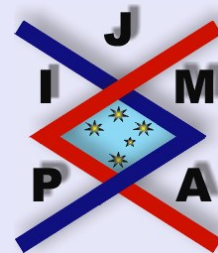
and (4.3) follows.

Since  $\frac{(x-a)^n}{n!}$  is an increasing function of  $x$  on  $[a, b]$ , then

$$\begin{aligned} M \int_a^{a+\lambda} \frac{(x-a)^n}{n!} dx + m \int_{a+\lambda}^b \frac{(x-a)^n}{n!} dx \\ \leq \int_a^b \frac{(x-a)^n}{n!} f^{(n+1)}(x) dx \\ \leq m \int_a^{b-\lambda} \frac{(x-a)^n}{n!} dx + M \int_{b-\lambda}^b \frac{(x-a)^n}{n!} dx, \end{aligned}$$

and (4.4) follows.

The proofs of (4.5) and (4.6) are similar and so are omitted.  $\square$



**Note On Inequalities Involving  
Integral Taylor's Remainder**

Zheng Liu

Title Page

Contents



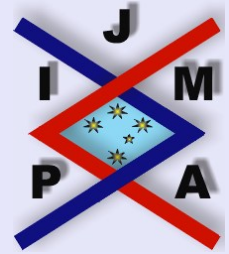
Go Back

Close

Quit

Page 12 of 14

**Remark 3.** *It should be mentioned that (4.5) and (4.6) have also been proved in [4]*



---

**Note On Inequalities Involving  
Integral Taylor's Remainder**

Zheng Liu

---

Title Page

Contents



Go Back

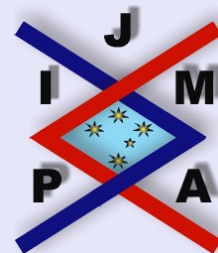
Close

Quit

Page 13 of 14

## References

- [1] L. BOUGOFFA, Some estimations for the integral Taylor's remainder, *J. Inequal. Pure and Appl. Math.*, **4**(5) (2003), Art. 86. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=327>].
- [2] P. CERONE, Generalised trapezoidal rules with error involving bounds of the nth derivative, *Math. Ineq. and Applic.*, **5**(3) (2002), 451–462.
- [3] X.L.CHENG AND J. SUN, A note on the perturbed trapezoid inequality, *J. Inequal. Pure and Appl. Math.*, **3**(2) (2002), Art. 29. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=181>].
- [4] H. GAUCHMAN, Some integral inequalities involving Taylor's remainder. I, *J. Inequal. Pure and Appl. Math.*, **3**(2) (2002), Art. 26. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=178>].
- [5] H. GAUCHMAN, Some integral inequalities involving Taylor's remainder. II, *J. Inequal. Pure and Appl. Math.*, **4**(1) (2003), Art. 1. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=237>].
- [6] M. MATIĆ, Improvement of some estimations related to the remainder in generalized Taylor's formula, *Math. Ineq. and Applic.*, **5**(4) (2002), 617–648.



---

### Note On Inequalities Involving Integral Taylor's Remainder

Zheng Liu

---

Title Page

Contents



Go Back

Close

Quit

Page 14 of 14